CHAPTER - 2

INVERSE SERIES RELATION I

2.1 INTRODUCTION

Having motivated by the inversion formula put forward recently by Singhal and S.Kumari [1], in the form :

$$(2.1.1)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/m]} y^k \binom{p-\lambda n+\lambda mk}{k} f(n-mk), \\ f(n) = \sum_{k=0}^{[n/m]} (-y^k) \dfrac{p-\lambda n+\lambda mk}{p-\lambda n+k} \binom{p-\lambda n+k}{k} F(n-mk), \end{cases}$$

an attempt has been made in this chapter to provide a further extension of these inverse series relations in the light of the Gould's inverse pair :

$$(2.1.2)\quad \begin{cases} F(a) = \sum_{k=0}^{M} (-1)^k \dfrac{a}{a+bk} \binom{a+bk}{k} f(a+bk-k), \\ f(a) = \sum_{k=0}^{M} \binom{a}{k} F(a+bk-k), \end{cases}$$

(see (1.2.16)).

In fact, the proposed general inverse series relation which is stated below as theorem-1 provides a very useful unification of the above mentioned pairs, for besides yielding the special instances of (2.1.1) and (2.1.2), it also gives rise to a large number of other inverse series relations including the Riordan's classes of inverse relations given in Tables 1 to 6.

THEOREM - 1

$$(2.1.3)\quad u(a) = \sum_{k=0}^{M} \frac{y^k \; V(a+bk)}{k!\, \Gamma(1+p-ar-brk-k)}$$

if, and only if

$$(2.1.4)\quad V(a) = \sum_{k=0}^{M} (-y)^k \; \frac{p-ar-brk}{k!} \; \Gamma(p-ar+k)\; u(a+bk)$$

where

$$M = \begin{cases} [-a/b], & \text{if 'a' is positive and b is a negative integer} \\ \infty, & \text{if 'a' and 'b' are positive integers.} \end{cases}$$

2.2 PROOF OF THE THEOREM

In the inverse pair (1.2.31) of Gould and Hsu, put $a_i = p-nr+i-1$, and $b_i = mr-1$, for all i. Then after a little simplification and modification one finds the following inverse relations.

$$(2.2.1)\quad \begin{cases} f_j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma(p-nr+mrk-k+j) \quad g_k , \\ g_j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \dfrac{p-nr+mrk}{\Gamma(1+p-nr+mrj-j+k)} \quad f_k . \end{cases}$$

This inverse pair provides a useful tool in the proof of theorem-1. Yet another (preliminary) result which will also be used in the proof, is that if p(x) is a polynomial in x of degree less than n, then .

$$(2.2.2)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} p(a+bk) = 0, \quad n \geq 1.$$

In the first place, employing the method due to Gould [3], the theorem will be proved by setting $a = n$ $(n=0,1,2 \ldots.,)$, and

$b = -m$ $(m=1,2,3,\ldots.)$ that is, when $M = [n/m]$.

For the sake of simplicity, $[n/m]$ will be denoted by N.

If σ denotes the right hand member of (2.1.3), then on making use of (2.1.4) (with $M = [n/m] = N$), one gets

$$\sigma = \sum_{k=0}^{N} \sum_{j=0}^{N-k} (-1)^j \, y^{k+j} \, \frac{(p-nr+mrk+mrj)\,\Gamma(p-nr+mrk+j)}{k!\; j!\; \Gamma(1+p-nr+mrk-k)} \; u(n-mk-mj),$$

which with the aid of an easily justifiable relation :

$$(2.2.3) \qquad \sum_{k=0}^{N} \sum_{j=0}^{N-k} A(k,j) = \sum_{j=0}^{N} \sum_{k=0}^{j} A(k,j-k),$$

takes the form :

$$(2.2.4)\ \sigma = \sum_{j=0}^{N} (-y)^j \, \frac{p-nr+mrj}{j!} \; u(n-mj) \sum_{j=0}^{k} (-1)^k \binom{j}{k} \cdot \frac{\Gamma(p-nr+mrk-k+j)}{\Gamma(p-nr+mrk-k+1)} .$$

Now since,

$$\frac{\Gamma(p-nr+mrk-k+j)}{\Gamma(p-nr+mrk-k+1)} = \prod_{i=1}^{j} (p-nr+mrk-k+j-i)$$
$$= \sum_{s=0}^{j-1} \alpha_s \, k^s,$$

(2.2.4) becomes

$$\sigma = u(n) + \sum_{j=1}^{N} (-y)^j \, \frac{p-nr+mrj}{j!} \; u(n-mj) . \sum_{k=0}^{j} (-1)^k \binom{j}{k} \sum_{s=0}^{j-1} \alpha_s \, k^s$$

which in view of (2.2.2) vanishes for all $j \geq 1$ and consequently, $\sigma = u(n)$.

With this the proof of the first part is completed.

In order to prove the converse part, take

$$\sum_{k=0}^{N} (-y)^k \frac{p-nr+mrk}{k!} \Gamma(p-nr+k)\, u(n-mk) = \mu .$$

Then on making use of (2.1.3), and (2.2.3) in succession one arrives at

$$(2.2.5)\quad \mu = \sum_{j=0}^{N} y^j \frac{V(n-mj)}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(p-nr+mrk)\,\Gamma(p-nr+k)}{\Gamma(1+p-nr+mrj-j+k)} .$$

It will now be shown that the inner series in (2.2.5) is equal to δ_{jo}. In fact, denoting this inner series by g_j, and replacing $\Gamma(p-nr+k)$ by f_k, one gets

$$(2.2.6)\quad g_j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{p-nr+mrk}{\Gamma(1+p-nr+mrj-j+k)} f_k .$$

In view of (2.2.1), the relation (2.2.6) is invertible in the form

$$(2.2.7)\quad f_j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma(p-nr+mrk-k+j)\, g_k .$$

wherein on setting

$$g_k = \binom{0}{k}$$

one finds

$$f_j = \Gamma(p-nr+j).$$

Whereas in (2.2.6), the same substitution yields the relation

$$\binom{0}{j} = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(p-nr+mrk)\,\Gamma(p-nr+k)}{\Gamma(1+p-nr+mrj-j+k)} .$$

Thus, (2.2.5) becomes

$$\mu = V(n) + \sum_{j=1}^{N} y^j \frac{V(n-mj)}{j!} \delta_{jo} ,$$

and thus

$$\mu = V(n);$$

which completes the proof of the converse part, and hence the proof of the theorem when M = [n/m].

The proof of theorem-1 corresponding to the case M= ∞ which runs almost parallel to the above, involves the use of the double infinite series manipulation in the form as stated below.

$$(2.2.8) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{N} B(n-k,k) .$$

In order to prove the first part it may be observed that in view of the relation (2.1.4), the right hand side of (2.1.3), denoted for brevity by Δ, can be expressed in the form :

$$\Delta = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j y^{k+j} \frac{(p-ar-brk-brj)\,\Gamma(p-ar-brk+j)}{k!\ j!\ \Gamma(1+p-ar-brk-k)} u(a+bk+bj).$$

This, with the help of (2.2.8) may be put in the form

$$\Delta = \sum_{j=0}^{\infty} (-1)^j \frac{p-ar-brj}{j!} u(a+bj) \sum_{k=0}^{j} (-1)^k \binom{j}{k} \cdot \frac{\Gamma(p-ar-brk+j-k)}{\Gamma(1+p-ar-brk-k)}$$

$$(2.2.9) \qquad = u(a) + \sum_{j=1}^{\infty} \frac{p-ar-brj}{j!} u(a+bj) \sum_{k=0}^{j} (-1)^k \binom{j}{k} \cdot \frac{\Gamma(p-ar-brk+j-k)}{\Gamma(1+p-ar-brk-k)} .$$

Since, the inner series in this last expression is same as the inner series occuring in (2.2.4), it follows that the expression (2.2.9) leads to the relation

$\Delta = u(a)$,

which completes the proof of the first part for the case $M=\infty$.

Conversely, put

$$\sum_{k=0}^{\infty} (-y)^k \frac{p-ar-brk}{k!} \Gamma(p-ar+k)\, u(a+bk) = \theta .$$

Then on making use of the relations (2.1.3), and (2.2.8) in turn, one arrives at

$$(2.2.10) \quad \theta = \sum_{j=0}^{\infty} y^j \frac{V(a+bj)}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (p-ar-brk) \cdot \frac{\Gamma(p-ar+k)}{\Gamma(1+p-ar-brj-j+k)} .$$

Again, it is easy to see that the inner series occuring in (2.2.10) is of the same form as that of (2.2.5). Thus, employing the method used to obtain the orth ogonal series relation corresponding to the inner series in (2.2.5), one finds the following orthogonal relation implied by the inner series of (2.2.10).

$$(2.2.11) \quad \binom{0}{j} = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(p-ar-brk)\,\Gamma(p-ar+k)}{\Gamma(1+p-ar-brj-j+k)} .$$

With this orthogonal relation, the expression (2.2.10) gives

$\theta = V(a)$.

This completes the proof of the second part when $M = \infty$, and hence the proof of the theorem.

2.3 ALTERNATIVE FORMS OF THEOREM-1

In this section several alternative forms of theorem-1, involving the binomial coefficients, are given which will be used in the next section in order to illustrate the various particular cases.

First see that theorem-1 when rewritten in terms of binomial coefficients, reads (when y=1)

$$(2.3.1)\quad \begin{cases} u(a) = \sum_{k=0}^{M} \binom{p-ar-brk}{k} V(a+bk), \\ V(a) = \sum_{k=0}^{M} (-1)^k \dfrac{p-ar-brk}{p-ar+k} \binom{p-ar+k}{k} u(a+bk) . \end{cases}$$

This pair of inverse relations enables one to obtain some more alternative pairs. For instance, on multiplying both the relations in (2.3.1) by p-ar, and putting $(p-ar)\,u(a)= u^*(a)$ and, $(p-ar)\,V(a)= V^*(a)$, one gets the pair

$$(2.3.2)\quad \begin{cases} u^*(a) = \sum_{k=0}^{M} \binom{p-ar-brk-1}{k} \dfrac{p-ar}{p-ar-brk-k} V^*(a+bk), \\ V^*(a) = \sum_{k=0}^{M} (-1)^k \binom{p-ar+k-1}{k} u^*(a+bk) . \end{cases}$$

Further, on replacing p by p+1 and, r by -r, (2.3.2) gets transformed to

$$(2.3.3)\quad \begin{cases} u^*(a) = \sum_{k=0}^{M} \binom{p+ar+brk}{k} \dfrac{p+ar+1}{p+ar+brk-k+1} V^*(a+bk), \\ V^*(a) = \sum_{k=0}^{M} (-1)^k \binom{p+ar+k}{k} u^*(a+bk) . \end{cases}$$

Again in view of the formula (Riordan [2,p.1])

(2.3.4.) $$\binom{-\alpha}{n} = (-1)^n \binom{\alpha+n-1}{n} .$$

the above pair (2.3.1) takes the form:

(2.3.5) $$\begin{cases} u(a) = \sum_{k=0}^{M} (-1)^k \binom{-p+ar+brk+k-1}{k} V(a+bk), \\ \\ V(a) = \sum_{k=0}^{M} \frac{p-ar-brk}{p-ar+k} \binom{-p+ar-1}{k} u(a+bk) . \end{cases}$$

Also, applying the formula (2.3.4) to the pair (2.3.2), and then replacing -p by p, one finds the relations :

(2.3.6) $$\begin{cases} u(a) = \sum_{k=0}^{M} (-1)^k \binom{p+ar+brk+k}{k} \frac{p+ar}{p+ar+brk+k} V(a+bk), \\ \\ V(a) = \sum_{k=0}^{M} \binom{p+ar}{k} u(a+bk) . \end{cases}$$

Lastly, the substitutions br=-1 and a=n (n=0,1,2,....) in theorem-1, give useful pair of relations

(2.3.7) $$u(n) = \sum_{k=0}^{M} \frac{V(n+bk)}{k!} \; ; \quad V(n) = \sum_{k=0}^{M} (-1)^k \frac{u(n+bk)}{k!} .$$

2.4 PARTICULAR CASES

It is not difficult to see that the pair (2.3.6) under the substitutions p=0 and r=1, readily yields the inversion pair (2.1.2); whereas theorem-1 with a=n, b=-m (n=0,1,2,...; m=1,2,3,...) and with r replaced by λ, reduces to the inverse pair (2.1.1). Thus, the polynomials $f_n^c(x,y,r,m)$ given by (1.1.12) and those associated to it namely, the polynomials of generalized Humbert (1.2.23), Humbert, Kinney etc. are contained in theorem-1 together with their inverse series relations (see (1.2.26) to (1.2.30)).

On the other hand, the inverse series relations classified into the simplest type pairs, the Gould classes, the simpler Chebyshev classes etc. (see section-1.3) may be deduced (directly) from theorem-1 or they may be obtained in a straightforward manner form the inverse pairs of relations given in section-2.3.

As an illustration, consider the pair (2.3.7) which is capable of yielding the "simplest inverse relations" given in table-1 (section-1.3).In fact, when b=-1 the inverse relations in (2.3.7) gets reduced to

$$(2.4.1) \qquad u(n) = \sum_{k=0}^{n} \frac{V(n-k)}{k!} \;,\; V(n) = \sum_{k=0}^{n} (-1)^k \frac{u(n-k)}{k!} .$$

If these series are reversed, then (2.4.1) reads as

$$(2.4.2) \qquad u(n) = \sum_{k=0}^{n} \frac{V(k)}{(n-k)!} \;,\; V(n) = \sum_{k=0}^{n} (-1)^{n-k} \frac{u(k)}{(n-k)!} .$$

This inversion pair, with u(n) replaced by $a_n/n!$ and, V(n) replaced by $b_n/n!$, transforms to

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \,, \qquad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k$$

which is the simplest pair No.(1) in Table-1 (of section-1.3).

Likewise, putting b=1, replacing k by k-n and , then putting $u(n)= n!a_n$, $V(n)=n!b_n$ in (2.3.7), one arrives at the second simplest pair (of Table-1) :

$$a_n = \sum_{k=n} \binom{k}{n} b_k , \qquad b_n = \sum_{k=n} (-1)^{n+k} \binom{k}{n} a_k .$$

The remaining pairs follow in a similar manner (see Table-7 at the end of this section).

In order to get an inverse series relation belonging to the Gould classes, set a=n a nonnegative integer, b=-1, and r=1-q in (2.3.1), so that

$$u(n) = \sum_{k=0}^{n} \binom{p-(1-q)n+(1-q)k}{k} V(n-k) ,$$

$$V(n) = \sum_{k=0}^{n} (-1)^k \frac{p-(1-q)n+(1-q)k}{p-(1-q)n+k} \binom{p-(1-q)n+k}{k} u(n-k) .$$

This may be put by reversing the series, in the form :

$$u(n) = \sum_{k=0}^{n} \binom{p+qk-k}{n-k} V(k) ,$$

$$v(n) = \sum_{k=0}^{n} (-1)^{k+n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{k} u(k),$$

which is the inversion pair No.(1) in Table-2.

In an analogous way, one can obtain all the inverse relations quoted in Table-1 to 6, by specializing the parameters b,r, and p involved in the pairs (2.3.1) to (2.3.7) as indicated in the following tables.

Table-7: *Reducibilities to simplest inverse pairs*

Inverse pair (citation)	b	u(n)	V(n)	pair No. in Table-1
(2.3.7)	-1	$a_n/n!$	$b_n/n!$	(1)
	1	$n!\,a_n$	$n!\,b_n$	(2)
	-1	$(p-n)!a_n$	$(p-n)!b_n$	(3)
	-1	$a_n/(p+n)!$	$b_n/(p+n)!$	(4)
	1	$(p+n)!a_n$	$(p+n)!b_n$	(5)
	-1	$a_n/n!(n-1)!$	$b_n/n!(n-1)!$	(6)

Table-8 : *Reducibilities to Gould classes*

Inversion pair (citation) with a=n	b	r	p	Pair No. in Table-2
(2.3.1)	-1	1-q	p	(1)
(2.3.3)	-1	q-1	p	(2)
(2.3.6)	1	q-1	p	(3)
(2.3.5)	1	q-1	-p-1	(4)

Table-9 : *Reducibilities to simpler Chebyshev classes*

Inversion pair (citation) with a=n	b	r	p	Pair No. in Table-3
(2.3.6)	-2	1	0	(1)
(2.3.5)	-2	1	-1	(2)
(2.3.1)	2	-1	0	(3)
(2.3.3)	2	1	0	(4)
(2.3.1)	-1	-1	0	(5)
(2.3.3)	-1	1	0	(6)

Table-10 : *Reducibilities to Chebyshev classes*

Inversion pair (citation) with a=n	b	r	p	Pair No. in Table-4
(2.3.6)	c	1	0	(1)
(2.3.5)	c	1	-1	(2)
(2.3.1)	c	-1	0	(3)
(2.3.3)	c	1	0	(4)

Table-11 : *Reducibilities to simpler Legendre classes*

Inversion pair (citation) with a=n	b	r	p	Pair No. in Table-5
(2.3.5)	-1	2	-p-1	(1)
(2.3.6)	-1	2	p	(2)
(2.3.3)	1	2	p	(3)
(2.3.1)	1	-2	p	(4)
(2.3.6)	-2	2	p	(5)
(2.3.5)	-2	2	-p-1	(6)

Table-12 : *Reducibilities to Legendre-Chebyshev classes*

Inversion pair (citation) with a=n	b	r	p	Pair No. in Table-6
(2.3.6)	-1	c	p	(1)
(2.3.6)	1	c	p	(2)
(2.3.1)	-1	-c	p	(3)
(2.3.1)	1	-c	p	(4)
(2.3.5)	-1	c	-p-1	(5)
(2.3.5)	1	c	-p-1	(6)
(2.3.3)	-1	c	p	(7)
(2.3.3)	1	c	p	(8)

It is to be pointed out here that if the various parameters involved in theorem-1 are particularized suitably in the light of the series expansion of the Bessel function :

$$(2.4.3) \quad J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \ (x/2)^{n+2k}}{\Gamma(n+k+1) \ k!} ,$$

leads to the (known) Neumann's expansion (Rainville [1], Watson [1,p.132]) :

$$(2.4.4) \quad (x/2)^n = \sum_{k=0}^{\infty} \frac{n+2k}{k!} \ \Gamma(n+k) \ J_{n+2k}(x)$$

(n is a nonnegative integer).