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A q-ANALOGUE OF A GENERAL CLASS OF POLYNOMIALS AND ITS INVERSE

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Recently, Singhal and Savita Kumari [*Indian J. pure appl. Math.* 13 (8) (1982), 907-11] established an elegant inversion theorem which provides an effective tool to obtain the inverse relation for a class of polynomials $\{g_n^c(x, r, s)\}$, introduced earlier by R. Panda [*Glasgow Math. J.* 18 (1977), 177-84].

We have defined in this paper a q-analogue of the class $\{g_n^c(x, r, s)\}$ and discussed the inverse relations under certain conditions on the parameters involved.

1. INTRODUCTION

A few years ago, some known polynomials like Jacobi, Laguerre, Hermite etc. were unified by the introduction of a new class of polynomials $\{g_n^c(x, r, s)\}$ by Panda[1].

These polynomials are defined by means of the explicit representation

$$g_n^c(x, r, s) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{-c-rk}{n-sk} \gamma_k x^k. \qquad ...(1.1)$$

The study of this class of polynomials was further extended by Singhal and Savita Kumari[4,6] who studied these polynomials from the viewpoint of inverse series relations. They proved that (1.1) admits the inverse relation

$$x^n = \frac{1}{\gamma_n} \sum_{k=0}^{sn} \frac{c+rk/s}{c+rn-sn+k} \binom{-c+sn-rn-k}{sn-k} g_k^c(x,r,s). \qquad ...(1.2)$$

In fact, the inverse pair of relations (1.1) and (1.2) are contained in the following general theorem due to Singhal and Savita Kumari[4].

Theorem —

$$F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} f(k) \qquad ...(1.3)$$

implies

$$f(n) = \sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k). \qquad ...(1.4)$$

In this paper we first define a q-analogue of (1.1) in the form

$$g_n^c\,(x,\, r,\, s|q) = \sum_{k=0}^{[n/s]} (-1)^{n-sk}\, q^{(sk(sk+1)-2snk)/2}$$

$$\times \frac{[q^{-c+sk-rk-n+1}]_\infty}{[q^{-c-rk+1}]_\infty\,[q]_{n-sk}}\,\delta_k\,x^k \qquad \text{...(1.5)}$$

and prove two inversion theorems suggested by the above cited theorem and thereby deduce the inverse relations for (1.5).

In what follows, we shall make use of the notations

$$[a]_n = \begin{cases} (1-a)\,(1-aq)\,\ldots\,(1-aq^{n-1}), & n = 1, 2, \ldots \\ 1, & n = 0 \\ \dfrac{[a]_\infty}{[aq^n]_\infty}, & n \text{ is arbitrary} \end{cases} \qquad \text{...(1.6)}$$

$$[a]_\infty = \prod_{n=0}^{n} (1-aq^n),\ 0 < q < 1 \qquad \text{...(1.7)}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[q]_n}{[q]_{n-k}\,[q]_k} \qquad \text{...(1.8)}$$

and the known result (Pólya and Szegö[2], p. 11, Problem 60.3)

$$\sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} (-x)^k = \prod_{k=1}^{n} (1-xq^{k-1}) \qquad \text{...(1.9)}$$

which is a particular case of the basic binomial theorem [Srivastava and Karlsson[8], p. 348, eqn. 9.4 (274)]

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{[q]_k}\,x^k = \frac{[\lambda x]_\infty}{[x]_\infty}\ (|x|<1). \qquad \text{...(1.10)}$$

2. Inversion Theorems

The inversion theorems that we propose to prove here are as given below.

Theorem I—

$$F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk}\, q^{sk(sk+1)/2-snk}\, \frac{[pq^{bsk-n+1}]_\infty}{[pq_1^{bsk-sk+1}]_\infty}\, \frac{G(k)}{[q]_{n-sk}} \qquad \text{...(2.1)}$$

implies

$$G(n) = \sum_{k=0}^{sn} q^{k(k-1)/2}\, \frac{[pq_1^{bsn-sn+1}]_\infty}{[pq^{bsn-k}]_\infty\,[q]_{sn-k}}\, F(k) \qquad \text{...(2.2)}$$

where $b \leqslant 1$, and q_1 is another base.

Theorem II–

$$f(n) = \sum_{k=0}^{n} (-1)^k q_1^{k(k+1)/2-nk} \frac{[cq^{rk}]_\infty}{[cq^{rk-k+n}]_\infty (q_1; q_1)_{n-k}} g(k) \quad ...(2.3)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^k q_1^{k(k-1)/2} \frac{(1 - cq^{rk}) [cq^{rn-n+k+1}]_\infty}{[cq^{rn}]_\infty (q_1 ; q_1)_{n-k}} f(k) \quad ...(2.4)$$

wherein $q_1 = q^{r-1}$, $r \neq 1$.

In order to prove Theorem I, we observe that in view of (2.1), the right hand member of (2.2), denoted for brevity by Φ, can be expressed in the form

$$\Phi = \sum_{k=0}^{sn} \frac{q^{k(k-1)/2} [pq_1^{bsn-sn+1}]_\infty}{[pq^{bsn-k}]_\infty [q]_{sn-k}} \sum_{j=0}^{[k/s]} (-1)^{k-sj} q^{sj(sj+1)/2-skj}$$

$$\times \frac{[pq^{bsj-k+1}]_\infty}{[pq_1^{bsj-sj+1}]_\infty [q]_{k-sj}} G(j)$$

$$= \sum_{j=0}^{n} \frac{[pq_1^{bsn-sn+1}]_\infty}{[pq_1^{bsn-sj+1}]_\infty} G(j) \sum_{k=0}^{sn-sj} (-1)^k q^{k(k-1)/2} \begin{bmatrix} sn - sj \\ k \end{bmatrix}$$

$$\times \frac{[pq^{bsj-sj-k+1}]_\infty}{[pq^{bsn-sj-k}]_\infty}$$

$$= \sum_{j=0}^{n} \frac{[pq_1^{bsn-sn+1}]_\infty G(j)}{[pq_1^{bsj-sj+1}]_\infty [q]_{sn-sj}} \sum_{m=0}^{bsn-bsj-1} (-1)^m A_m$$

$$\times \sum_{k=0}^{sn-sj} (-1)^k q^{k(k-1)/2} \begin{bmatrix} sn - sj \\ k \end{bmatrix} q^{-mk}$$

where $A_0 = 1$.

Now making an appeal to (1.9), we get

$$\Phi = \sum_{j=0}^{n} \frac{[pq_1^{bsn-sn+1}]_\infty G(j)}{[pq_1^{bsj-sj+1}]_\infty [q]_{sn-sj}} \sum_{m=0}^{bsn-bsj-1} (-1)^m A_m \prod_{k=1}^{sn-sj} (1-q^{-m+k-1})$$

$$= \begin{cases} 0, \text{ if } b \leq 1, \text{ and } j \neq n \\ G(n), \text{ when } j=n, \end{cases}$$

which completes the proof of Theorem I.

Theorem II is a special case of our result given in Theorem 1 of Singhal and Dave[7] therefore we merely give the outlines of its proof as follows.

Since the matrix corresponding to (2.3) has its all diagonal elements nonzero, it has unique inverse and because (2.4) satisfies (2.3), it follows that (2.3) and (2.4) are inverses of each other.

3. Particular Cases

On making the substitutions $b = (s-r)/s$, $q_1 = q$, $p = -c$, and $G(k) = \delta_k x^k$ in (2.1) and comparing with the defining relation (1.5), we get $F(n) = g_n^c(x, r, s|q)$, and consequently (2.2) would yield its inverse relation in the form

$$x^n = \frac{1}{\delta_n} \sum_{k=0}^{sn} \frac{q^{k(k-1)/2} [q^{-c-rn+1}]_\infty}{[q^{-c+sn-rn-k}]_\infty [q]_{sn-k}} g_k^c(x, r, s|q). \quad ...(3.1)$$

The particular case of (3.1) corresponding to the reducibility of (1.5) under the substitutions $c = 1+\alpha$, $r = s = 1$ and $\delta_k = (-1)^k q^{-k\alpha-k(k+1)/2} \{[q]_k\}^{-1}$, viz.

$$L_n^{(\alpha)}(x|q) = \sum_{k=0}^{n} \frac{q^{k(k+1)/2-nk} [\alpha q]_n}{[\alpha q]_k [q]_{n-k} [q]_k} (-x)^k \quad ...(3.2)$$

may be given as

$$x^n = [q]_n \sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)/2} [\alpha q]_n}{[\alpha q]_k [q]_{n-k}} L_k^{(\alpha)}(x|q) \quad ...(3.3)$$

where $L_n^{(\alpha)}(x|q)$ is a basic Laguerre polynomial.

In fact, (3.2) holds if and only (3.3) holds.

On the other hand, taking $s=1$, $r=2$ and replacing x by $4x$ in (1.5) and comparing it with (2.3) one readily gets the following pair.

$$f_n(x|q) = \sum_{k=0}^{n} (-1)^k q^{k(k+1)/2-nk} \frac{[cq^{2k}]_{n-k}}{[q]_{n-k}} \delta_k (4x)^k \quad ...(3.4)$$

if and only if

$$x^n = \frac{[c]_{2n}}{\delta_n 4^n} \sum_{k=0}^{n} (-1)^k \frac{q^{k(k-1)/2} (1-cq^{2k})}{[c]_{n+k+1} [q]_{n-k}} f_k(x|q) \quad ...(3.5)$$

in which $f_n(x|q)$ is a q-analogue of the polynomial $f_n(x|q)$ considered by Rainville[3] (p. 137).

Finally, we consider the polynomial

$$f_n^c(x, y, r, m) = \sum_{k=0}^{[n/m]} y^k \binom{-c-rn+mk}{n-mk} \gamma_{n-mk} x^{n-mk}. \quad ...(3.6)$$

introduced by Singhal and Savita Kumari[5]. If we define its q-analogue for $m=1$

as follows :

$$f_n^c\ (x,\ y,\ r,\ 1|q) = \sum_{k=0}^{n} y^{n-k}\ q_1^{(k-n)(k+n-1)/2}\ \frac{[cq^{rn-rk}]_k}{(q_1\ ;\ q_1)_k} \times \alpha_{n-k}\ x^{n-k} \qquad \ldots(3.7)$$

where $q_1 = q^{r-1}$, $r \neq 1$, then it is easy to see that the choice $g(k) = \alpha_k\ x^k$ in Theorem II yields the interesting relation

$$f_n^c\ (x,\ -1,\ r,\ 1|q) = g_n^c\ (x,\ r,\ 1|q).$$

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ON A q-INVERSION THEOREM

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ABSTRACT

In this paper, an attempt has been made to establish a q-inversion theorem, which on particularization yields the inverse relations for several classes of polynomials including the q-Hahn polynomials due to W. Hahn [3], and the q-analogue of the polynomials $W_n^{(\alpha,\beta)}(x\,;k)$ considered by J. P. Singhal and Savita Kumari [5].

1. INTRODUCTION

As long ago as 1949, W. Hahn [3] introduced a new class of polynomials known as q-Hahn polynomials by means of the relation

$$(1{\cdot}1)\quad Q_n(x\,;\alpha,\beta,N\mid q)=\sum_{j=0}^{n}\frac{[q^{-n}]_j\,[\alpha\beta q^{n+1}]_j\,[q^{-x}]_j}{[\alpha q]_j\,[q^{-N}]_j\,[q]_j}\,q^j.$$

When $q\to 1$, $Q_n(x\,;\alpha,\beta,N\mid q)$ reduces to the ordinary Hahn polynomials $Q_n(x\,;\alpha,\beta,N)$ represented by

$$(1{\cdot}2)\quad Q_n(x\,;\alpha,\beta,N)=\sum_{j=0}^{n}\frac{(-n)_j\,(\alpha+\beta+n+1)_j(-x)_j}{(\alpha+1)_j\,(-N)_j\,j\,!}.$$

where $Re(\alpha,\beta)>-1$, N a non-negative integer, and $n=0,1\ldots,N$. These polynomials include Jacobi polynomials, Meixner, Krowtchouk, Charlier, Laguerre polynomials and Bessel functions as special cases.

G. Gasper [2] obtained the inverse relation of (1·2) in the form

$$(1{\cdot}3)\quad (-x)_n=(-N)_n\,(\alpha+1)_n\sum_{j=0}^{n}\frac{(-n)_j\,(\alpha+\beta+1+2j)\,Q_j(x\,;\alpha,\beta,N)}{(\alpha+\beta+1+n+j)\,(\alpha+\beta+1+j)_n\,j\,!}.$$

It would be of interest to look for an analogous inverse relation of (1·1). Our attempt in this direction led us to a general q-inversion theorem which not only yields the inverse relation of (1·1) but also includes the inverse relation of another interesting class of q-polynomials defined as

$$(1\cdot4) \quad W_n^{(\alpha,\beta)}(x\,;\,k \mid q)=\frac{[\alpha q]_n}{(q^k\,;\,q^k)_n}\sum_{j=0}^{n}\frac{(q^{-kn}\,;\,q^k)_j\,[\alpha\beta q^{n+1}]_{kj}}{[\alpha q]_{kj}\,(q^k\,;\,q^k)_j}\cdot\left(\frac{q(1-x)}{2}\right)^{kj},$$

where $Re(\alpha, \beta) > -1$, and k is a positive integer ; which defines the q-analogue of the polynomials $W_n^{(\alpha,\beta)}(x\,;\,k)$ representable in the form

$$(1\cdot5) \quad W_n^{(\alpha,\beta)}(x\,;\,k)=\frac{(\alpha+1)_n}{n\,!}\sum_{j=0}^{n}\frac{(-n)_j\,(\alpha+\beta+n+1)_{kj}}{(\alpha+1)_{kj}\,j\,!}\left(\frac{1-x}{2}\right)^{kj},$$

where $Re(\alpha, \beta) > -1$, and k is a positive integer;

The polynomial set $\{W_n^{(\alpha,\beta)}(x\,;\,k)\}$ and its companion set $\{X_n^{(\alpha,\beta)}(x;\,k)\}$ which form a pair of biorthogonal polynomials associated with Jacobi weight function, were introduced by Singhal and Savita Kumari [5] (see also [4]) who also gave the inverse of (1·5) in the form

$$(1\cdot6) \quad \left(\frac{1-x}{2}\right)^{kn}=(\alpha+1)_{kn}\sum_{j=0}^{n}\frac{(-n)_j\,(\alpha+\beta+1+kj+j)}{(\alpha+1)_j\,(\alpha+\beta+1+j)_{kn+1}}\,W_j^{(\alpha,\beta)}(x\,;\,k).$$

For obtaining the general q-inversion theorem we consider the following extensions of (1·1) and (1·4).

$$(1\cdot7) \quad Q_{n,\nu}^{\mu,\lambda,\delta}(x\,;\,\alpha,\beta,\mathrm{N} \mid q)=\sum_{j=0}^{n}\frac{(q^{-\mu n}:\,q^{\mu})_j\,[\beta q^n]_\infty\,[\alpha q^{\lambda j}]_\infty\,[q^{-N+\delta j}]_\infty}{[\alpha]_\infty\,[\beta q^{n+\mu j}]_\infty\,[q^{-N}]_\infty\,(q^{\mu}\,;\,q^{\mu})_j}\,\frac{[q^{-x}]_\infty}{[q^{-x+\nu j}]_\infty}\,q^{\mu j},$$

where $Re(\alpha, \beta, \mu) > 0$, and $n=0, 1, \ldots, N$.

$$(1\cdot8) \quad P_{n,\nu}^{\mu,\lambda}(\alpha,\beta,\gamma,\delta\,;\,x \mid q)=\frac{[\alpha]_\infty}{[\alpha q^{\lambda n}]_\infty\,(q^{\mu}\,;\,q^{\mu})_n}\sum_{j=0}^{n}\frac{(q^{-\mu n}\,;\,q^{\mu})_j}{(q^{\mu}\,;\,q^{\mu})_j}\cdot\frac{[\gamma q^{\delta j}]_\infty\,[\beta q^n]_\infty\,q^{\mu j}}{[\gamma]_\infty\,[\beta q^{\mu j+n}]_\infty}\left(\frac{1-x}{2}\right)^{\nu j}.$$

where $Re(\alpha, \beta, \gamma, \mu) > 0$.

Obviously, (1·7) would reduce to (1·1) when $\lambda=\nu=\mu=\delta=1$ and α and β are replaced by αq and $\alpha\beta q$, respectively; whereas (1·8) would reduce to (1·4) when $\nu=\mu=\delta=k$, $\lambda=1$, and both α and γ are replaced by αq, and β is replaced by $\alpha\beta q$.

In what follows, we shall make use of the following notations :

$$(1\cdot 9) \quad (a\,;\,q)_n=[a]_n=\begin{cases}(1-a)\,(1-aq)\ldots(1-aq^{n-1}), & n=1,2,\ldots\\ 1, & n=0\\ \dfrac{[a]_\infty}{[aq^n]_\infty}\,, & n \text{ is arbitrary complex,}\end{cases}$$

where $[a]_\infty=\prod_{n=0}^{\infty}(1-aq^n),\ 0<q<1.$

$$(1\cdot 10) \qquad \begin{bmatrix} n \\ k \end{bmatrix}_\delta=\frac{(1-q^{n\delta})\,(1-q^{(n-1)\delta})\ldots(1-q^{(n-k+1)\delta})}{(1-q^{\delta})\,(1-q^{2\delta})\ldots(1-q^{k\delta})},\ \delta>0$$

and $\begin{bmatrix} n \\ k \end{bmatrix}_\delta=\begin{bmatrix} n \\ k \end{bmatrix}$, when $\delta=1$.

$$(1\cdot 11) \qquad \sum_{k=0}^{n} q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix} x^k=\prod_{k=1}^{n}(1+xq^{k-1}),$$

$$(1\cdot 12) \qquad \Gamma_q(x)=\frac{[q]_\infty}{[q^x]_\infty}\,(1-q)^{1-x},\ 0<q<1.$$

2. THE INVERSION THEOREM

The general inversion theorem that we propose to prove may be stated in the form

Theorem 1. For $n=0, 1, \ldots\ldots,$

$$(2\cdot 1) \qquad F(n)=\sum_{j=0}^{n}(-1)^j\,q^{\mu j(j+1)/2-\mu nj}\,\frac{(\alpha q_1^{\lambda j}\,;\,q_1)_r\,(\gamma q_2^{\delta j}\,;\,q_2)_s}{[\beta q^{\mu j+n}]_{m-n+j}\,(q^{\mu}\,;\,q^{\mu})_{n-j}}\,f(j)$$

if and only if

$$(2\cdot 2) \qquad f(n)=\sum_{j=0}^{n}(-1)^j\,q^{\mu j(j-1)/2}\,\frac{1-\beta q^{\mu j+j}}{1-\beta q^{\mu n+j}}\,\frac{[\beta q^{\mu n+j}]_{m+n-j}}{(\alpha q_1^{\lambda n};\,q_1)_r(\gamma q_2^{\delta n}\,;\,q_2)_s}\,\frac{F(j)}{(q^{\mu};q^{\mu})_{n-j}},$$

where $Re(\alpha, \beta, \gamma, \mu)>0.$

The proof of this theorem runs parallel to the method used by Carlitz [1] which we summarize as below :

As the diagonal elements

$$(-1)^n\,q^{-\mu n(n-1)/2}\,\frac{(\alpha q_1^{\lambda n}\,;\,q_1)_r\,(\gamma q_2^{\delta n}\,;\,q_2)_s}{[\beta q^{\mu n+n}]_m}$$

of the matrix formed by (2·1), are all non-zero $\forall\ n=0, 1, \ldots$, it has unique inverse. So, if we show that (2·2) satisfies (2·1), then (2·1) and (2·2) would become inverse of each other.

On substituting (2·2) into the right hand member of (2·1), which we denote as ψ, we get

$$\psi=\sum_{j=0}^{n}(-1)^{j}\,q^{\mu j(j+1)/2-\mu nj}\,\frac{(\alpha q_1^{\lambda j};q_1)_r\,(\gamma q_2^{\delta j};q_2)_s}{[\beta q^{\mu j+n}]_{m-n+j}\,(q^{\mu};q^{\mu})_{n-j}}$$

$$\cdot\sum_{l=0}^{j}(-1)^{l}\,q^{\mu l(l-1)/2}\,\frac{1-\beta q^{\mu l+l}}{1-\beta q^{\mu j+l}}\;\frac{[\beta q^{\mu j+l}]_{m+j-l}}{(\alpha q_1^{\lambda j};q_1)_r\,(\gamma q_2^{\delta j};q_2)_s}\cdot\frac{F(l)}{(q^{\mu};q^{\mu})_{j-l}},$$

which may be put in the form

$$(2{\cdot}3)\quad \psi=F(n)+\sum_{l=0}^{n-1}q^{\mu l^2-\mu ln}\,\frac{1-\beta q^{\mu l+l}}{(q^{\mu};q^{\mu})_{n-l}}\,F(l)\cdot$$

$$\cdot\sum_{j=0}^{n-l}(-1)^{j}\,q^{\mu j(j-1)/2+\mu j(l-n+1)}\begin{bmatrix}n-l\\ j\end{bmatrix}_{\mu}\cdot\frac{[\beta q^{\mu j+\mu l+l+1}]_{m+j-1}}{[\beta q^{\mu j+\mu l+n}]_{m-n+j+l}}.$$

Since,

$$\frac{[\beta q^{\mu j+\mu l+l+1}]_{m+j-1}}{[\beta q^{\mu j+\mu l+n}]_{m-n+j+l}}=1-\{q^{\beta+\mu l}\,(q^{l+1}+q^{l+2}+\ldots+q^{n-1})\}\,q^{\mu j}+$$
$$+\{q^{2\beta+2\mu l}\,(q^{2l+3}+q^{2l+5}+\ldots+q^{2n-3})\}\,q^{2\mu j}+\ldots+(-1)^{n-l-1}$$
$$\cdot\{q^{(n-l-1)(\beta+\mu l)+(n-l-1)(l+1+l+2\ldots+(n-1))}\}\,q^{(n-l-1)\mu j}$$

$$=\sum_{i=0}^{n-l-1}(-1)^{i}\,A_i^{\beta,\mu,n,l}\,q^{\mu ij},$$

where $A_0^{\beta,\mu,n,l}=1$, consequently the right hand member of (2·3) assumes the form

$$F(n)+\sum_{l=0}^{n-l}q^{\mu l^2-\mu ln}\,\frac{1-\beta q^{\mu l+l}}{(q^{\mu};q^{\mu})_{n-1}}\,F(l)\sum_{i=0}^{n-l-1}(-1)^{i}\,A_i^{\beta,\mu,n,l}.$$

$$\sum_{j=0}^{n-l}(-1)^{j}\,q^{\mu j(j-1)/2}\begin{bmatrix}n-l\\ j\end{bmatrix}_{\mu}q^{(l-n+1+i)\mu j},$$

wherein the use of (1·11) transforms its inner-most sum to

$$\prod_{j=1}^{n-l}(1-q^{(l-n+i+j)\mu}),$$

which vanishes when $i=0, 1, \ldots, n-l-1$.

Hence the theorem.

The rotated version of this theorem may be stated as follows.

Theorem 2. For $n=0, 1, \ldots,$

$$(2\cdot4) \qquad F(n)=\sum_{j=n} (-1)^n q^{\mu n(n+1)/2-\mu nj} \frac{(\alpha q_1{}^{\lambda n};\, q_1)_r\, (\gamma q_2{}^{\delta n};\, q_2)_s}{[\beta q^{\mu n+j}]_{m+n-j}} \cdot \frac{f(j)}{(q^\mu\,;\, q^\mu)_{j-n}}$$

if and only if

$$(2\cdot5) \quad f(n)=\sum_{j=n} (-1)^n q^{\mu n(n-1)/2} \frac{1-\beta\, q^{\mu n+n}}{1-\beta\, q^{\mu j+n}} \frac{[\beta\, q^{\mu j+n}]_{m-n+j}}{(\alpha\, q_1{}^{\lambda j};q_1)_r} \cdot$$

$$\cdot \frac{F(j)}{(\gamma\, q_2{}^{\delta j};q_2)_s\, (q^\mu\,;\, q^\mu)_{j-n}},$$

where Re $(\alpha, \beta, \gamma, \mu) > 0$.

3. PARTICULAR CASES

On making m, r and s tend to infinity, and substituting $\gamma=-N$, $q_1=q_2=q$, in theorem 1, we obtain

$$(3\cdot1) \quad F(n)=\sum_{j=0}^{n} (-1)^j q^{\mu j(j+1)/2-\mu nj} \frac{[\alpha\, q^{\lambda j}]_\infty\, [q^{-N+\delta j}]_\infty}{[\beta\, q^{\mu+\mu j}]_\infty\, (q^\mu\,;\, q^\mu)_{n-j}} f(j)$$

if and only if

$$(3\cdot2) \quad f(n)=\sum_{j=0}^{n} (-1)^j q^{\mu j\,(j-1)/2} \frac{1-\beta q^{\mu j+j}}{1-\beta q^{\mu n-j}} \frac{[\beta q^{\mu n+j}]_\infty}{[\alpha q^{\lambda n}]_\infty\, [q^{-N+\delta n}]_\infty} \frac{F(j)}{(q^\mu\,;q^\mu)_{n-j}}.$$

Choosing $F(n)=\dfrac{[q^{-N}]_\infty\, [\alpha]_\infty}{[\beta q^n]_\infty\, (q^\mu\,;\, q^\mu)_n}\; Q_{n,\,\nu}^{\mu,\,\lambda,\,\delta}(x;\, \alpha,\, \beta,\, N|q)$,

and $f(n)=\dfrac{[q^{-x}]_\infty}{[q^{-x+\nu n}]_\infty\, (q^\mu\,;\, q^\mu)_n}$, we are led to the inverse of (1·7) in the form

$$(3\cdot3) \quad \frac{[q^{-x}]_\infty}{[q^{-x+\nu n}]_\infty} = \frac{[q^{-N}]_\infty\, [\alpha]_\infty}{[q^{-N+\delta n}]_\infty [\alpha q^{\lambda n}]_\infty} \sum_{j=0}^{n} q^{\mu nj} \frac{1-\beta q^{\mu j+j}}{1-\beta q^{\mu n+j}} \cdot$$

$$\cdot \frac{(q^{-\mu n}\,;\, q^\mu)_j\, [\beta q^{\mu n+j}]_\infty}{[\beta q^j]_\infty\, (q^\mu\,;\, q^\mu)_j}\; Q_{j,\nu}^{\mu,\lambda,\delta}\; (x\,;\, \alpha,\, \beta,\, N \mid q),$$

which, on setting $\nu=\mu=\lambda=\delta=1$, and replacing α and β by αq and $\alpha\beta q$ respectively, simplifies to the inverse of (1·1) in the form

$$(3\cdot4) \quad [q^{-x}]_n=[q^{-N}]_n\, [\alpha q]_n \sum_{j=0}^{n} q^{nj} \frac{[q^{-n}]_j\, (1-\alpha\beta q^{2j+1})}{[\alpha\beta q^{j+1}]_{n+1}\, [q]_j} \cdot Q_j\, (x\,;\, \alpha,\, \beta,\, N \mid q).$$

Next, if we take $r=0$, $q_2=q$, and make s and m tend to infinity in theorem 1, we obtain

$$(3{\cdot}5)\quad F(n)=\sum_{j=0}^{n}(-1)^j\, q^{\lambda j(j+1)/2-\mu nj}\,\frac{[\gamma q^{\delta j}]_\infty}{[\beta q^{\mu j+n}]_\infty\,(q^\mu\,;\,q^\mu)_{n-j}}\,f(j)$$

if and only if

$$(3{\cdot}6)\quad f(n)=\sum_{j=0}^{n}(-1)^j\, q^{\mu j(j-1)/2}\,\frac{1-\beta q^{\mu j+j}}{1-\beta q^{\mu n+j}}\,\frac{[\beta q^{\mu n+j}]_\infty}{[\gamma q^{\delta n}]_\infty\,(q^\mu\,;\,q^\mu)_{n-j}}\,F(j).$$

For $F(n)=\dfrac{[\alpha q^{\lambda n}]_\infty\,[\gamma]_\infty}{[\alpha]_\infty\,[\beta q^n]_\infty}\,P_{n,\nu}^{\mu,\lambda}(\alpha,\beta,\gamma,\delta\,;x\mid q)$ and

$f(n)=\dfrac{(1-x)^{\nu n}}{2^{\nu n}\,(q^\mu\,;\,q^\mu)_n}$, (3·5) gets transformed into (1·8), whereas

(3·6) yields the corresponding inverse relation

$$(3{\cdot}7)\quad \left(\frac{1-x}{2}\right)^{\nu n}=\frac{[\gamma]_\infty}{[\gamma q^{\delta n}]_\infty}\sum_{j=0}^{n} q^{\mu nj}\,\frac{(q^{-\mu n}\,;\,q^\mu)_j\,(1-\beta q^{\mu j+j})}{[\beta q^j]_\infty\,[\alpha]_\infty}\,.$$

$$[\beta q^{\mu n+j+1}]_\infty[\alpha q^{\lambda j}]_\infty\,P_{j,\nu}^{\mu,\lambda}(\alpha,\beta,\gamma,\delta\,;x\mid q).$$

(3·7) can be further particularized by taking $\lambda=1$, $\mu=\delta=\nu=k$, and replacing α and γ both by αq, and β by $\alpha\beta q$, to the inverse series of $W_n^{(\alpha,\beta)}(x\,;k\mid q)$ as given below :

$$(3{\cdot}8)\quad \left(\frac{1-x}{2}\right)^{kn}=[\alpha q]_{kn}\sum_{j=0}^{n} q^{knj}\,\frac{(q^{-kn}\,;\,q^k)_j\,(1-\alpha\beta q^{kj+j+1})}{[\alpha\beta q^{j+1}]_{kn+1}\,[\alpha q]_j}\,.$$

$$\cdot\, W_j^{(\alpha,\beta)}(x\,;k\mid q).$$

4. LIMITING CASE

Theorem 1, may be stated in a slightly different form as given below :

Theorem 3. For $n=0,1,\ldots,$

$$(4{\cdot}1)\quad F(n)=\sum_{j=0}^{n}(-1)^j\, q^{\mu j(j+1)/2-\mu nj}\,\frac{(\alpha q_1^{\lambda j}\,;\,q_1)_r\,(\gamma q_2^{\delta j}\,;\,q_2)_s}{[\beta q^{\mu j+n}]_{m-n+j}}\,.$$

$$\frac{(1-q_1)^{\lambda j}\,(1-q_2)^{\delta j}}{(1-q)^{\mu j}\,(q^\mu\,;\,q^\mu)_{n-j}}\,f(j)$$

if and only if

$$(4\cdot2)\quad f(n)=\sum_{j=0}^{n}(-1)^j q^{\mu j(j-1)/2}\frac{1-\beta q^{\mu j+j}}{1-\beta q^{\mu n+j}}\frac{[\beta q^{\mu n+j}]_{m+n-j}}{(\alpha q_1^{\lambda n};q_1)_r\,(1-q_1)^{\lambda n}}\cdot\frac{(1-q)^{\mu n}}{(\gamma q_2^{\delta n};q_2)_s\,(1-q_2)^{\delta n}}F(j).$$

Replacing $F(n)$ and $f(n)$ by

$$\frac{(1-q)^{\beta+n-1}[q_1]_\infty[q_2]_\infty F(n)}{(1-q_1)^{\alpha-1}(1-q_2)^{\gamma-1}[q]_\infty (q^\mu;q^\mu)_n}\text{ and }\frac{f(n)}{(q^\mu;q^\mu)_n}$$

respectively, and then making r, s and m tend to infinity we, on making use of (1·12) get

$$(4\cdot3)\quad F(n)=\sum_{j=0}^{n}q^{\mu j}\frac{(q^{-\mu n};q^\mu)_j\,\Gamma q(\beta+n+\mu j)}{\Gamma q_1(\alpha+\lambda j)\,\Gamma q_2(\gamma+\delta j)\,(q^\mu;q^\mu)_j}f(j)$$

if and only if

$$(4\cdot4)\quad f(n)=\sum_{j=0}^{n}q^{\mu jn}\frac{1-\beta q^{\mu j+j}}{1-\beta q^{\mu n+j}}\frac{(q^{-\mu n};q^\mu)_j\,\Gamma q_1(\alpha+\lambda n)}{(q^\mu;q^\mu)_j\,\Gamma q(\beta+\mu n+j)}\cdot\Gamma q_2(\gamma+\delta n)\,F(j),$$

which on making q_1, q_2 and q tend to 1, further simplify to

Theorem 4. For $n=0, 1, \ldots$

$$(4\cdot5)\quad F(n)=\sum_{j=0}^{n}\frac{(-n)_j\,\Gamma(\beta+n+\mu j)}{\Gamma(\alpha+\lambda j)\,\Gamma(\gamma+\delta j)\,\Gamma(j+1)}f(j)$$

if and only if

$$(4\cdot6)\quad f(n)=\Gamma(\alpha+\lambda n)\,\Gamma(\gamma+\delta n)\sum_{j=0}^{n}\frac{(-n)_j\,(\beta+\mu j+j)}{\Gamma(\beta+\mu n+j+1)\,\Gamma(j+1)}F(j),$$

where $Re(\alpha,\beta,\gamma,\mu)>0$.

By choosing $F(n)$ and $f(n)$ appropriately, the above pair would yield the inverse relations for the limiting cases $q\to1$ of the polynomials defined by (1·7) and (1·1).

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AN EXPANSION FORMULA AND ASSOCIATED GENERATING RELATIONS

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Following the work of L. Carlitz [1], and of S. Kumari and J.P. Singhal [2], we derive an expansion for the product of several generalized Laguerre polynomials which are included in the general class of polynomials $\{g_n^c(x,r,s)\}$ due to R. Panda [3]. We also obtain the associated generating relation for the coefficients occurring in the expansion.

1. INTRODUCTION

Let

$$(1.1)\qquad \Gamma_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} (-y)^k \frac{(c+rn-rmk)_k}{(n-mk)!\,k!} x^{n-mk}$$

denote the generalized Laguerre polynomial of degree n which is a particular case of the class of polynomials $\{f_n^c(x,y,r,m)\}$ introduced by Singhal and Kumari [4]. With a view to providing an extension of a result due to Carlitz [1], Kumari and Singhal [2] derived a generating function for the coefficients $D_s^{(n_1,\dots,n_p)}$ which occur in the expansion formula

$$(1.2)\qquad \Gamma_{n_1}^{c_1}(a_1x, y_1, r_1, m) \dots \Gamma_{n_p}^{c_p}(a_px, y_p, r_p, m)$$

$$= \sum_{s=0}^{K^*} D_s^{(n_1,\dots,n_p)} \Gamma^c_{K+ms-mK^*}(x,y,r,m)$$

where $K^* = [K/m]$. The close resemblance of the definition (1.1) with that of another generalization of the Laguerre polynomials given by (see also [3]);

(1.3) $$\lambda_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \frac{(c+rk)_{n-sk}}{(n-sk)!\,k!} x^k$$

suggests that it would be of interest to obtain an analogous expansion formula and the associated generating relation. It should be mentioned here that the polynomials $\lambda_n^c(x,r,s)$ are included, as a particular case, in the class of polynomials $\{g_n^c(x,r,s)\}$ introduced by Panda [3]. We also mention that the expansion formula, and the generating relation derived here, provide an extension of the corresponding results due to Carlitz [1] as well as of the important particular case m=1 of the results of Kumari and Singhal [2].

The expansion formula that we prove in this paper is as given below :

(1.4) $$\lambda_{n_1}^{c_1}(a_1x,r_1,s_1) \ldots \lambda_{n_p}^{c_p}(a_px,r_p,s_p) = \sum_{m=0}^{\bar{n}} G_m^{(n_1,\ldots,n_p)} \lambda_m^c(x,r,s),$$

where $\bar{n} = [n_1/s_1] + \ldots + [n_p/s_p]$,

and the coefficients $G_m^{(n_1,\ldots,n_p)}$ as given by (2.2) are generated by the relation

(1.5) $$\sum_{n_1,\ldots,n_p=0}^{\infty} G_m^{(n_1,\ldots,n_p)} w_1^{n_1} \ldots w_p^{n_p} = (1-w_1)^{-c_1} \ldots (1-w_p)^{-c_p} Z^{m/s} V^{c+rm/s},$$

where

(1.6) $$Z = \frac{a_1 w_1^{s_1}}{(1-w_1)^{r_1}} + \ldots + \frac{a_p w_p^{s_p}}{(1-w_p)^{r_p}}$$ and

(1.7) $$V = 1 - (ZV^r)^{1/s}.$$

2. DERIVATION OF (1.4) AND (1.5)

In view of the pair of inverse sevies relations given by Singhal and Kumari [5], it is easy to deduce that

$$(2.1)\quad x^k = k! \sum_{m=0}^{sk} \frac{(-c - rm/s)}{(sk - m)!} (1-c-rk)_{sk-m-1} \lambda_m^c(x,r,s).$$

Now making use of the notations

$$\begin{cases} [n_i/s_i] = n_i^*, \quad i = 1,2,\ldots,p \\ n_1^* + n_2^* + \ldots + n_p^* = \bar{n} \\ k_1 + k_2 + \ldots + k_p = K \\ m^* = [m/s] \end{cases},$$

it is easy to see from (1.3) and (2.1) that

$$\lambda_{n_1}^{c_1}(a_1x,r_1,s_1) \ldots \lambda_{n_p}^{c_p}(a_px,r_p,s_p)$$

$$= \sum_{k_1,\ldots,k_p=0}^{n_1^*,\ldots,n_p^*} \left\{ \prod_{i=1}^{p} \frac{(c_i+r_ik_i)_{n_i - s_ik_i}}{(n_i - s_ik_i)!\; k_i!} a_i^{k_i} \right\} \sum_{m=0}^{sK} \frac{K!}{(sK - m)!} \cdot$$

$$\cdot (-c - rm/s)\ (1-c-rK)_{sK-m-1}\ \lambda_m^c(x,r,s),$$

which, in view of the easily establishable relation

$$\sum_{k_1,\ldots,k_p = 0}^{n_1^*,\ldots,n_p^*} \sum_{m=0}^{sK} \frac{B(m,k_1,\ldots,k_p)}{(sK - m)!}$$

$$= \sum_{m=0}^{s\bar{n}} \sum_{k_1,\ldots,k_p = 0}^{n_1^*,\ldots,n_p^*} \frac{B(m,k_1,\ldots,k_p)}{(sK - m)!},$$

leads us to (1.4), where

$$(2.2)\quad G_m^{(n_1,\ldots,n_p)} = \sum_{k_1,\ldots,k_p=0}^{n_1^*,\ldots,n_p^*} \left\{ \prod_{i=1}^{p} \frac{(c_i+r_ik_i)_{n_i-s_ik_i}}{(n_i-s_ik_i)!\,k_i!} a_i^{k_i} \right\} \cdot$$

$$\cdot \frac{K!\,(-c-rm/s)\,(1-c-rK)_{sK-m-1}}{(sK-m)!}$$

Now to obtain the generating relation (1.5), we observe that

$$\sum_{n_1,\ldots,n_p=0}^{\infty} G_m^{(n_1,\ldots,n_p)} w_1^{n_1} \ldots w_p^{n_p}$$

$$= (1-w_1)^{-c_1} \ldots (1-w_p)^{-c_p} \sum_{k_1,\ldots,k_p=0}^{\infty} \frac{K!\,(-c-rm/s)(1-c-rK)_{sK-m-1}}{k_1!\ldots k_p!\,(sK-m)!}$$

$$\cdot \left(\frac{a_1w_1^{s_1}}{(1-w_1)^{r_1}}\right)^{k_1} \ldots \left(\frac{a_pw_p^{s_p}}{(1-w_p)^{r_p}}\right)^{k_p}$$

$$= (1-w_1)^{-c_1} \ldots (1-w_p)^{-c_p} \sum_{k=m^+}^{\infty} \frac{(-c-rm/s)\,(1-c-rk)_{sk-m-1}}{(sk-m)!} Z^k$$

where Z is given by (1.6). The right-hand side of the above expression may be put in the form

$$Z^{m/s} (1-w_1)^{-c_1} \ldots (1-w_p)^{-c_p} \sum_{j=0}^{\infty} \frac{(c+rm/s)}{(c+rm/s+rj/s)} \binom{c+rm/s+rj/s}{j} \cdot (-Z)^{j/s}$$

which can be further simplified with the help of the relation (cf. [6; p. 355, eq.(9)])

$$x^a = \sum_{k=0}^{\infty} \frac{a}{a+bk} \binom{a+bk}{k} z^k,$$

$$z = (x-1)\,x^{-b}$$

to yield the generating relation (1.5).

3. SPECIAL CASES

When s = 1, the expansion formula (1.4) and generating relation (1.5), in view of

$$\lambda_n^c(x,r,1) = \Gamma_n^c(x, -1, r, 1),$$

would yield the expansion formula and the associated generating relation derived earlier by Kumari and Singhal [2, p.58, eq.(1.6) and (1.7)].

On the other hand, the application of the relation

$$L_n^{(\alpha)}(x) = \lambda_n^{\alpha+1}(-x,1,1)$$

would reduce the generating relation (1.5) to Carlitz's result (see [1,p.395, eq.(6)]).

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