CHAPTER - 4

INVERSE SERIES RELATIONS AND EXTENSIONS OF CERTAIN POLYNOMIALS

4.1 EXTENSIONS OF CERTAIN POLYNOMIALS AND RIORDAN'S INVERSE RELATIONS

The inversion formula (1.2.35) quoted in chapter-1, namely,

$$(4.1.1)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/s]} \binom{p+qsk-sk}{n-sk} f(k) \\ \text{if, and only if} \\ f(n) = \sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k), \\ \text{and} \\ \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k) = 0,\ n \neq ms, \end{cases}$$

(m=1,2,3,...) was proved by Singhal and S.Kumari, with the help of which they obtained the inverse relation of the polynomial

$$(4.1.2)\quad g_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \frac{(c+rk)_{n-sk}}{(n-sk)!} \gamma_k x^k$$

in the form

$$(4.1.3)\quad \gamma_n x^n = \sum_{k=0}^{sn} \frac{(-1)^{sn-k}(c+(rk/s))}{(c)_{rn-sn+k+1}\,(sn-k)!} g_k^c(x,r,s).$$

Here, a particular case of (4.1.2) viz. the extended Jacobi polynomial (H.M.Srivastava [5]) :

(4.1.4) $$F^{(c)}_{n,\ell,s}\,[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q\ :x] =$$

$$= \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}\ (c+n)_{\ell k}\ (\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\ \dots(\beta_q)_k\qquad k!}\ x^k$$

is worth mentioning, which occurs when $r-s$ $(=\ell)$ is a positive integer and,

$$\gamma_k = (-1)^{sk}\ (c)_{sk+\ell k}\ \frac{(\alpha_1)_k\ \dots(\alpha_p)_k}{(\beta_1)_k\ \dots(\beta_q)_k}\ .$$

The inverse series relation of this polynomial is easily obtainable in the form :

(4.1.5) $$\frac{(sn)!\ (\alpha_1)_n\ \dots(\alpha_p)_n}{n!\ (\beta_1)_n\ \dots\ (\beta_q)_n}\ x^n = \sum_{k=0}^{sn} \frac{(-sn)_k\ (c+k+(\ell k/s))}{(c+k)_{\ell n+1}\ k!}\ .$$

$$F^{(c)}_{k,\ell,s}[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q:x],$$

by making use of the substitutions mentioned above, in the relation (4.1.3).

This explicit representation of the extended Jacobi polynomial suggests that an analogous extension could also be carried out for the Hahn polynomial $Q_n(x;\alpha,\beta,N)$, the Racah polynomial $R_n(x(x+\gamma+\delta+1); \alpha,\beta,\gamma,\delta)$, and for the Wilson polynomial $P_n(x^2)$ (cited in (1.1.8,9,10)).

In fact, it is quite interesting to see that the pair of inverse relations (4.1.1) when written in an alternative form :

$$(4.1.6)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/s]} (-n)_{sk}\ (c+n)_{\ell k}\ f(k) \\ \text{if, and only if} \\ f(n) = \sum_{k=0}^{sn} \dfrac{(-sn)_k\ (c+k+(\ell k/s))}{(c+k)_{\ell n+1}\ (sn)!\ k!}\ F(k) \\ \text{and} \\ \sum_{k=0}^{n} \dfrac{(-n)_k\ (c+k+(\ell k/s))}{(c+k)_{\ell n+1}\ k!}\ F(k) = 0, \quad n \neq s, 2s, 3s, \ldots \end{cases}$$

$$(\ell = r-1)$$

enables one to carry out the extensions of the above mentioned polynomials. The proposed extensions as obtained below by means of the pair (4.1.6), are denoted respectively by $Q_{n,\ell,s}(x;\alpha,\beta,N)$, $R_{n,\ell,s}(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)$, and $P_{n,\ell,s}(x^2)$.

In order to obtain an extended form of the Hahn polynomial, set $c = 1+\alpha+\beta$, and

$$f(k) = \frac{(-x)_k}{(\alpha+1)_k\ (-N)_k\ k!}$$

in (4.1.6). In this case, one gets the following explicit representation of extended Hahn polynomial:

$$(4.1.7)\quad Q_{n,\ell,s}(x;\alpha,\beta,N) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}\ (\alpha+\beta+n+1)_{\ell k}}{(\alpha+1)_k\ (-N)_k\ k!}\ (-x)_k .$$

The inverse series relation of this follows from the pair (4.1.6) in the form :

$$(4.1.8)\quad \frac{(sn)!\ (-x)_n}{(\alpha+1)_n (-N)_n\ n!} = \sum_{k=0}^{sn} \frac{(-sn)_k\ (\alpha+\beta+1+k+(\ell k/s))}{(\alpha+\beta+k+1)_{\ell n+1}\ k!}\ Q_{k,\ell,s}(x;\alpha,\beta,N).$$

When $c = \alpha+\beta+1$, and

$$f(k) = \frac{(-x)_k \ (x+\gamma+\delta+1)_k}{(\alpha+1)_k \ (\beta+\delta+1)_k \ (\gamma+1)_k \ k!},$$

then the pair (4.1.6) provides an extension of the Racah polynomial in the form :

(4.1.9) $$R_{n,\ell,s}(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)$$

$$= \sum_{k=0}^{[n/s]} \frac{(-n)_{sk} \ (\alpha+\beta+n+1)_{\ell k} \ (-x)_k \ (x+\gamma+\delta+1)_k}{(\alpha+1)_k \ (\beta+\delta+1)_k \ (\gamma+1)_k \ k!}$$

along with its inverse relation :

(4.1.10) $$\frac{(-x)_n \ (x+\gamma+\delta+1)_n \ (sn)!}{(\alpha+1)_n \ (\beta+\delta+1)_n \ (\gamma+1)_n \ n!}$$

$$= \sum_{k=0}^{sn} (-sn)_k \ \frac{\alpha+\beta+k+1+(\ell k/s)}{(\alpha+\beta+k+1)_{\ell n+1} \ k!} \ R_{k,\ell,s}(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta).$$

Similarly an extended version of the Wilson polynomial together with its inverse series relation follow from (4.1.6) with 'c' = a+b+c+d+n-1, and

$$f(k) = \frac{(a+ix)_n \ (a-ix)_n \ (sn)!}{(a+b)_n \ (a+c)_n \ (a+d)_n \ n!},$$

which are as given below.

$$(4.1.11)\begin{cases} \dfrac{P_{n,\ell,s}(x^2)}{(a+b)_n(a+c)_n(a+d)_n} = \sum_{k=0}^{[n/s]} \dfrac{(-n)_{sk}\ (a+b+c+d+n-1)_{\ell k}}{(a+b)_k\ (a+c)_k\ (a+d)_k\ k!} . (a+ix)_k(a-ix)_k, \\ \dfrac{(a+ix)_n(a-ix)_n(sn)!}{(a+b)_n(a+c)_n(a+d)_n} = \sum_{k=0}^{sn} \dfrac{(-sn)_k(a+b+c+d+k-1+(\ell k/s))}{(a+b+c+d+k-1)_{\ell n+1}\ \ (a+b)_k} . \dfrac{P_{k,\ell,s}(x^2)}{(a+c)_k\ (a+d)_k\ k!} \end{cases}$$

Besides yielding the aforementioned extended polynomials, the inverse pair (4.1.1) also provides an effective tool for carrying out the extensions of certain inverse series relations belonging to the Riordan's classification. As a matter of fact, the Gould classes (1) and (2) (Table-2), the simpler Legendre classes (1) and (2) (Table-5) and, the Legendre-Chebyshev classes (1), (3), (5), and (7) (Table-6), admit extensions in the light of the formula (4.1.1) and its several alternative forms which are deduced below.

First, replacing F(n), f(n), and p by F(n)/p+qn-n+1, f(n)/p+qsn-sn+1, and p+1 respectively, in (4.1.1), one gets

$$(4.1.12)\begin{cases} F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \dfrac{p+qn-n+1}{p+qsk-n+1} \binom{p+qsk-sk}{n-sk} f(k), \\ f(n) = \sum_{k=0}^{sn} \binom{p+qsn-k}{sn-k} F(k). \end{cases}$$

Next, with the aid of the formula

$$\binom{-\alpha}{n} = (-1)^n \binom{\alpha+n-1}{n} ,$$

and with p replaced by $-p-1$, the inverse relations in (4.1.1) assume the forms:

$$(4.1.13)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/s]} \binom{p+n-qsk}{n-sk} f(k), \\ f(n) = \sum_{k=0}^{sn} (-1)^{sn-k} \dfrac{p-qk+k+1}{p-qsn+k+1} \binom{p-qsn+sn}{sn-k} F(k). \end{cases}$$

In this, replacing $F(n)$ by $F(n)/p-qn+n+1$, $f(n)$ by $f(n)/p-qsn+sn+1$, and p by $p-1$, one finds the pair:

$$(4.1.14)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/s]} \dfrac{p-qn+n}{p-qsk+n} \binom{p-qsk+n}{n-sk} f(k), \\ f(n) = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{p-qsn+sn}{sn-k} F(k). \end{cases}$$

On choosing the parameter q appropriately in these inverse series relations, one arrives at the extended versions of the above mentioned classes in the forms which are given in the following table.

Table-20 *Extensions of Riordan's inverse relations*

$$F(n) = \sum_{k=0}^{[n/s]} A_{n,k} f(k) \; ; \; f(n) = \sum_{k=0}^{sn} (-1)^{sn-k} B_{n,k} f(k)$$

Citation	q	$A_{n,k}$	$B_{n,k}$	Extension of class (No.)as in Tables-2,5,6
(4.1.1)	q	$\binom{p+qsk-sk}{n-sk}$	$\frac{p+qk-k}{p+qsn-k}\binom{p+qsn-k}{sn-k}$	Gould class(1) Table-2
(4.1.12)	q	$\frac{p+qn-n+1}{p+qsk-n+1}\binom{p+qsk-sk}{n-sk}$	$\binom{p+qsn-k}{sn-k}$	Gould class(2) Table-2
(4.1.13)	-1	$\binom{p+n+sk}{n-sk}$	$\frac{p+2k+1}{p+sn+k+1}\binom{p+2sn}{sn-k}$	simpler Legendre Class (1) Table-5
(4.1.14)	-1	$\frac{p+2n}{p+n+sk}\binom{p+n+sk}{n-sk}$	$\binom{p+2sn}{sn-k}$	simpler Legendre Class (2) Table-5
(4.1.14)	-c+1	$\frac{p+cn}{p+sck}\binom{p+n+sck-sk-1}{n-sk}$	$\binom{p+scn}{sn-k}$	Legendre-Chebyshev Class (1) Table-6
(4.1.1)	c+1	$\binom{p+sck}{n-sk}$	$\frac{p+ck}{p+scn}\binom{p+scn+sn-k-1}{sn-k}$	Legendre-Chebyshev class (3) Table-6
(4.1.13)	-c+1	$\binom{p+n+sck-sk}{n-sk}$	$\frac{p+ck+1}{p+scn-sn+k+1}\binom{p+scn}{sn-k}$	Legendre-Chebyshev class (5) Table-6
(4.1.12)	c+1	$\frac{p+cn+1}{p-n+sck+sk+1}\binom{p+sck}{n-sk}$	$\binom{p+scn+sn-k}{sn-k}$	Legendre-Chebyshev Class (7) Table-6

The result appearing above in the forms of the generalizations of various polynomials and the inverse series relations, can also be viewed as the limiting cases of their corresponding basic analogues (also called q-extensiions), which put them into further extended forms. In fact, an attempt made in obtaining the proposed basic analogues led to certain interesting and seemingly new results which are incorporated in the following sections.

4.2 A BASIC INVERSE RELATION

In this section, a basic analogue of the polynomial $g_n^c(x,r,s)$ will be defined first whose inverse series will be obtained by proving a more general pair of inverse series relation.

Let,

$$(4.2.1)\quad g_n^c(x,r,s|q)=\sum_{k=0}^{[n/s]}(-1)^{n-sk}\,\frac{q^{sk(sk-2n+1)/2}\,[q^{-c+sk-rk-n+1}]_\infty}{[q^{-c-rk+1}]_\infty\;[q]_{n-sk}}\,.\delta_k\,x^k$$

which may be considered to define a basic analogue of the explicit representation (4.1.2) of the polynomial $g_n^c(x,r,s)$.

Then in the light of the first relation occuring in (4.1.1), it is not difficult to see that the polynomial defined above is contained in a more general expression given by

$$F(n)=\sum_{k=0}^{[n/s]}(-1)^{n-sk}\,q^{sk(sk-2n+1)/2}\,\frac{[q^{p+bsk-n+1}]_\infty\quad G(k)}{[q^{p+bsk-sk+1}]_\infty\;[q]_{n-sk}}\,.$$

Thus in order to obtain an inverse relation of the polynomial (4.2.1), it would be worthwhile to prove an inverse relation corresponding to the above general expression. In fact, having guided by the related work of Singhal and S.Kumari [2], a basic inverse relation is proved here in the form of the following relations.

$$(4.2.2)\quad F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk}\, q^{sk(sk-2n+1)/2} \frac{[q^{p+bsk-n+1}]_\infty\, G(k)}{[q^{p+bsk-sk+1}]_\infty [q]_{n-sk}}$$

implies

$$(4.2.3)\quad G(n) = \sum_{k=0}^{sn} q^{k(k+1)/2} \frac{[q^{p+bsn-sn}]_\infty\, F(k)}{[q^{p+bsn-k}]_\infty\, [q]_{sn-k}},$$

where the positive integer $bs \le s$.

The proof, as given below, is based on the technique used by Carlitz [3].

Let,

$$\sum_{k=0}^{sn} q^{k(k-1)/2} \frac{[q^{p+bsn-sn}]_\infty\, G(k)}{[q^{p+bsn-k}]_\infty\, [q]_{sn-k}} = \phi .$$

Then on making use of the relation (4.2.2), one gets

$$\phi = \sum_{k=0}^{sn} q^{k(k-1)/2} \frac{[q^{p+bsn-sn}]_\infty}{[q^{p+bsn-k}]_\infty\, [q]_{sn-k}} \sum_{j=0}^{[k/s]} (-1)^{k-sj} .$$

$$. q^{sj(sj-2k+1)/2} \frac{[q^{p+bsj-k+1}]_\infty\, G(j)}{[q^{p+bsj-sj+1}]_\infty [q]_{k-sj}}$$

$$= \sum_{j=o}^{n} \frac{[q^{p+bsn-sn}]_\infty \quad G(j)}{[q^{p+bsj-sj+1}]_\infty [q]_{sn-sj}} \sum_{k=0}^{sn-sj} (-1)^k q^{k(k-1)/2} \left[{sn-sj \atop k} \right] \cdot$$

$$\cdot \left\{ \frac{[q^{p+bsj-sj-k+1}]_\infty}{[q^{p+bsn-sj-k}]_\infty} \right\}.$$

If bs is a positive integer then the expression in braces assumes its equivalent series form

$$\sum_{m=0}^{bsn-bsj-1} A_m \; q^{-mk}$$

with which one further gets

$$\phi = G(n) + \sum_{j=0}^{n-1} \frac{[q^{p+bsn-sn}]_\infty \quad G(j)}{[q^{p+bsj-sj+1}]_\infty [q]_{sn-sj}} \sum_{m=0}^{bsn-bsj-1} A_m \cdot$$

$$\cdot \sum_{k=0}^{sn-sj} (-1)^k q^{k(k-1)/2} \left[{sn-sj \atop k} \right] q^{-mk}.$$

Now on making an appeal to the formula (Carlitz [3])

$$\sum_{k=0}^{n} q^{k(k-1)/2} \left[{n \atop k} \right] x^k = \prod_{k=1}^{n} (1+xq^{k-1}),$$

the above expression gets transformed to

$$\phi = G(n) + \sum_{j=0}^{n-1} \frac{[q^{p+bsn-sn}]_\infty \quad G(j)}{[q^{p+bsj-sj+1}]_\infty [q]_{sn-sj}} \sum_{m=0}^{bsn-bsj-1} A_m \cdot$$

$$\cdot \prod_{m=0}^{sn-sj} (1-q^{-m+k-1})$$

$$= \begin{cases} 0, & \text{if } j \neq n \\ G(n), & \text{if } j=n \end{cases},$$

provided $bs \le s$; thus (4.2.2) implies (4.2.3).

Since the relation (4.2.2), under the substitutions $bs=s-r$, $p=-c$, and $G(k)= \delta_k\ x^k$ readily yields the basic polynomial $g_n^c(x,r,s|q)$, one finds by employing the same substitutions in (4.2.3), its inverse relation in the form:

$$(4.2.4)\quad \delta_n\ x^n = \sum_{k=0}^{sn} \frac{q^{k(k-1)/2}\ [q^{-c-rn}]_\infty}{[q^{-c+sn-rn-k}]_\infty\ [q]_{sn-k}}\ g_k^c(x,r,s|q).$$

A worth mentioning particular case of the basic polynomial (4.2.1) is a basic analogue of the Laguerre polynomial given by (M.A.Khan [1]):

$$(4.2.5)\quad {}_qL_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{q^{k(k-2n+1)/2}\ [\alpha q]_n}{[\alpha q]_k\ [q]_{n-k}\ [q]_k}\ (-x)^k$$

to which it would reduce when $c=1+\alpha$, $r=s=1$, and $\delta_k=(-1)^k\ q^{-k(k-2\alpha+1)/2}\{[q]_k\}^{-1}$. An inverse relation of this basic Laguerre polynomial follows from (4.2.4) under the same substitutions, which is as given below :

$$(4.2.6)\quad x^n = [q]_n \sum_{k=0}^{n} \frac{(-1)^k\ q^{k(k-1)/2}\ [\alpha q]_n}{[\alpha q]_k\ \ [q]_{n-k}}\ {}_qL_k^{(\alpha)}(x).$$

In view of the inverse series relations proved in the form of theorem-2 (chapter-3), it is quite natural to examine whether the converse of the relations (4.2.2) and (4.2.3) holds true. In fact, an attempt made in this direction led us to consider a slight variant of the defining relation (4.2.1) in the form :

$$(4.2.7)\quad g_n^c(x,r,s;q) = \sum_{k=0}^{[n/s]} q^{(r-s)k(sk-2n+1)/2} \frac{[q^{c+rk}]_{n-sk}\,\delta_k}{(q^{R-1};q^{R-1})_{n-sk}} x^k,$$

where $R = r/s$, $r \neq s$.

In order to obtain an inverse series relation of the basic polynomial (4.2.7), a more general inversion formula will be proved ; wherein the converse of the series relations would also hold. This forms the subject matter of section-4.3.

4.3 A GENERAL PAIR OF INVERSE SERIES RELATIONS

The proposed general inverse series relations which will be proved in this section, is stated below as

THEOREM -3. If $b \neq 0$, and $s = 2,3,4,\ldots$, then

$$(4.3.1)\quad F_n = \sum_{K=0}^{[n/s]} (-1)^{n-sk} q^{bsk(sk-2n+1)/2} \frac{[q^{p+bsk-n+1}]_\infty}{(q^b;q^b)_{n-sk}} G_k$$

if and only if

$$(4.3.2)\quad G_n = \sum_{k=0}^{sn} q^{bk(k-1)/2} \frac{(1-q^{p+bk-k})}{[q^{p+bsn-k}]_\infty} \frac{F_k}{(q^b;q^b)_{sn-k}}$$

and

$$(4.3.3)\quad \sum_{k=0}^{n} q^{bk(k-1)/2} \frac{(1-q^{p+bk-k})}{[q^{p+bn-k}]_\infty} \frac{F_k}{(q^b;q^b)_{n-k}} = 0,\ n \neq ms,$$

$m = 1,2,3,\ldots$.

For $s=1$, and $b \neq 0$, the following relations hold true.

$$(4.3.4)\quad f_n = \sum_{k=0}^{n} (-1)^{n-k} q^{bk(k-2n+1)/2} \frac{[q^{p+bk-n+1}]_\infty}{(q^b;q^b)_{n-k}} g_k$$

if and only if

$$(4.3.5)\quad g_n = \sum_{k=0}^{n} q^{bk(k-1)/2} \frac{(1-q^{p+bk-k}) \qquad f_k}{[q^{p+bn-k}]_\infty \quad (q^b;q^b)_{n-k}} .$$

The proof of theorem-3 as given below, is based on the methods due to Gould [4], and Singhal and S.Kumari [4]. The proof also makes use of a particular case of the Carlitz's inverse relations (1.5.6), viz. the pair

$$(4.3.6)\quad \begin{cases} f(sn-sj) = \sum\limits_{k=0}^{sn-sj} (-1)^k q^{bk(k-2sn+2sj+1)/2} \left[{sn-sj \atop k} \right]_b \cdot \\ \qquad\qquad \cdot [q^{p+bsj-sn+bk+1}]_\infty \, g(k), \\ \\ g(sn-sj) = \sum\limits_{k=0}^{sn-sj} (-1)^k q^{bk(k-1)/2} \left[{sn-sj \atop k} \right]_b \cdot \\ \qquad\qquad \cdot \dfrac{1-q^{p+bsj-sj+bk+k}}{[q^{p+bsn-sj-k}]_\infty} f(k) \end{cases}$$

which follows readily from (1.5.6) under the substitutions $a_i=1$, $b_i= -q^{p+bsj-sj-i+1}$, and with λ and n replaced by b and $sn-sj$, respectively.

In order to prove the first part, consider the series

$$\sum_{k=0}^{sn} q^{bk(k-1)/2} \frac{(1-q^{p+bk-k}) \qquad F_k}{[q^{p+bsn-k}]_\infty \, (q^b;\, q^b)_{sn-k}} = \omega \ , \ (\text{say}).$$

Then on substituting the relation (4.3.1) in this, one gets

$$\omega = \sum_{k=0}^{sn} \sum_{j=0}^{[k/s]} (-1)^{k-sj}\, q^{bk(k-2sj-1)/2 + bsj(sj+1)/2} \,.$$

$$\cdot \frac{(1-q^{p+bk-k})\ [q^{p+bsj-k+1}]_\infty \quad G_j}{[q^{p+bsn-k}]_\infty\ (q^b;q^b)_{sn-k}\ (q^b;q^b)_{k-sj}},$$

which in view of an easily establishable relation

$$(4.3.7)\quad \sum_{k=0}^{sn} \sum_{j=0}^{[k/s]} A(k,j) = \sum_{j=0}^{n} \sum_{k=0}^{sn-sj} A(k+sj,j),$$

assumes the form :

$$(4.3.8)\ \omega = G_n + \sum_{j=0}^{n-1} \frac{G_j}{(q_1;q_1)_{n-j}} \sum_{k=0}^{sn-sj} (-1)^k\, q_1^{k(k-1)/2} \left[{sn-sj \atop k}\right]_b \,.$$

$$\cdot \frac{1-q^{p+bk+bsj-k-sj}}{[q^{p+bsn-k-sj}]_\infty}\ [q^{p+bsj-k-sj+1}]_\infty \,,$$

wherein $q_1 = q^b$ $(b\neq 0)$.

Now, in order to show that (4.3.1) implies (4.3.2), it suffices to show that the inner series in (4.3.8) is equal to

$$\left[{0 \atop sn-sj}\right].$$

In fact, in (4.3.8), replacing $[q^{p+bsj-k-sj+1}]_\infty$ by $f(k)$, and denoting the inner series by $g(sn-sj)$, one gets

$$(4.3.9)\ g(sn-sj) = \sum_{k=0}^{sn-sj} (-1)^k\, q_1^{k(k-1)/2} \left[{sn-sj \atop k}\right]_b \frac{(1-q^{p+bk-k+bsj-sj})}{[q^{p+bsn-k-sj}]_\infty} \cdot f(k);$$

whose inverse companion follow from (4.3.6) in the form :

$$(4.3.10)\quad f(sn-sj) = \sum_{k=0}^{sn-sj} (-1)^k q_1^{k(k-2sn+2sj+1)/2} \begin{bmatrix} sn-sj \\ k \end{bmatrix}_b \cdot$$

$$\cdot\ [q^{p+bk+bsj-sn+1}]_\infty\, g(k).$$

In this last (inverse) relation, setting

$$g(k) = \begin{bmatrix} 0 \\ k \end{bmatrix},$$

one finds

$$f(k) = [q^{p+bsj-k-sj+1}]_\infty .$$

With these f(k) and g(k), (4.3.9) yields the orthogonality relation :

$$(4.3.11)\quad \sum_{k=0}^{sn-sj} (-1)^k q_1^{k(k-1)/2} \begin{bmatrix} sn-sj \\ k \end{bmatrix}_b \frac{(1-q^{p+bk+bsj-k-sj})}{[q^{p+bsn-k-sj}]_\infty} \cdot$$

$$\cdot [q^{p+bsj-k-sj+1}]_\infty = \begin{bmatrix} 0 \\ sn-sj \end{bmatrix},$$

by means of which, the expression in (4.3.8) gets reduced to

$$\omega = G_n + \sum_{j=0}^{n-1} \frac{G_j}{(q_1;q_1)_{n-j}} \begin{bmatrix} 0 \\ sn-sj \end{bmatrix}$$

$$= G_n .$$

Thus, (4.3.1) implies (4.3.2).

In order to show that (4.3.1) also implies (4.3.3), put

$$(4.3.12)\quad \mu(n) = \sum_{k=0}^{n} q^{bk(k-1)/2} \frac{(1-q^{p+bk-k})}{[q^{p+bn-k}]_\infty\, (q_1;q_1)_{n-k}}\, F_k .$$

where, as before, $q_1 = q^b$.

Then in view of the relation (4.3.1), this can be expressed as

$$(4.3.13)\quad \mu(n) = \sum_{j=0}^{[n/s]} \frac{G_j}{(q_1;q_1)_j} \sum_{k=0}^{n-sj} (-1)^k q_1^{k(k-1)/2} \left[{n-sj \atop k}\right]_b \cdot \frac{[q^{p+bsj-k-sj+1}]_\infty}{[q^{p+bsj-k-sj-bn}]_\infty} .$$

Now, following the method employed in obtaining the orthogonality relation (4.3.11), it can be shown that the inner series in (4.3.13) equals to

$$\left[{0 \atop n-sj}\right] ,$$

as a result of which (4.3.13) gets reduced to

$$\mu(n) = \sum_{j=0}^{[n/s]} \frac{G_j}{(q_1;q_1)_j} \left[{0 \atop n-sj}\right] .$$

If n/s is not an integer i.e. $n \neq ms$, $m = 1,2,3,\ldots$, then the right hand member of the last expression given above vanishes and thus, (4.3.1) implies (4.3.3); which completes the proof of the first part.

for proving the converse part, assume that the relations (4.3.2) and (4.3.3) viz.

$$G_n = \sum_{k=0}^{sn} q^{bk(k-1)/2} \frac{(1-q^{p+bk-k})}{[q^{p+bsn-k}]_\infty (q^b;q^b)_{sn-k}} F_k ,$$

and

$$\sum_{k=0}^{n} q^{bk(k-1)/2} \frac{1-q^{p+bk-k}}{[q^{p+bn-k}]_\infty (q^b;q^b)_{n-k}} F_k = 0,\ n \neq ms,$$

where $m = 1,2,3,\ldots$, hold true.

Now, in view of (4.3.12) and (4.3.3) one readily gets

$$(4.3.14)\qquad \mu(n)=0,\ n \neq sm,$$

and also, by comparing (4.3.12) with (4.3.2), one finds a useful relation:

(4.3.15) $\mu(sn) = G_n$.

Since, the inversion pair (4.3.6) with j=0 and s=1, reduces to the result (with $g(n) = \mu(n)$, and $f(n) = F_n$) :

$$(4.3.16)\quad \begin{cases} \mu(n) = \sum_{k=0}^{n} q_1^{k(k-1)/2} \dfrac{(1-q^{p+bk-k}) \qquad F_k}{[q^{p+bn-k}]_\infty \ (q_1;q_1)_{n-k}} \\ \text{implies} \\ F_n = \sum_{k=0}^{n} (-1)^k \ q_1^{k(k-2n+1)/2} \ \dfrac{[q^{p+bk-n+1}]_\infty}{(q_1;q_1)_{n-k}} \ \mu(k), \end{cases}$$

it follows from (4.3.14) and (4.3.15) that

$$(4.3.17)\quad \begin{cases} \mu(sn) = \sum_{k=0}^{sn} q_1^{k(k-1)/2} \dfrac{(1-q^{p+bk-k}) \qquad F_k}{[q^{p+bsn-k}]_\infty \ (q_1;q_1)_{sn-k}} \\ \text{implies} \\ F_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \ q_1^{sk(sk-2n+1)/2} \ \dfrac{[q^{p+bsk-n+1}]_\infty}{(q_1;q_1)_{n-sk}} \ \mu(sk), \end{cases}$$

where $\mu(sn) = G_n$.

Thus, the relation (4.3.2) with $\mu(n)=0$ ($n \neq ms$), implies the relation (4.3.1), which proves the converse part, and hence the theorem.

The pair of inverse relations (4.3.4) and (4.3.5) are contained in the pair (4.3.6) (with j=0 and s=1), and therefore, its proof is omitted here.

4.4 PARTICULAR CASES : POLYNOMIALS

Amongst the several interesting particular cases of theorem-3, the basic analogues of the polynomial $g_n^c(x,r,s)$ and its special case viz. the extended Jacobi polynomial (see(4.1.4)) will be obtained along with their inverse series relations. As for the other consequences of theorem-3, the basic analogues of the extended polynomials $Q_{n,\ell,s,}(x;\alpha,\beta,N)$, $R_{n,\ell,s}(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)$, and $P_{n,\ell,s}(x^2)$ will be obtained together with the inverse series representations of each of them. The basic analogues of these extended polynomials will be denoted by $Q_{n,\ell,s}(x;\alpha,\beta,N|q)$, $R_{n,\ell,s}(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta|q)$, and $P_{n,\ell,s}(x^2|q)$ respectively.

Now in order to get the basic polynomial $g_n^c(x,r,s;q)$ defined by (4.2.7), put bs=s-r, p=-c, and $G_k=\delta_k x^k$ in the relation (4.3.1) of theorem-3. In this case, one obtains with the help of the formula

$$(4.4.1)\qquad (q^{-1};q^{-1})_N = (-1)^N q^{-N(N+1)/2} [q]_N ,$$

the polynomial $g_n^c(x,r,s;q)$ as defined in (4.2.7), whose inverse series relation as given below follows from (4.3.2) under the same substitutions.

$$(4.4.2)\ \delta_n x^n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{q^{(r-s)k(k-1)/2s}}{[q^c]_{rn-sn+k+1}} \frac{[q^c]_{rn}\ (1-q^{c+Rk})}{(q^{R-1};q^{R-1})_{sn-k}} \cdot g_k^c(x,r,s;q),$$

where, as before, R=r/s (r≠s) (cf.(4.1.3)).

The alternative forms of $g_n^c(x,r,s;q)$ and its inverse relation (4.4.2) may be obtained by transforming the relations (4.3.1) and (4.3.2) of theorem-3, appropriately.

In fact, on making use of the formulas (4.4.1),

$$(4.4.3)\qquad [a]_{-N}\,[q/a]_N = (-1)^N\, q^{N(N-2a+1)/2},$$

and

$$(4.4.4)\qquad [q^{-N}]_m\,[q]_{N-m} = (-1)^m\, q^{m(m-2N-1)/2}\,[q]_N$$

in theorem-3 and then using the substitutions bs=s-r, p=-c, and $F_n=[q^{-c-n+1}]_\infty F_n^*$, one arrives at the following alternative forms of (4.3.1) and (4.3.2).

$$(4.4.5)\begin{cases} F_n^* = \sum_{k=0}^{[n/s]} q^{(r-s)k}\,(q_2^{-n};q_2)_{sk}\,[q^{c+n}]_{rk-sk}\,G_k\,, \\[2ex] G_n = \sum_{k=0}^{sn} q^{(r-s)nk}\,\dfrac{(q_2^{-sn};q_2)_k\,(1-q^{c+Rk})\qquad F_k^*}{[q^{c+k}]_{rn-sn+1}\,(q_2;q_2)_k\,(q_2;q_2)_{sn}}\,, \end{cases}$$

wherein $q_2 = q^{R-1}$, $R = r/s$.

In the pair (4.4.5), putting $G_k=\delta_k x^k$ one gets yet another alternative versions of $g_n^c(x,r,s;q)$ and its inverse in the forms

$$(4.4.6)\begin{cases} g_n^c(x,r,s;q) = \sum_{k=0}^{[n/s]} q^{(r-s)k}\,(q^{-n(R-1)};q^{R-1})_{sk}[q^{c+n}]_{rk-sk}\,\delta_k\,x^k, \\[2ex] \delta_n x^n = \sum_{k=0}^{sn} q^{(r-s)nk}\,\dfrac{(q^{-sn(R-1)};q^{R-1})_k\,(1-q^{c+Rk})\,g_k^c(x,r,s;q)}{[q^{c+k}]_{rn-sn+1}\,(q^{R-1};q^{R-1})_k(q^{R-1};q^{R-1})_{sn}}. \end{cases}$$

It is to be mentioned here that the polynomial $g_n^c(x,r,s;q)$ given by (4.2.7) when considered in the form $(R-1)^n\, g_n^c(x(R-1)^{-s},r,s;q)$, readily approches, as $q\to 1$, to the 'ordinary' polynomial $g_n^c(x,r,s)$ given in (4.1.2). However, the limits of the expressions (4.3.1) and (4.3.2) of theorem-3, when $q \to 1$, may be examined by converting them into following forms with the aid of the basic Gamma function :

$$\Gamma_q(x) = \frac{[q]_\infty}{[q^x]_\infty} (1-q)^{1-x} \qquad (|q|<1)$$

(quoted in section - 1.4).

$$(4.4.7)\begin{cases} F_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \dfrac{q^{bsk(sk-2n+1)/2}\,[q]_\infty (1-q)^{-p+n-bsk}}{\Gamma_q(1+p-n+bsk)\quad (q^b;q^b)_{n-sk}} G_k, \\ G_n = \sum_{k=0}^{sn} \dfrac{q^{bk(k-1)/2}}{(q^b;q^b)_{sn-k}} \dfrac{1-q^{p+bk-k}}{1-q^{p+bsn-k}} \dfrac{\Gamma_q(1+p+bsn-k)}{[q]_\infty (1-q)^{-p-bsn+k}} F_k. \end{cases}$$

A more convenient form of this pair may be obtained by replacing G_k by

$$\frac{\Gamma_q(1+p+bsk-sk)\; G_k}{[q]_\infty (1-q)^{-p-bsk}\; b^{sk}},$$

and, F_k by $(1-q)^k\, b^{-k}\, F_k$. In this case, the pair (4.4.7) yields the corresponding ordinary forms mentioned in (4.1.1), as $q \to 1$. Similarly, it can be shown that the equation (4.3.3) when $q \to 1$, gives the corresponding equation appearing in (4.1.1).

Now in order to illustrate a basic analogue of the extended Jacobi polynomial (4.1.4), put

$$\delta_k = \frac{[\alpha_1]_k \cdots [\alpha_i]_k}{[\beta_1]_k \cdots [\beta_j]_k\, [q]_k}$$

in (4.4.6). If $r-s(=\ell)$ denotes a positive integer, then (4.4.6) gives rise to the following inverse pair of a basic extended Jacobi polynomial which is denoted here by

$F^{(c)}_{n,\ell,s}[\alpha_1,\ldots,\alpha_i;\beta_1,\ldots,\beta_j:x|q]$.

$$(4.4.8)\quad \begin{cases} F^{(c)}_{n,\ell,s}[\alpha_1,\ldots,\alpha_i;\beta_1,\ldots,\beta_j:x|q] = \sum_{k=0}^{[n/s]} q^{\ell k}\,(q_2^{-n};q_2)_{sk}\,\cdot \\ \qquad\qquad \cdot\,[q^{c+n}]_{\ell k}\,\dfrac{[\alpha_1]_k\cdots[\alpha_i]_k\quad x^k}{[\beta_1]_k\cdots[\beta_j]_k\quad [q]_k}, \\ \dfrac{[\alpha_1]_n\cdots[\alpha_i]_n\quad x^n}{[\beta_1]_n\cdots[\beta_j]_n\quad [q]_n} = \sum_{k=0}^{sn} q^{\ell nk}\,\dfrac{(q_2^{-sn};q_2)_k\,(1-q^{c+Lk+k})}{[q^{c+k}]_{\ell n+1}\quad (q_2;q_2)_k} \\ \qquad\qquad \cdot\,\dfrac{F^{(c)}_{k,\ell,s}(\alpha_1,\ldots,\alpha_i;\beta_1,\ldots,\beta_j:x|s)}{(q_2;q_2)_{sn}}, \end{cases}$$

wherein $q_2 = q^{\ell/s}$, and $L = \ell/s$.

It can be seen that the extended polynomials of Hahn, Racah, and of Wilson ((4.1.7), (4.1.9), (4.1.11)) in the light of the basic pair (4.4.5), admit q-extensions in the forms as deduced below.

In the first place put $c = \alpha+\beta+1$, and

$$G_k = \frac{[q^{-x}]_k}{[\alpha q]_k\,[q^{-N}]_k\,[q]_k}$$

in (4.4.5). Then with $r-s=\ell$, one finds a basic analogue of the polynomial $Q_{n,\ell,s}(x;\alpha,\beta,N)$ in the form :

$$(4.4.9)\quad Q_{n,\ell,s}(x;\alpha,\beta,N|q) = \sum_{k=0}^{[n/s]} q^{\ell k}\,\frac{(q_2^{-n};q_2)_{sk}\,[\alpha\beta q^{n+1}]_{\ell k}\,[q^{-x}]_k}{[\alpha q]_k\,[q^{-N}]_k\,[q]_k}\,.$$

whose inverse relation may be expressed by

$$(4.4.10)\quad \frac{[q^{-x}]_n}{[q^{-N}]_n[\alpha q]_n[q]_n} = \sum_{k=0}^{sn} \frac{q^{\ell nk}\,(q_2^{-sn};q_2)_k\,(1-\alpha\beta q^{Lk+k+1})}{[\alpha\beta q^{k+1}]_{\ell n+1}\,(q_2;q_2)_k\,(q_2;q_2)_{sn}} .$$

$$\cdot \Omega_{k,\ell.s}(x;\alpha,\beta,N|q)\ ,$$

where $q_2 = q^{\ell/s}$, and $L = \ell/s$.

Also, when $c=\alpha+\beta+1$, and

$$G_k = \frac{[q^{-x}]_k\ [\gamma\delta q^{x+1}]_k}{[\alpha q]_k\ [\beta\delta q]_k\ [\gamma q]_k\ [q]_k}\ ,$$

the pair (4.4.5) provides a basic analogue of the extended Racah polynomial which is representable in the form :

$$(4.4.11)\quad R_{n,\ell,s}(\mu(x);\alpha,\beta,\gamma,\delta|q) = \sum_{k=0}^{[n/s]} \frac{q^{\ell k}\,(q_2^{-n};q_2)_{sk}\,[\alpha\beta q^{n+1}]_{\ell k}}{[\alpha q]_k\ [\beta\delta q]_k\ [\gamma q]_k\ [q]_k} .$$

$$\cdot\ [q^{-x}]_k\ [\gamma\delta q^{x+1}]_k$$

along with its inverse series relation:

$$(4.4.12)\quad \frac{[q^{-x}]_n\ [\gamma\delta q^{x+1}]_n}{[\alpha q]_n[\beta\delta q]_n[\gamma q]_n[q]_n} = \sum_{k=0}^{sn} q^{\ell nk}\,\frac{(q_2^{-sn};q_2)_k\,(1-\alpha\beta q^{Lk+k+1})}{[\alpha\beta q^{k+1}]_{\ell n+1}\,(q_2;q_2)_k} .$$

$$\cdot\ \frac{R_{k,\ell.s}(\mu(x);\alpha,\beta,\gamma,\delta|q)}{(q_2;q_2)_{sn}}\ ,$$

in which $\mu(x)=q^{-x}+\gamma\delta q^{x+1}$, $q_2=q^{\ell/s}$, and $L=\ell/s$.

Similarly, with 'c' = a+b+c+d-1, and

$$G_k = \frac{[ae^{i\theta}]_k\ \ [ae^{-i\theta}]_k}{[ab]_k\ [ac]_k\ [ad]_k\ [q]_k}\ ,$$

the relation in (4.4.5) results in an inverse pair of basic analogue of $P_{n,\ell,s}(x^2)$ which may be expressed as

$$(4.4.13)\quad \begin{cases} \dfrac{P_{n,\ell,s}(x;a,b,c,d|q)}{[ab]_n\ [ac]_n\ [ad]_n} = \displaystyle\sum_{k=0}^{[n/s]} \dfrac{(q_2^{-\ell n};q_2)_{sk}\ [abcdq^{n-1}]_{\ell k}}{[ab]_k\ [ac]_k\ [ad]_k\ [q]_k} \,.\, q^{\ell k}\ [ae^{i\theta}]_k\ [ae^{-i\theta}]_k, \\[2ex] \dfrac{(q_2;q_2)_{sn}[ae^{i\theta}]_n[ae^{-i\theta}]_n}{[q]_n\ [ab]_n\ [ac]_n\ [ad]_n} = \displaystyle\sum_{k=0}^{sn} \dfrac{(q_2^{-sn};q_2)_k(1-abcdq^{Lk+k-1})}{[abcdq^{k-1}]_{\ell n+1}[ab]_k[ac]_k} \,.\, q^{\ell nk}\ \dfrac{P_{k,\ell,s}(x;a,b,c,d|q)}{[ad]_k\ (q_2;q_2)_k}, \end{cases}$$

wherein $x = \cos\theta$, and as usual $q_2 = q^L$, $L = \ell/s$.

It may be observed that the polynomials deduced above in (4.4.9), (4.4.11) and in (4.4.13), besides providing basic analogues of the corresponding ordinary polynomials, also provide extensions of polynomials of basic Hahn, basic Racah, and of Asky-Wilson (cf.(1.5.9),(1.5.10),(1.5.11)).

4.5 q-EXTENSIONS OF CERTAIN INVERSE RELATIONS

It is interesting to remark here that theorem-3 and a few of its alternative forms also lead to the extensions of the inverse series relations given in table-20. With a view to obtain these extensions, it may be observed that on making use of the formula

$$(4.5.1) \qquad (q^{-1};\, q^{-1})_N = (-1)^N q^{-N(N-1)/2} [q]_N .$$

and by replacing G_k by $[q]_{p+bsk-sk} G_k / [q]_\infty$, the relations (4.3.1) and (4.3.2) (i.e. theorem-3) get transformed to the forms

$$(4.5.2) \begin{cases} F_n = \sum_{k=0}^{[n/s]} q^{-bsk(sk-1)/2} \dfrac{[q]_{p+bsk-sk} \quad G_k}{[q]_{p+bsk-n} \; (q^{-b};q^{-b})_{n-sk}} , \\ G_n = \sum_{k=0}^{sn} (-1)^{sn+k} q^{-bk(k-2sn+1)/2} \; . \\ \qquad \cdot \dfrac{(1-q^{p+bk-k}) \; [q]_{p+bsn-k-1}}{[q]_{p+bsn-sn} \; (q^{-b};q^{-b})_{sn-k}} F_k . \end{cases}$$

This pair may also be put in the following alternative forms.

$$(4.5.3) \begin{cases} F_n = \sum_{k=0}^{[n/s]} q^{-bsk(sk-1)/2} \dfrac{(1-q^{p+bn-n+1}) [q]_{p+bsk-sk}}{[q]_{p+bsk-n+1} (q^{-b};q^{-b})_{n-sk}} G_k , \\ G_n = \sum_{k=0}^{sn} (-1)^{sn+k} q^{-bk(k-2sn+1)/2} \; . \\ \qquad \cdot \dfrac{[q]_{p+bsn-k} \quad F_k}{[q]_{p+bsn-sn} \; (q^{-b};q^{-b})_{sn-k}} ; \end{cases}$$

$$(4.5.4) \begin{cases} F_n = \sum_{k=0}^{[n/s]} q^{bsk(sk-1)/2} \dfrac{[q]_{-p+n-bsk-1} \quad G_k}{[q]_{-p-bsk+sk-1} \; (q^{b};q^{b})_{n-sk}} , \\ G_n = \sum_{k=0}^{sn} (-1)^{sn+k} q^{bk(k-2sn+1)/2} \; . \\ \qquad \cdot \dfrac{(1-q^{-p-bk+k}) \; [q]_{-p-bsn+sn-1}}{[q]_{-p-bsn+k} \; (q^{b};q^{b})_{sn-k}} F_k , \end{cases}$$

and

$$
(4.5.5)\begin{cases}
F_n = \sum_{k=0}^{[n/s]} q^{bsk(sk-1)/2} \dfrac{(1-q^{-p-bn+n})\,[q]_{-p+n-bsk-1}}{[q]_{-p-bsk+sk}\,(q^b;q^b)_{n-sk}} G_k, \\
G_n = \sum_{k=0}^{sn} (-1)^{sn+k} q^{bk(k-2sn+1)/2} \; . \\
\qquad \cdot \dfrac{[q]_{-p-bsn+sn}\; F_k}{[q]_{-p-bsn+k}\,(q^b;q^b)_{sn-k}} .
\end{cases}
$$

The form (4.5.3) is obtained from (4.5.2) by replacing first p by p+1, and then replacing F_n by $F_n\ /1-q^{p+bn-n+1}$, and G_n by $G_n\ /1-q^{p+bsn-sn+1}$. Whereas to obtain (4.5.4), the base q in (4.5.2) is inverted first and then, G_k is replaced by $G_k\ /(q^{-p-bsk+sk-1};q^{-1})_\infty$. If in (4.5.4) F_n is replaced by $F_n\ /1-q^{-p-bn+n}$, then it gets transformed to (4.5.5).

It is obvious that the inverse pair (4.5.3), (with b=m) provides an extension of the basic Gould class (2) :

$$
(4.5.6)\begin{cases}
F(n) = \sum_{k=0}^{n} q^{-mk(k-1)/2} \dfrac{(1-q^{p+mn-n+1})[q]_{p+mk-k}}{[q]_{p+mk-n+1}(q^{-m};q^{-m})_{n-k}} g(k), \\
G(n) = \sum_{k=0}^{n} (-1)^{n+k} q^{-mk(k-2n+1)/2} \; . \\
\qquad \cdot \dfrac{[q]_{p+mn-k}\; F(k)}{[q]_{p+mn-n}\,(q^{-m};q^{-m})_{n-k}}
\end{cases}
$$

of Table-15 (Ch.3). Also, when b is replaced by c+1, the pair (4.5.3) provides extension of the basic Legendre-Chebyshev class

(7) of Table-19 (Ch.3), i.e.

$$(4.5.7)\quad \begin{cases} F(n) = \sum_{k=0}^{n} q^{-(c+1)k(k-1)/2} \dfrac{(1-q^{p+cn+1})\,[q]_{p+ck}\; G(k)}{[q]_{p+ck+k-n+1}(q^{-c-1};q^{-c-1})_{n-k}} , \\[2ex] G(n) = \sum_{k=0}^{n} (-1)^{n+k}\, q^{-(c+1)k(k-2n+1)/2} \,. \\[1ex] \qquad\qquad \cdot \dfrac{[q]_{p+cn+n-k}\; F(k)}{[q]_{p+cn}\;(q^{-c-1};q^{-c-1})_{n-k}} . \end{cases}$$

Similarly, the extensions of other pairs of Table-20 may be obtained. A complete list of these extensions is given in the following table.

Table-21 *Basic extended inverse relations*

$$F_n = \sum_{k=0}^{[n/s]} q^{\alpha sk(sk-1)/2} A_{n,k} \, G_k \; ; \quad G_n = \sum_{k=0}^{sn} (-1)^{sn-k} \, q^{\alpha k(k-2sn+1)/2} \, B_{n,k} \, F_k$$

Inverse pair (citation)	h	p	α	$A_{n,k}$	$B_{n,k}$	Extension of class (No) in Table-No.
(4.5.2)	m	p	-m	$\frac{[q]_{p+msk-sk}}{[q]_{p+msk-n}\,(q^{-m};q^{-m})_{n-sk}}$	$\frac{(1-q^{p+mk-k})\,[q]_{p+msn-k-1}}{[q]_{p+msn-sn}\,(q^{-m};q^{-m})_{sn-k}}$	Basic Gould class (1). Table-15
(4.5.3)	m	p	-m	$\frac{(1-q^{p+mn-n+1})[q]_{p+msk-sk}}{[q]_{p+msk-n+1}\,(q^{-m};q^{-m})_{n-sk}}$	$\frac{[q]_{p+msn-k}}{[q]_{p+msn-sn}\,(q^{-m};q^{-m})_{sn-k}}$	Basic Gould class (2). Table-15
(4.5.4)	-1	-p-1	-1	$\frac{[q]_{p+n+sk}}{[q]_{p+2sk}\,(q^{-1};q^{-1})_{n-sk}}$	$\frac{(1-q^{p+2k+1})\,[q]_{p+2sn}}{[q]_{p+sn+k+1}\,(q^{-1};q^{-1})_{sn-k}}$	Basic Simpler Legendre class (1). Table-18
(4.5.5)	-1	-p	-1	$\frac{[q]_{p+n+sk-1}\,(1-q^{p+2n})}{[q]_{p+2sk}\,(q^{-1};q^{-1})_{n-sk}}$	$\frac{[q]_{p+2sn}}{[q]_{p+sn+k}\,(q^{-1};q^{-1})_{sn-k}}$	Basic Simpler Legendre class (2). Table-18
(4.5.5)	-c+1	-p	-c+1	$\frac{[q]_{p+n+sck-sk-1}\,(1-q^{p+cn})}{[q]_{p+sck}\,(q^{-c+1};q^{-c+1})_{n-sk}}$	$\frac{[q]_{p+scn}}{[q]_{p+scn-sn+k}\,(q^{-c+1};q^{-c+1})_{sn-k}}$	Basic Legendre-Chebyshev class (1). Table-19
(4.5.2)	c+1	p	-c-1	$\frac{[q]_{p+sck}}{[q]_{p+sck+sk-n}\,(q^{-c-1};q^{-c-1})_{n-sk}}$	$\frac{(1-q^{p+ck})\,[q]_{p+scn+sn-k-1}}{[q]_{p+scn}\,(q^{-c-1};q^{-c-1})_{sn-k}}$	Basic Legendre-Chebyshev class (3). Table-19
(4.5.4)	-c+1	-p-1	-c+1	$\frac{[q]_{p+n+sck-sk}}{[q]_{p+sck}\,(q^{-c+1};q^{-c+1})_{n-sk}}$	$\frac{(1-q^{p+ck+1})\,[q]_{p+scn}}{[q]_{p+scn-sn+k+1}\,(q^{-c+1};q^{-c+1})_{sn-k}}$	Basic Legendre-Chebyshev class (5). Table-19
(4.5.3)	c+1	p	-c-1	$\frac{[q]_{p+sck}\,(1-q^{p+cn+1})}{[q]_{p+sck+sk-n+1}\,(q^{-c-1};q^{-c-1})_{n-sk}}$	$\frac{[q]_{p+scn+sn-k}}{[q]_{p+scn}\,(q^{-c-1};q^{-c-1})_{sn-k}}$	Basic Legendre-Chebyshev class (7). Table-19