CHAPTER - 7

CERTAIN RESULTS INVOLVING THE POLYNOMIAL $f_n^c(x,y,r,m|q)$

7.1 APPLICATION OF AN INVERSE RELATION

In chapter-3, a basic analogue of a general class of polynomials $\{f_n^c(x,y,r,m)\}$ was obtained in the form

$$(7.4.1)\quad f_n^c(x,y,r,m|q) = \sum_{k=0}^{[n/m]} y^k \frac{[q^{-c-nr+mrk-k+1}]_\infty \, \alpha_{n-mk} \, x^{n-mk}}{[q^{-c-nr+mrk+1}]_\infty \, (q^{mr-1};\, q^{mr-1})_k}$$

together with its inverse series relation :

$$(7.1.2)\quad \alpha_n x^n = \sum_{k=0}^{[n/m]} (-y)^k \, q^{(mr-1)(k-1)k/2} \frac{(1-q^{-c-nr+mrk})[q^{-c-nr+1}]_\infty}{[q^{-c-nr+k}]_\infty (q^{mr-1};q^{mr-1})_k} \cdot f_{n-mk}^c(x,y,r,m|q) .$$

If c=d and r=r′, then (7.1.2) reads as

$$(7.1.3)\quad \alpha_n x^n = \sum_{k=0}^{[n/m]} (-y)^k \, q^{(mr'-1)(k-1)k/2} \frac{(1-q^{-d-nr'+mr'k})}{(q^{mr'-1};q^{mr'-1})_k} \cdot \frac{[q^{-d-nr'+1}]_\infty}{[q^{-d-nr'+k}]_\infty} .$$

It is interesting to see that the inverse relation (7.1.3) when combined with the polynomial

$$(7.1.4)\quad g_n^c(x,r,s;q) = \sum_{k=0}^{[n/s]} q^{(r-s)k(sk-2n+1)/2} \frac{[cq^{rk}]_{n-sk}}{(q_1;q_1)_{n-sk}} \delta_k \, x^k$$

$$(q_1 = q^{(r-s)/s}),$$

gives

$$g_n^c(x,r,s;q) = \sum_{k=0}^{[n/s]} \sum_{j=0}^{[k/s]} (-1)^j \, q^{((r-s)k(sk-2n+1)+(mr'-1)j(j-1))/2} \, .$$

$$\cdot \frac{[cq^{rk}]_{n-sk} \, (1-q^{-d-kr'+mr'j}) \quad \delta_k}{[q^{-d-kr'+j}]_\infty \, (q_1;q_1)_{n-sk} \, (q_2;q_2)_k \, \alpha_k} \, f^d_{k-mj}(x,y,r',m|q)$$

$$(q = q_1^{(r-s)/s}, q_2 = q^{mr'-1}).$$

This, with the help of the relation

$$\sum_{k=0}^{[n/s]} \sum_{j=0}^{[k/s]} A(k,j) = \sum_{k=0}^{[n/s]} \sum_{j=0}^{[(n-sk)/sm]} A(k+mj,j)$$

may be expressed as

$$(7.1.7) \quad g_n^c(x,r,s;q) = \sum_{k=0}^{[n/s]} q^{(r-s)k(sk-2n+1)/2} \, \sigma_k \quad f_k^d(x,y,r',m|q),$$

in which

$$\sigma_k = \sum_{j=0}^{[(n-sk)/sm]} q_1^{s^2kmj-smnj+smj(smj+1)/2} \, q_2^{j(j-1)/2} \, \frac{(-y)^j}{\alpha_{k+mj}} \, .$$

$$\cdot \frac{(1-q^{-d-r'k}) \qquad [cq^{rk+rmj}]_{n-smj-sk} \quad \delta_{k+mj}}{[q^{-d-kr'-mr'j+j}]_\infty \, (q_1;q_1)_{n-smj-sk} \, (q_2;q_2)_{k+mj}}$$

where, as before, $q_1 = q^{(r-s)/s}$, $q_2 = q^{mr'-1}$.

The expansion formula (7.1.7) serves as a tool through which a basic polynomial belonging to the class $\{g_n^c(x,r,s;q)\}$ may be expanded in a series of a polynomial which is contained in the class $\{f_n^c(x,y,r,m|q)\}$.

As an illustration, set $s=y=r'=1$, $r=m=2$, $d=1/2$, and also

$$\delta_k = \frac{(-1)^k}{[\alpha q]_k [q]_k}, \quad \text{and} \quad \alpha_k = \frac{(-1)^k (1+q)^k}{[q]_k},$$

and make $c \longrightarrow \infty$. Since, under the above substitutions, $g_n^c(x,r,s;q)$ reduces to the polynomial ${}_qL_n^{(\alpha)}(x)$ given by (Khan [1]).

$${}_qL_n^{(\alpha)}(x) = \frac{[\alpha q]_n}{[q]_n} \sum_{k=0}^{n} \frac{[q^{-n}]_k \, x^k}{[\alpha q]_k \, [q]_k},$$

and the inverse relation (7.1.3) gets reduced to the corresponding inverse relation of basic Legendre polynomial in the form (cf.(3.1.9)) :

$$(1+q)^n \, (-x)^n = \sum_{k=0}^{[n/2]} (-1)^k \, q^{k(k-1)/2} \, \frac{1-q^{-(1/2)-n+2k}}{[q^{-(1/2)-n+k}]_\infty [q]_k} \, P_{n-2k}(x|q),$$

it follows from (7.1.7) that

$$(7.1.8) \quad {}_qL_n^{(\alpha)}(xq) = [\alpha q]_n \sum_{k=0}^{n} q^{k(k-2n+1)/2} \, \sigma_k \, P_k(x|q),$$

where

$$(7.1.9) \quad \sigma_k = \frac{1-q^{-k-(1/2)}}{[\alpha q]_k \, [q]_{n-k} \, (1+q)^k} \sum_{j=0}^{[\frac{n-k}{2}]} (-1)^j \, q^{j(j+3)/2} \cdot \frac{[q^{-n+k}]_{2j}}{[\alpha q^{k+1}]_j \, [q^{-(1/2)-k-j}]_\infty \, [q]_j (1+q)^{2j}}.$$

Now since,

$$\frac{1}{[q^{-(1/2)-k-j}]_\infty} = \frac{[q^{-(1/2)-k}]_\infty}{[q^{-(1/2)-k-j}]_\infty \, [q^{-(1/2)-k}]_\infty} = \frac{[q^{-(1/2)-k}]_{-j}}{[q^{-(1/2)-k}]_\infty},$$

and

$$[a]_{-n} = (-1)^n q^{n(n-2a+1)/2} / [q/a]_n .$$

therefore (7.1.9) becomes

$$(7.1.10)\quad \sigma_k = \frac{1-q^{-(1/2)-k}\quad (1+q)^{-k}}{[q^{-(1/2)-k}]_\infty [\alpha q]_k\ [q]_{n-k}} \sum_{j=0}^{[\frac{n-k}{2}]} q^{j^2+kj+(5j/2)} .$$

$$\cdot \frac{(q^{-n+k};q^2)_j\ (q^{-n+k+1};\ q^2)_j}{(\alpha q^{k+1};q^2)_j\ (\alpha q^{k+2};q^2)_j\ [q^{k+(3/2)}]_j\ (1+q)^{2j}} .$$

Notice that

$$\frac{[q^{-1/2}]_\infty\ (1-q^{-k-(1/2)})}{[q^{-(1/2)-k}]_\infty\ [q^{-1/2}]_\infty} = \frac{1-q^{-(1/2)-k}}{1-q^{-1/2}}\ \frac{[q^{-1/2}]_{-k}}{[\sqrt{q}]_\infty}$$

$$= \frac{(-1)^k\ q^{k^2/2}\ (1-q^{k+(1/2)})}{[q^{3/2}]_k\ [\sqrt{q}]_\infty\ (1-\sqrt{q})} .$$

Thus,

$$\sigma_k = \frac{(-1)^k\ q^{k^2/2}\ (1-q^{k+(1/2)})}{[q^{3/2}]_k\ [\sqrt{q}]_\infty\ [\alpha q]_k\ [q]_{n-k}\ (1+q)^k\ (1-\sqrt{q})}$$

$$\cdot \sum_{j=0}^{[\frac{n-k}{2}]} \frac{q^{j^2+kj+(5j/2)}\ (q^{-n+k};q^2)_j\ (q^{-n+k+1};\ q^2)_j}{(\alpha q^{k+1};q^2)_j\ (\alpha q^{k+2};q^2)_j\ [q^{k+(3/2)}]_j (1+q)^{2j}}$$

and therefore (7.1.8) results in

$$(7.1.11)\quad {}_qL_n^{(\alpha)}(xq) = \sum_{k=0}^{n} \frac{(-1)^k\, q^{k(2k-2n+1)/2}\, [\alpha q]_n\, (1-q_1^{2k+1})}{[\alpha q]_k\, [q_1^3]_k\, [q_1]_\infty\, [q]_{n-k}\, (1-q_1)(1+q)^k} \cdot$$

$$\cdot {}_4\phi_5 \left[\begin{matrix} q_1^{-n+k},\ -q_1^{-n+k},\ q_1^{-n+k+1},\ -q_1^{-n+k+1};\ \dfrac{q^{k+(7/2)}}{(1+q)^2} \\ q_1^{\alpha+k+1}, -q_1^{\alpha+k+1}, q_1^{\alpha+k+2}, -q_1^{\alpha+k+2}, q_1^{k+(3/2)}; \end{matrix} \right] \cdot P_k(x|q),$$

$$(q_1=\sqrt{q}),$$

which provides a basic analogue of the known series expansion formula (Rainville [1,p.216]) :

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} {}_2F_3 \left[\begin{matrix} (-n+k)/2,\ (-n+k+2)/2\ ;\ 1/4 \\ k+(3/2),\ (\alpha+k+1)/2,\ (\alpha+k+2)/2\ ; \end{matrix} \right] \cdot$$

$$\cdot \frac{(-1)^k\, (1+\alpha)_n\, (2k+1)\, P_k(x)}{2^k\, (n-k)!\, (3/2)_k\, (1+\alpha)_k} .$$

Likewise, an expansion of the little q-Jacobi polynomial $p_n(x;\alpha,\beta,q)$ in a series of basic Legendre polynomial $P_n(x|q)$ ((3.1.9)) may be obtained from (7.1.7) on making use of the substitutions

$$\delta_k = \frac{(-1)^k\, [\alpha\beta q]_{2k}}{[\alpha q]_k\, [q]_k}, \quad \alpha_k = \frac{(-1)^k\, (1+q)^k}{[q]_k},$$

and the fact that

$$g_n^{1+\alpha+\beta}(x,2,1;q) = \frac{[\alpha\beta q]_{2n}}{[q]_n}\, p_n(x;\alpha,\beta;q), \qquad \text{and}$$

$$f_n^{1/2}\,(x,1,1,2|q) = \frac{P_n(x|q)}{[\sqrt{q}]_\infty} .$$

The resulting expansion formula may be stated in the form :

$$(7.1.12)\quad p_n(x;\alpha,\beta;q) = \sum_{k=0}^{n} \frac{(-1)^k\,[q]_n\,[\alpha\beta q^{n+1}]_k\,(1-q^{k+(1/2)})}{[\alpha q]_k\,[q]_{n-k}\,[q^{3/2}]_k\,[\sqrt{q}]_\infty\,(1-\sqrt{q})}$$

$$\cdot\ {}_8\phi_5\left[\begin{matrix} q_1^{-n+k},\ -q_1^{-n+k},\ q_1^{-n+k+1},\ -q_1^{-n+k+1},\ q_1^{\delta+n+k},\ -q_1^{\delta+n+k} \\ q_1^{\alpha+k+1},\ -q_1^{\alpha+k+1},\ q_1^{\alpha+k+2},\ -q_1^{\alpha+k+2} \end{matrix}\right.$$

$$\left.\begin{matrix} q_1^{\delta+n+k+1},\ -q_1^{\delta+n+k+1}\ ;\ \dfrac{q^{k+(3/2)}}{(1+q)^2} \\ q^{k+(3/2)}\ ;\ q^2 \end{matrix}\right] \frac{q^{k^2-nk+(3k/2)}}{(1+q)^k}\,P_k(x|q),$$

wherein $q_1 = \sqrt{q}$, $\delta = 1+\alpha+\beta$, and

$${}_A\phi_B\left[\begin{matrix}(a_A)\ ;\ z \\ (b_B)\ ;\ q^\lambda\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[(a_A)]_n\,z^n}{[(b_B)]_n\,[q]_n}\,q^{\lambda n(n+1)/2}$$

defines (for $\lambda>0$) a basic hypergeometric series (R.Y.Denis[1]).

7.2 CERTAIN ADDITIONAL RESULTS

Let

$$(7.2.1)\quad \Gamma_n^c(x,y,r,m|q) = \sum_{k=0}^{[n/m]} \frac{[q^{-c-nr+mrk-k+1}]_\infty\,y^k(1-q^m)^{n-mk}}{(q^{mr-1};q^{mr-1})_k\,[q]_{n-mk}}\,x^{n-mk}$$

denotes a particular case of a general class of polynomials $\{f_n^c(x,y,r,m|q)\}$ for $\alpha_k=(1-q^m)^k/[q]_k$. This polynomial when operated by the q-derivative operator $\mathcal{D}_q$ defined by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(xq)}{x(1-q)}$$

as before, yields certain interesting summation formulae which are as obtained below.

With $f(x) = \Gamma_n^c(x,y,r,m|q)$ which will be abbreviated by $\Gamma_n^c(x|q)$, one gets

$$(7.2.2) \qquad x(1-q)\, \mathcal{D}_q\, \Gamma_n^c(x|q) = \Gamma_n^c(x|q) - \Gamma_n^c(xq|q).$$

But since

$$\mathcal{D}_q\, \Gamma_n^c(x|q) = \sum_{k=0}^{[\frac{n-1}{m}]} y^k \frac{[q^{1-c-r-(n-1)r+mrk-k}]_\infty\, (x(1-q))^{n-1-mk}}{(q^{mr-1};\, q^{mr-1})_k\, [q]_{n-1-mk}}$$

$$= \Gamma_{n-1}^{c+r}(x|q),$$

the expression in (7.2.2) becomes

$$(7.2.3) \qquad x(1-q)\, \Gamma_{n-1}^{c+r}(x|q) = \Gamma_n^c(x|q) - \Gamma_n^c(xq|q) \quad (n\geq 1).$$

By repeated application of (7.2.3), one obtains

$$(7.2.4) \qquad \Gamma_N^c(x|q) = \sum_{k=0}^{N} \Gamma_{N-k}^{c+k+r}(xq|q)\, (x(1-q))^k,\ N=1,2,3,\ldots\ .$$

Further, it may be observed that

$$x^p(1-q)^p\, \mathcal{D}_q^p\, \Gamma_n^c(x|q) = (-1)^p\, q^{-p(p-1)/2} \sum_{k=0}^{p} (-1)^k\, q^{k(k-1)/2}\ .$$

$$.\ \begin{bmatrix} p \\ k \end{bmatrix}\ \Gamma_n^c(xq^{p-k}|q)\ ,$$

which in view of

$$(7.2.5) \qquad \mathcal{D}_q^p\, \Gamma_n^c(x|q) = \Gamma_{n-p}^{c+pr}(x|q)$$

furnishes the summation formula :

$$(7.2.9)\quad x^{p}(1-q)^{p}\,\Gamma^{c+pr}_{n-p}(x|q) = \sum_{k=0}^{p} (-1)^{k}\, q^{k(k-2p+1)/2} \left[{p \atop k} \right] \Gamma^{c}_{n}(xq^{k}|q).$$

This evidently provides a generalization of (7.2.3).

It is also seen easily that (7.2.4) and (7.2.5) yield the formula :

$$(7.2.7)\quad \Gamma^{c+pr}_{N-p}(x|q) = \sum_{k=0}^{N-p} \Gamma^{c+pr+kr}_{N-p-k}(xq|q)\ (x(1-q))^{k}.$$