C H A P T E R - 5

FURTHER EXTENSIONS OF CERTAIN GENERAL INVERSE PAIRS

5.1 INTRODUCTION

Amongst the known inverse series relations that are referred to in the previous chapters, the most elegant and possessing the potential for further useful generalizations is the pair

$$(5.1.1)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left\{ \prod_{i=1}^{n} (a_i+kb_i) \right\} g(k), \\ g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \dfrac{a_{k+1} + k\, b_{k+1}}{\prod_{i=1}^{k+1} (a_i+nb_i)} f(k) \end{cases}$$

due to Gould and Hsu [1].

A basic analogue of this inversion pair was given by Carlitz [3] in the form :

$$(5.1.2)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \left\{ \prod_{i=1}^{n} (a_i+q^{-k} b_i) \right\} g(k), \\ g(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \dfrac{a_{k+1}+q^{-k}\, b_{k+1}}{\prod_{i=1}^{k+1} (a_i+q^{-n} b_i)} f(k). \end{cases}$$

These two pairs which encompass quite a good number of inversion formulas (quoted in sections- 1.2 and 1.5) , were encountered during the process of providing extensions of various known inverse series relatios. While making an attempt of extending the aforementioned pairs, it was observed that the forms in which they admit extensions are analogous to the forms of the following pair of relations due to Singhal and S.Kumari[4].

$$(5.1.3)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} f(k) \\ \text{if and only if} \\ f(n) = \sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k) \\ \text{and} \\ \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k)=0, \quad n\neq ms,\ m=1,2,3,\ldots . \end{cases}$$

The proposed extensions of the pairs (5.1.1) and (5.1.2) which will be proved in the sections 5.2 and 5.3 respectively, may be stated in the forms of following theorems.

THEOREM - 4. For $s = 2,3,4,\ldots,$and for $m=1,2,3,\ldots,$

$$(5.1.4)\quad W_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \left\{ \prod_{i=1}^{n} (a_i+skb_i) \right\} \frac{V_k}{(n-sk)!}$$

if and only if

$$(5.1.5)\quad V_n = \sum_{k=0}^{sn} (a_{k+1}+kb_{k+1}) \left\{ \prod_{i=1}^{k+1} (a_i+snb_i) \right\}^{-1} \frac{W_k}{(sn-k)!}$$

and

$$(5.1.6)\quad \sum_{k=0}^{n} (a_{k+1}+kb_{k+1}) \left\{ \prod_{i=1}^{k+1} (a_i+nb_i) \right\}^{-1} \frac{W_k}{(n-k)!} = 0, \text{ when } n\neq ms.$$

THEOREM - 5. For $s = 2,3,4,\ldots,$and for $m=1,2,3,\ldots,$

$$(5.1.7)\quad U_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} q^{sk(sk-1)/2} \left\{ \prod_{i=1}^{n} (a_i+q^{-sk} b_i) \right\} \frac{T_k}{[q]_{n-sk}}$$

if and only if

$$(5.1.8)\quad T_n = \sum_{k=0}^{sn} q^{k(k-2sn+1)/2} \frac{a_{k+1} +q^{-k} b_{k+1}}{\prod_{i=1}^{k+1} (a_i+q^{-sn} b_i)} \frac{U_k}{[q]_{sn-k}}$$

and

$$(5.1.9)\quad \sum_{k=0}^{n} q^{k(k-2n+1)/2} \frac{a_{k+1}+q^{-k} b_{k+1}}{\prod_{i=1}^{k+1} (a_i +q^{-n} b_i)} \frac{U_k}{[q]_{n-k}} = 0. \text{ when } n\neq ms.$$

It may be pointed out here that the pairs (5.1.1) and (5.1.2) also admit alternative extensions which are stated in the forms of theorems 6 and 7.

THEOREM - 6. For $s = 2,3,4,\ldots,$and for $m=0,1,2,\ldots,$

$$(5.1.10)\quad A_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \frac{a_{sk+1}+sk\ b_{sk+1}}{\prod_{i=1}^{sk+1} (a_i+nb_i)} \frac{B_k}{(n-sk)!}$$

if and only if

$$(5.1.11)\quad B_n = \sum_{k=0}^{sn} \left\{ \prod_{i=1}^{sn} (a_i+kb_i) \right\} \frac{A_k}{(sn-k)!}$$

and

$$(5.1.12)\quad \sum_{k=0}^{n} \left\{ \prod_{i=1}^{n} (a_i+kb_i) \right\} \frac{A_k}{(n-k)!} = 0\ , \quad n > ms.$$

THEOREM - 7. For $s = 2,3,4,\ldots$, and for $m=0,1,2,\ldots$,

$$(5.1.13)\quad C_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \frac{q^{sk(sk-2n+1)/2}}{[q]_{n-sk}} \frac{(a_{sk+1}+q^{-sk} b_{sk+1})}{\prod_{i=1}^{sk+1} (a_i+q^{-n}b_i)} D_k.$$

if and only if

$$(5.1.14)\quad D_n = \sum_{k=0}^{sn} q^{k(k-1)/2} \left\{ \prod_{i=1}^{sn}(a_i+q^{-k} b_i)\right\} \frac{C_k}{[q]_{sn-k}},$$

and

$$(5.1.15)\quad \sum_{k=0}^{n} q^{k(k-1)/2} \left\{ \prod_{i=1}^{n}(a_i+q^{-k} b_i)\right\} \frac{C_k}{[q]_{n-k}} = 0, \quad n>ms.$$

The proofs of these theorems are given in sections 5.4, and 5.5 respectively.

The object of studying the extensions stated above as theorems 4 to 7 is that by appropriately choosing the parameters a_i and b_i, it becomes possible to carry out extensions of certain polynomials as well as certain classes of inverse series relations belonging to the Gould classes, simpler Legendre classes, and the Legendre - Chebyshev classes, as is discussed in sections 5.6 and 5.7.

5.2 PROOF OF THEOREM-4

While proving theorem-4, the following orthogonality relation will be used; which is supplied by the pair (5.1.1) under the replacement of j, n, and n-j by sj, sn and sn-sj(=N) respectively.

$$(5.2.1)\quad \sum_{k=0}^{N} (-1)^k \binom{N}{k} \frac{(a_{k+sj+1}+(k+sj)b_{k+sj+1})}{\prod_{i=1}^{k+sj+1} (a_i+(N+sj)b_i)} \prod_{i=1}^{k+sj} (a_i+sjb_i) = \binom{0}{N}$$

(Gould and Hsu [1,p.887]).

Now, in order to prove the 'only if' part, it will be first shown that (5.1.4) implies (5.1.5).

If the right hand side of (5.1.5) be denoted by V^*, then in view of the relation (5.1.4) it can be seen that

$$V^* = \sum_{k=0}^{sn} \sum_{j=0}^{[k/s]} (-1)^{k+sj} \frac{(a_{k+1}+k\, b_{k+1})}{(sn-k)!\,(k-sj)!} \frac{\prod_{i=1}^{k} (a_i+sj\, b_i)}{\prod_{i=1}^{k+1} (a_i+sn\, b_i)} V_j$$

$$= V_n + \sum_{j=0}^{n-1} \frac{1}{(sn-sj)!} V_j \sum_{k=0}^{sn-sj} (-1)^k \binom{sn-sj}{k} \frac{\prod_{i=1}^{k+sj} (a_i+sjb_i)}{\prod_{i=1}^{k+sj+1} (a_i+snb_i)} \cdot (a_{k+sj+1}+(k+sj)b_{k+sj+1}).$$

Here, on comparing the inner series with the orthogonality relation (5.2.1), one readily gets

$$V^* = V_n + \sum_{j=1}^{n-1} \frac{1}{(sn-sj)!} \binom{0}{sn-sj} V_j ,$$

whence it follows that

$$V^* = V_n$$

and thus, (5.1.4) implies (5.1.5).

For completing the proof of 'only if'- part, it remains to show that the relation (5.1.4) also implies (5.1.6) when n/s is not an integer. For this, put

$$(5.2.2)\quad \sum_{k=0}^{n} (a_{k+1}+k\, b_{k+1}) \left\{ \prod_{i=1}^{k+1}(a_i+n\, b_i)\right\}^{-1} \frac{W_k}{(n-k)!} = \bar{V}_n ,$$

then on making use of the relation (5.1.4) and proceeding as above, one arrives at

$$(5.2.3)\quad \bar{V}_n = \sum_{j=0}^{[n/s]} \frac{1}{(n-sj)!} \binom{0}{n-sj} V_j .$$

Here, if n/s is not an integer that is, $n \neq ms$ (m=1,2,3...) then $\bar{V}_n$ in (5.2.3) vanishes, which completes the proof of 'only if' part.

The proof of 'if' part runs as follows.

First, it is to be noted that the inverse series relation of (5.2.2) is given by (Gould and Hsu [1]):

$$(5.2.4)\quad W_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{ \prod_{i=1}^{n}(a_i+kb_i)\right\} \frac{\bar{V}_k}{(n-k)!} .$$

Since the relation (5.1.6) holds, therefore $\bar{V}_n=0$ when $n \neq ms$. If n=ms, then it follows from (5.1.5) that $\bar{V}_{ms}(=\bar{V}_{sm})=V_m$. Thus, the inverse relations given in (5.2.2) and (5.2.4) will assume the forms

$$V_n = \sum_{k=0}^{sn} (a_{k+1}+kb_{k+1}) \left\{ \prod_{i=1}^{k+1} (a_i+snb_i)\right\}^{-1} \frac{W_k}{(sn-k)!}$$

implies

$$W_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \left\{ \prod_{i=1}^{n}(a_i+skb_i)\right\} \frac{V_k}{(n-sk)!} ,$$

with this the proof of the 'if' part is completed, and hence the theorem.

5.3 PROOF OF THEOREM - 5

The proof of theorem-5 runs parallel to that of previous theorem in which the following orthogonality relation will be used.

$$(5.3.1)\quad \sum_{k=0}^{N} (-1)^k q^{k(k-2N+1)/2} \left[{N \atop k} \right] (a_{k+sj+1}+q^{-k-sj} b_{k+sj+1}) \cdot \left\{ \prod_{i=1}^{k+sj} (a_i+q^{-sj}b_i) \right\} \left\{ \prod_{i=1}^{k+sj+1} (a_i+q^{-N-sj} b_i) \right\}^{-1} = \left[{0 \atop N} \right].$$

In order to prove the first part, i.e. (5.1.7) implies both (5.1.8) and (5.1.9), consider the left hand side of (5.1.8) and denote it by T^*, i.e.,

$$T^* = \sum_{k=0}^{sn} q^{k(k-2sn+1)/2} (a_{k+1}+q^{-k}b_{k+1}) \left\{ \prod_{i=1}^{k+1} (a_i+q^{-sn}b_i) \right\}^{-1} \frac{U_k}{[q]_{n-k}}.$$

In this, if the relation (5.1.7) is substituted for U_k, then one obtains

$$T^* = \sum_{k=0}^{sn} \sum_{k=0}^{[k/s]} (-1)^{k-sj} q^{(k(k-2sn+1)+sj(sj-1))/2} \frac{a_{k+1}+q^{-k}b_{k+1}}{[q]_{sn-k}\,[q]_{k-sj}} \cdot \left\{ \prod_{i=1}^{k} (a_i+q^{-sj} b_i) \right\} \left\{ \prod_{i=1}^{k+1} (a_i+q^{-sn} b_i) \right\}^{-1} T_j$$

$$= T_n + \sum_{j=0}^{n-1} \frac{q^{s^2j^2-s^2nj}\, T_j}{[q]_{sn-sj}} \sum_{k=0}^{sn-sj} (-1)^k q^{k(k-2sn+2sj+1)/2} \left[{sn-sj \atop k} \right] \cdot (a_{k+sj+1}+q^{-k-sj} b_{k+sj+1}) \frac{\prod_{i=1}^{k+sj} (a_i+q^{-sj} b_i)}{\prod_{i=1}^{k+sj+1} (a_i+q^{-sn} b_i)}.$$

In this last expression, on making use of the relation

(5.3.1), one finds

$$T^* = T_n + \sum_{j=0}^{n-1} q^{s^2j^2-s^2nj} \frac{T_j}{[q]_{sn-sj}} \left[{0 \atop sn-sj} \right] .$$

which ultimately simplifies to

$$T^* = T_n ,$$

and thus, (5.1.7) implies (5.1.8).

Now to show that the relation (5.1.7) also implies (5.1.9), put

$$\bar{T}_n = \sum_{k=0}^{n} q^{k(k-2n+1)/2} (a_{k+1}+q^{-k} b_{k+1}) \left\{ \prod_{i=1}^{k+1} (a_i+q^{-n} b_i) \right\}^{-1} \frac{U_k}{[q]_{n-k}},$$

wherein the use of the relation (5.1.7) gives

$$\bar{T}_n = \sum_{k=0}^{n} \sum_{j=0}^{[k/s]} (-1)^{k-sj} \frac{q^{(k(k-2n+1)+sj(sj-1))/2}}{\left\{ \prod_{i=1}^{k+1} (a_i+q^{-n} b_i) \right\} [q]_{n-k} [q]_{k-sj}} .(a_{k+1}+q^{-k}b_{k+1}) \left\{ \prod_{i=1}^{k} (a_i+q^{-sj} b_i) \right\} T_j$$

(5.3.2)
$$= \sum_{j=0}^{[n/s]} q^{s^2j^2-s^2nj} \frac{T_j}{[q]_{n-sj}} \sum_{k=0}^{n-sj} (-1)^k \frac{q^{(k-2n+2sj+1)k/2}}{\prod_{i=1}^{k+sj+1} (a_i+q^{-n}b_i)} . \left[{n-sj \atop k} \right] (a_{k+sj+1}+q^{-k-sj} b_{k+sj+1}) \left\{ \prod_{i=1}^{k+sj} (a_i+q^{-sj} b_i) \right\}.$$

Here, the orthogonality relation (5.3.1) with $N = n-sj$ transforms (5.3.2) in the form

(5.3.3) $$\bar{T}_n = \sum_{j=0}^{[n/s]} q^{s^2j^2-s^2nj} \left[{0 \atop n-sj} \right] \frac{T_j}{[q]_{n-sj}} ,$$

from which it follows that if $n \neq ms$ $(m = 1,2,3,....)$, then

$$\bar{T}_n = 0,$$

which completes the proof of the first part.

It is to be noted that when n = ms, (5.3.3) gives

$$\bar{T}_{ms} = \sum_{j=0}^{m} q^{s^2j^2-s^2mj} \left[{0 \atop sm-sj}\right] \frac{T_j}{[q]_{sm-sj}}$$

$$= T_m + \sum_{j=0}^{m-1} q^{s^2j^2-s^2mj} \left[{0 \atop sm-sj}\right] \frac{T_j}{[q]_{sm-sj}}$$

(5.3.4) $\quad = T_m.$

Now, in order to prove the converse part i.e., the relations (5.1.8) and (5.1.9) imply (5.1.7), we make an appeal to the pair (5.1.2) to get the inverse companion of

(5.3.5) $$\bar{T}_n = \sum_{k=0}^{n} q^{(k-2n+1)/2} \left[{n \atop k}\right] (a_{k+1}+q^{-k} b_{k+1}) \left\{\prod_{i=1}^{k+1} (a_i+q^{-n} b_i)\right\}^{-1} U_k$$

in the form

(5.3.6) $$U_n = \sum_{k=0}^{n} (-1)^{n-k} q^{k(k-1)/2} \left[{n \atop k}\right] \left\{\prod_{i=1}^{n} (a_i+q^{-k} b_i)\right\} \bar{T}_k .$$

Since the relation (5.1.9) holds, it follows that

(5.3.7) $\quad \bar{T}_n = 0$ if $n \neq ms$.

Also from (5.3.4),

(5.3.8) $\quad \bar{T}_{ms} = \bar{T}_m$ if $n = ms$.

Thus, in view of (5.3.7) and (5.3.8) , the inverse relations (5.3.5) and (5.3.6) assume the forms :

$$\bar{T}_{ns} = \sum_{k=0}^{sn} q^{k(k-2sn+1)/2} \left[{sn \atop k}\right] (a_{k+1}+q^{-k}b_{k+1}) \left\{\prod_{i=1}^{k+1} (a_i+q^{-sn} b_i)\right\}^{-1} U_k$$

implies

$$U_n = \sum^{[n/s]} (-1)^{n-sk} q^{k(k-1)/2} \left[{n \atop sk}\right] \left\{\prod_{i=1}^{n} (a_i+q^{-sk} b_i)\right\} \bar{T}_{ks} ,$$

which readily leads to

$$T_n = \sum_{k=0}^{sn} q^{k(k-2sn+1)/2} (a_{k+1}+q^{-k}b_{k+1}) \left\{ \prod_{i=1}^{k+1} (a_i+q^{-sn}b_i) \right\}^{-1} \frac{U_k}{[q]_{sn-k}}$$

implies

$$U_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} q^{k(k-1)/2} \left\{ \prod_{i=1}^{n} (a_i+q^{-sk} b_i) \right\} \frac{T_k}{[q]_{sn-k}} .$$

if $\overline{T}_n=0$. Thus, (5.1.8) and (5.1.9) imply (5.1.7) which completes the proof of the theorem.

5.4 PROOF OF THEOREM-6

It is well known that if $P_n(x)$ is a polynomial of degree n<N, then

$$(5.4.1) \sum_{k=0}^{N} (-1)^k \binom{n}{k} P_n(a+bk) = 0, \ N \geq 1.$$

This preliminary result provides a useful tool in proving the 'only if' part of theorem-6.

In fact, if the left hand side of (5.1.11) is denoted by B, then in view of the relation (5.1.10), one gets

$$B = \sum_{k=0}^{sn} \sum_{j=0}^{[k/s]} (-1)^{k-sj} \frac{a_{sj+1}+sjb_{sj+1}}{(k-sj)!\ (sn-k)!} \frac{\prod_{i=1}^{sn}(a_i+kb_i)}{\prod_{i=1}^{sj+1}(a_i+kb_i)} B_j$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{sn-sj} (-1)^k \frac{a_{sj+1}+sjb_{sj+1}}{k!\ (sn-sj-k)!} \frac{\prod_{i=1}^{sn}(a_i+(k+sj)b_i)}{\prod_{i=1}^{sj+1}(a_i+(k+sj)b_i)} B_j$$

$$= B_n + \sum_{j=0}^{n-1} \sum_{k=0}^{sn-sj} \frac{(a_{sj+1}+sjb_{sj+1})}{(sn-sj)!} B_j \sum_{k=0}^{n-sj} (-1)^k \binom{sn-sj}{k} \cdot \left\{ \prod_{i=sj+2}^{sn} (a_i+(k+sj)b_i) \right\} .$$

Since, the product terms in this last expression represents the polynomial of degree sn-sj-1 in k, it follows from (5.4.1) that the inner series vanishes for all j = 0,1,2,...,n-1, and thus,

$$B = B_n ;$$

which proves that (5.1.10) implies (5.1.11).

In order to prove that (5.1.10) also implies (5.1.12), put

$$\alpha_n = \sum_{k=0}^{n} \left\{ \prod_{i=1}^{n} (a_i+kb_i) \right\} \frac{A_k}{(n-k)!} .$$

Then with the aid of the relation (5.1.10), and proceeding as above, one finds

$$\alpha_n = \sum_{j=0}^{[n/s]} \frac{(a_{sj+1}+sjb_{sj+1})}{(n-sj)!} B_j \sum_{k=0}^{n-sj} (-1)^k \binom{n-sj}{k} \frac{\prod_{i=1}^{n} (a_i+(k+sj)b_i)}{\prod_{i=1}^{sj+1} (a_i+(k+sj)b_i)} .$$

It is not difficult to see that when n-sj≥1, that is n >sj, j = 0,1,2,..., then the inner series in the above expression vanishes in view of the result (5.4.1). Thus, (5.1.10) implies (5.1.12).

Assume now that both the relations (5.1.11) and (5.1.12) hold true; and start with the relation

$$(5.4.2) \quad \alpha_n = \sum_{k=0}^{n} \left\{ \prod_{i=1}^{n} (a_i+kb_i) \right\} \frac{A_k}{(n-k)!} .$$

The inverse series of this is readily obtainable from the pair (5.1.1), in the form :

$$(5.4.3)\quad A_n = \sum_{k=0}^{n} (-1)^{n-k} (a_{k+1}+kb_{k+1}) \left\{\prod_{i=1}^{k+1} (a_i+nb_i)\right\}^{-1} \frac{\alpha_k}{(n-k)!} .$$

Since, (5.1.12) holds, hence

$$(5.4.4)\quad \alpha_n = 0 \text{ for } n > sm,\ m = 0,1,2,\dots .$$

When n =sm, then the relation corresponding to (5.4.2) will be non-zero; in fact, in this case, one finds

$$(5.4.5)\quad \alpha_{sm} = B_m = \sum_{k=0}^{sm} \left\{\prod_{i=1}^{sm} (a_i+kb_i)\right\} \frac{A_k}{(sm-k)!} .$$

Thus, the inverse series relations (5.4.2) and (5.4.3), in view of (5.4.4) and (5.4.5) will become

$$B_m = \sum_{k=0}^{sm} \left\{\prod_{i=1}^{sm} (a_i+kb_i)\right\} \frac{A_k}{(sm-k)!}$$

implies

$$A_m = \sum_{k=0}^{[m/s]} (-1)^{m-sk} (a_{sk+1}+skb_{sk+1}) \left\{\prod_{i=1}^{sk+1} (a_i+mb_i)\right\}^{-1} \frac{\alpha_k}{(m-sk)!}$$

which completes the 'if'-part and hence, the proof of the theorm.

5.5 PROOF OF THEOREM-7

In order to prove the first part, i.e. (5.1.13) implies (5.1.14) and (5.1..15), it may be seen that with an appeal to the relation (5.1.13), the left hand member of (5.1.14) which is denoted for brevity by D, can be expressed as

$$D = \sum_{k=0}^{sn} \sum_{j=0}^{[k/s]} (-1)^{k+sj} q^{(k(k-1)+sj(sj-2k+1))/2} \frac{a_{sj+1}+q^{-sj} b_{sj+1}}{[q]_{sn-k} [q]_{k-sj}} .$$

$$\cdot\left\{\prod_{i=1}^{sn}(a_i+q^{-k}\,b_i)\right\}\left\{\prod_{i=1}^{sj+1}(a_i+q^{-k}\,b_i)\right\}^{-1} D_j$$

$$= D_n + \sum_{j=0}^{n-1}(a_{sj+1}+q^{-sj}\,b_{sj+1})\,\frac{D_j}{[q]_{sn-sj}}\sum_{k=0}^{sn-sj}(-1)^k q^{k(k-1)/2}\,.$$

$$.\left[{sn-sj \atop k}\right]\left\{\prod_{i=1}^{n}(a_i+q^{-k-sj}\,b_i)\right\}\left\{\prod_{i=1}^{sj+1}(a_i+q^{-k-sj}\,b_i)\right\}^{-1}.$$

Here, the ratio of the product terms represents a polynomial of degree (sn-sj-1) in q^{-k}, and therefore on making use of the known formula (Carlitz [3]) :

$$(5.5.1)\quad \sum_{k=0}^{N} q^{k(k-1)/2}\left[{N \atop k}\right] x^k = \prod_{k=1}^{N}(1+xq^{k-1})$$

the inner series in the above expression becomes representable in the form:

$$\sum_{k=0}^{sn-sj}(-1)^k q^{k(k-1)/2}\left[{sn-sj \atop k}\right]\sum_{\ell=0}^{n-sj-1} c_\ell\, q^{-\ell k}\ ,$$

and hence one gets

$$D = D_n + \sum_{j=0}^{n-1}(a_{sj+1}+q^{-sj}b_{sj+1})\,\frac{D_j}{[q]_{sn-sj}}\sum_{\ell=0}^{n-sj-1} c_\ell \prod_{k=1}^{n-sj}(1-q^{-\ell+k-1})$$

$$= D_n\ ,$$

which shows that (5.1.13) implies (5.1.14).

The relation (5.1.13) also implies (5.1.15), which can be proved as follows.

If

$$\phi_n = \sum_{k=0}^{n} q^{k(k-1)/2}\left\{\prod_{i=1}^{n}(a_i+q^{-k}\,b_i)\right\}\frac{C_k}{[q]_{n-k}}\ ,$$

then on substituting the relation (5.1.13) for C_k, and proceeding as above, one obtains

$$\phi_n = \sum_{j=0}^{[n/s]} (a_{sj+1}+q^{-sj}b_{sj+1}) \frac{D_j}{[q]_{n-sj}} \sum_{k=0}^{n-sj} (-1)^k q^{k(k-1)/2} .$$

$$\cdot \left[{n-sj \atop k}\right] \left\{\prod_{i=1}^{n}(a_i+q^{-k-sj}b_i)\right\} \left\{\prod_{i=1}^{sj+1}(a_i+q^{-k-sj}b_i)\right\}^{-1} .$$

In this, the ratio of the product terms when replaced by its equivalent polynomial form :

$$\sum_{r=0}^{n-sj-1} c_r q^{-rk} ,$$

leads to the following expression.

$$\phi_n = \sum_{j=0}^{[n/s]} (a_{sj+1}+q^{-sj}b_{sj+1}) \frac{D_j}{[q]_{n-sj}} \sum_{r=0}^{n-sj-1} c_r \sum_{k=0}^{n-sj} q^{k(k-1)/2} .$$

$$\cdot \left[{n-sj \atop k}\right] (-q^{-r})^k ,$$

wherein the inner-most sum with the help of the formula (5.5.1) assumes the form:

$$\prod_{k=1}^{n-sj} (1-q^{-r+k-1}) = (1-q^{-r}) (1-q^{-r+1}) \ldots (1-q^{-r+n-sj-1}),$$

which ultimately gives

$\phi_n = 0$ when $n>sj$, $j = 0,1,2,\ldots$.

Hence, (5.1.13) implies (5.1.15), and with this the proof of the first part is completed.

Conversely, if the relations (5.1.14) and (5.1.15) hold true, then in order to show that these relations imply (5.1.13), consider the relation

(5.5.2) $$\phi_n = \sum_{k=0}^{n} q^{k(k-1)/2} \left\{\prod_{i=1}^{n}(a_i+q^{-k}\, b_i)\right\} \frac{C_k}{[q]_{n-k}}.$$

But this implies that (cf.(5.1.2))

(5.5.3) $$C_n = \sum_{k=0}^{n} (-1)^{n-k}\, q^{k(k-2n+1)/2}\, \frac{(a_{k+1}+q^{-k}\, b_{k+1})}{\prod_{i=1}^{k+1} (a_i+nb_i)} \quad \frac{\phi_k}{[q]_{n-k}} .$$

Since (5.1.15) holds, therefore

(5.5.4) $\phi_n = 0$ if $n>sm$, $m = 0,1,2,\ldots,$

and when $n = sm$, then on comparing the expression corresponding to ϕ_{sm} with (5.1.14), one finds

(5.5.5) $\phi_{sm} = D_m$.

Thus, the inverse relations (5.5.2) and (5.5.3), in view of (5.5.4) and (5.5.5), get changed to

$$D_m = \sum_{k=0}^{sm} q^{k(k-1)/2} \left\{\prod_{i=1}^{sm}(a_i+q^{-k}b_i)\right\} \frac{C_k}{[q]_{sm-k}}$$

implies

$$C_m = \sum_{k=0}^{[m/s]} (-1)^{m-sk}\, q^{sk(sk-2m+1)/2} \quad \frac{a_{sk+1}+q^{-sk}\, b_{sk+1}}{\prod_{i=1}^{k+1}(a_i+q^{-m}\, b_i)} \quad \frac{D_k}{[q]_{m-sk}} ,$$

which completes the proof of the converse part and hence the proof of theorem-7.

5.6 SOME MORE EXTENSIONS

The inverse series relations proved in the forms of theorems 4 to 7 suggest some more inverse relations including their alternative versions which are given below as theorems 4A, 5A, 6A and 7A.

In the first place, replacing W_n by $W_n/(a_{n+1}+n\,b_{n+1})$ in theorem-4, one obtains

THEOREM - 4A.

$$W_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} (a_{n+1}+nb_{n+1}) \left\{\prod_{i=1}^{n} (a_i+skb_i)\right\} \frac{V_k}{(n-sk)!}$$

if and only if

$$V_n = \sum_{k=0}^{sn} \left\{\prod_{i=1}^{k+1} (a_i+snb_i)\right\}^{-1} \frac{W_k}{(sn-k)!}$$

and

$$\sum_{k=0}^{n} \left\{\prod_{i=1}^{k+1} (a_i+nb_i)\right\}^{-1} \frac{W_k}{(n-k)!} = 0, \quad \text{when } n\neq ms,\ m=1,2,3,\ldots\ .$$

Next, replacing B_n by $B_n/(a_{sn+1}+snb_{sn+1})$, in theorem-6, one easily gets

THEORAM - 6A.

$$A_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \left\{\prod_{i=1}^{sk+1} (a_i+nb_i)\right\}^{-1} \frac{B_k}{(n-sk)!}$$

if and only if

$$B_n = \sum_{k=0}^{sn} (a_{sn+1}+snb_{sn+1}) \left\{\prod_{i=1}^{sn} (a_i+kb_i)\right\} \frac{A_k}{(sn-k)!}$$

and

$$\sum_{k=0}^{n} \left\{\prod_{i=1}^{n} (a_i+kb_i)\right\} \frac{A_k}{(n-k)!} = 0, \quad n>ms,\ m=0,1,2,\ldots\ .$$

Likewise the alternative forms of the theorems 5 and 7 may also be obtained ; however the following inverse relations are worth mentioning which follow from these theorems with q replaced by $q^{-\lambda}$.

In fact, theorem - 5 with q replaced by $q^{-\lambda}$, assumes the form :

$$U_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} q^{-\lambda sk(sk-1)/2} \left\{\prod_{i=1}^{n} (a_i + q^{\lambda sk} b_i)\right\} \frac{T_k}{(q^{-\lambda};q^{-\lambda})_{n-sk}}$$

if and only if

$$T_n = \sum_{k=0}^{sn} q^{-\lambda k(k-2sn+1)/2} \frac{a_{k+1}+q^{\lambda k} b_{k+1}}{\prod_{i=1}^{k+1} (a_i+q^{\lambda sn} b_i)} \frac{U_k}{(q^{-\lambda};q^{-\lambda})_{sn-k}}$$

and

$$\sum_{k=0}^{n} q^{-\lambda k(k-2sn+1)/2} \frac{a_{k+1}+q^{\lambda k} b_{k+1}}{\prod_{i=1}^{k+1} (a_i+q^{\lambda n} b_i)} \frac{U_k}{(q^{-\lambda};q^{-\lambda})_{n-k}} = 0,$$

when $n \neq ms$, $m=1,2,3,\ldots$.

In this, on making use of the formula

$$(q^{-1};q^{-1})_{N-m} = (-1)^{N-m} q^{-N(N+1)/2} q^{-m(m-2N-1)/2} [q]_{N-m},$$

and then replacing U_n by $q^{\lambda n(n+1)/2} U_n$, and T_n by $q^{\lambda sn(sn+1)/2} T_n$, one arrives at

THEOREM - 5A.

$$U_n = \sum_{k=0}^{[n/s]} q^{\lambda sk(sk-2n+1)/2} \left\{\prod_{i=1}^{n} (a_i + q^{\lambda sk} b_i)\right\} \frac{T_k}{(q^{\lambda};q^{\lambda})_{n-sk}}$$

if and only if

$$T_n = \sum_{k=0}^{sn} (-1)^{sn-k} q^{\lambda k(k-1)/2} \frac{a_{k+1}+q^{\lambda k} b_{k+1}}{\prod_{i=1}^{k+1} (a_i+q^{\lambda sn} b_i)} \frac{U_k}{(q^{\lambda};q^{\lambda})_{sn-k}}$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} q^{\lambda k(k-1)/2} \frac{a_{k+1}+q^{\lambda k} b_{k+1}}{\prod_{i=1}^{k+1} (a_i+q^{\lambda n} b_i)} \frac{U_k}{(q^{\lambda};q^{\lambda})_{n-k}} = 0$$

when $n \neq ms$, $m = 1,2,3,\ldots$.

In a similar manner, it can be shown that theorem-7 with q replaced by $q^{-\lambda}$, gets transformed to

THEOREM - 7A.

$$C_n = \sum_{k=0}^{[n/s]} q^{\lambda sk(sk-1)/2} \frac{a_{sk+1}+q^{\lambda sk} b_{sk+1}}{\prod_{i=1}^{sk+1} (a_i+q^{\lambda n} b_i)} \frac{D_k}{(q^\lambda;q^\lambda)_{n-sk}}$$

if and only if

$$D_n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{q^{\lambda k(k-2sn+1)/2}}{(q^\lambda;q^\lambda)_{sn-k}} \left\{\prod_{i=1}^{sn} (a_i+q^{\lambda k} b_i)\right\}^{-1} C_k$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{q^{\lambda k(k-2n+1)/2}}{(q^\lambda;q^\lambda)_{n-k}} \left\{\prod_{i=1}^{n} (a_i+q^{\lambda k} b_i)\right\} C_k = 0, \text{ when } n>ms,$$

$m = 0,1,2,\ldots$.

5.7 PARTICULAR CASES OF THEOREMS 4 TO 7A

Since the inverse relations in (5.1.1) include as special cases the inverse pairs (1.2.13) and (1.2.17), and also the inverse series of various polynomials like Jacobi, Hahn, Racah and that of Wilson (quoted in section-1.2 as (1.2.3) to (1.2.8), (1.2.32), (1.2.33)), it becomes straightforward to provide extensions of all these particular cases by means of theorem-4. For example, the inverse pair (1.2.13) may be extended to the form :

$$
(5.7.1)\quad \begin{cases} f(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{a+bsk}{n} \dfrac{g(k)}{(n-sk)!} , \\[2ex] g(n) = \sum_{k=0}^{sn} \binom{a+bsn}{sn}^{-1} A_{sn-k}(a+bk-k,b)\ f(k). \end{cases}
$$

Similarly, the pair (1.2.17) assumes the extension in the form :

$$
(5.7.2)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{a+n+bsk}{n} \dfrac{G(k)}{(n-sk)!} , \\[2ex] G(n) = \sum_{k=0}^{sn} \binom{a+bsn+k}{k}^{-1} \dfrac{a+bk+k+1}{a+bsn+k+1} \dfrac{F(k)}{(sn-k)!} . \end{cases}
$$

It is to be noted that the extensions of the aforementioned polynomials provided by theorem-4 are of the same forms as those considered in chapter-4 ((4.1.7) to (4.1.11)), and therefore, they are omitted here for brevity. However, it is worth mentioning here that the relations in (5.1.1) can also be particularized to obtain certain inversion pairs that appear in the classification due to John Riordan. They are Gould class (1), simpler Legendre class (1), and the Legendre - Chebyshev classes (3), and (5). In fact, their extended versions which are deduced in chapter-4 (Table-20), can also be obtained from theorem-4.

Theorem-4A on the other hand, provides the extensions of some other pairs belonging to the above mentioned classes; that is, the Gould class(2), simpler Legendre class(2), and the Legendre-Chebyshev classes (1) and (7). These extended forms are also embodied in Table-20.

The specializations of theorems 6 and 6A are in fact, more interesting because the forms in which the inverse pairs (1.2.13), (1.2.17) and those of Gould classes, simpler Legendre classes etc. (mentioned above) admit extensions by means of these theorems, are different from those of Table-20.

For example, by setting $a_i = p-i+1$ and, $b_i = q$ for all i, in theorem-6, one finds after a little simplification, the following pair :

$$a_n = \sum_{k=0}^{sn} \binom{p+qk-k}{sn-k} b_k \ ; \ b_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \frac{p+qsk-sk}{p+qn-sk} \binom{p+qn-sk}{n-sk} a_k .$$

It may be seen that this pair provides an extension of Gould class (1) of Table-2. Likewise, with $a_i = p-i+2$, and $b_i = q$, theorem-6A yields the relations :

$$a_n = \sum_{k=0}^{sn} \frac{p+qsn-sn+1}{p+qk-sn+1} \binom{p+qk-k}{sn-k} b_k \ ; \ b_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qn-sk}{n-sk} a_k ,$$

which evidently extends the Gould class (2) of Table-2.

In a similar manner, the other inverse pairs may be put in the extended forms by choosing a_i and b_i appropriately. The following table embodies all these extensions along with the particular choice of a_i and b_i.

Table-22 : *Extensions of Riordan's inverse relations*

$$G(n) = \sum_{k=0}^{sn} c_{n,k}\, g(k) \; ; \; g(n) = \sum_{k=0}^{[n/s]} (-1)^{n-sk}\, d_{n,k}\, G(k)$$

Theorem-No.	a_1	b_1	$c_{n,k}$	$d_{n,k}$	Extension of class (No.) in Table-No.
Theorem-6	p-i+1	q	$\binom{p+qk-k}{sn-k}$	$\frac{p+qsk-sk}{p+qn-sk}\binom{p+qn-sk}{n-sk}$	Gould class (1), Table-2
Theorem-6A	p-i+2	q	$\frac{p+qsn-sn+1}{p+qk-sn+1}\binom{p+qk-k}{sn-k}$	$\binom{p+qn-sk}{n-sk}$	Gould class (2), Table-2
Theorem-6	p+i	1	$\binom{p+sn+k}{sn-k}$	$\frac{p+2sk+1}{p+n+sk+1}\binom{p+2n}{n-sk}$	simpler Legendre Class (1), Table-5
Theorem-6A	p+i-1	1	$\frac{p+2sn}{p+sn+k}\binom{p+sn+k}{sn-k}$	$\binom{p+2n}{n-sk}$	simpler Legendre Class (2), Table-5
Theorem-6A	p+i-1	c-1	$\frac{p+scn}{p+ck}\binom{p+ck-k+sn-1}{sn-k}$	$\binom{p+cn}{n-sk}$	Legendre-Chebyshev Class (1), Table-6
Theorem-6	p-i+1	c+1	$\binom{p+ck}{sn-k}$	$\frac{p+sck}{p+cn}\binom{p+cn+n-sk-1}{n-sk}$	Legendre-Chebyshev Class (3), Table-6
Theorem-6	p+i	c-1	$\binom{p+ck-k+sn}{sn-k}$	$\frac{p+sck+1}{p+cn-n+sk+1}\binom{p+cn}{n-sk}$	Legendre-Chebyshev Class (5), Table-6
Theorem-6A	p-i+2	c+1	$\frac{p+scn+1}{p+ck+k-sn+1}\binom{p+ck}{sn-k}$	$\binom{p+cn+n-sk}{n-sk}$	Legendre-Chebyshev Class (7), Table-6

Coming to the specializations of theorems 5, 5A, 7 and 7A, it may be observed that the basic pairs cited in (1.5.4) and (1.5.5) and also the basic polynomials of Laguerre, Jacobi, Hahn etc. are contained in the pair (5.1.2) and therefore, their extensions may be carried out in the forms as given below.

If $a_i=1$ and $b_i=-q^{a+\lambda i}$, then theorem-5A yields an extension of (1.5.4) in the following pair.

$$(5.7.3)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/s]} q^{\lambda sk(sk-2n+1)/2} \begin{bmatrix} a+n+\lambda sk \\ n \end{bmatrix} \dfrac{[q]_n}{(q^\lambda;q^\lambda)_{n-sk}} G(k), \\ G(n) = \sum_{k=0}^{sn} (-1)^{sn-k} q^{\lambda k(k-1)/2} \dfrac{1-q^{a+\lambda k+k+1}}{1-q^{a+\lambda sn+k+1}} \dfrac{[q]_k}{(q^\lambda;q^\lambda)_{sn-k}} \cdot \begin{bmatrix} a+\lambda sn+k \\ k \end{bmatrix}^{-1} F(k). \end{cases}$$

An extended version of (1.5.5) may be obtained similarly. Further, in order to carry out extension of the basic Laguerre polynomial cited in (1.5.13), set $a_i=1$, and $b_i=0$ in theorem-5A. Then on replacing T_k by

$$q^{\lambda sk(sk+2\alpha+1)} \frac{(1-q)^{sk} x^k}{[q]_{sk} [\alpha q]_{sk}},$$

and denoting the polynomial thus obtained, by $L_{n,s,\lambda}^{(\alpha)}(x|q)$, one finds

$$L_{n,s,\lambda}^{(\alpha)}(x|q) = \sum_{k=0}^{[n/s]} q^{\lambda sk(sk+\alpha)} \frac{(1-q)^{sk} x^k}{[\alpha q]_{sk} (q^\lambda;q^\lambda)_{n-sk} [q]_{sk}}$$

(cf. (1.5.13) with $\lambda=1$, $s=1$),

whose inverse relation may be expressed as

$$\frac{(1-q)^{sn}\,x^n}{[\alpha q]_{sn}} = \sum_{k=0}^{sn} (-1)^{sn-k}\, q^{(\lambda sn(sn+1)+\lambda k(k-1))/2}\, \frac{[q]_{sn}\, L^{(\alpha)}_{k,s,\lambda}(x|q)}{[\alpha q]_k\, (q^\lambda;q^\lambda)_{sn-k}}.$$

The polynomials of Wall and of stieltjes-Wigert may be extended in a like manner. Also, the extensions of basic polynomials of Jacobi, Hahn etc. which are obtainable from theorem-5A, are of the same forms as those of chapter-4 (4.4.8) to ((4.4.13)), and therefore, they are omitted here. It may be seen that the polynomial $g_n^c(x,r,s;q)$, when r-s is a positive integer, is also contained in theorem-5A.

As noted in section-3.4, the pair (5.1.2) yields the basic analogues of certain classes of inverse relations viz. Gould classes (1),(2); simpler Legendre classes (1),(2); etc., thus by means of theorem-5A, it becomes straightforward to provide extensions of these classes of inverse series relations. In fact, these extended versions coincide with those recorded in Table-21 (section-4.5) and therefore, they are not repeated here for the sake of brevity.

However, the special instances of theorems 7 and 7A corresponding to the basic Gould classes (1),(2); basic simpler Legendre classes(1),(2);and the basic Legendre-Chebyshev classes (1),(3),(5),(7) are more interesting because they provide the extensions of all these inverse relations in the forms which are different from those listed in Table-21. This is illustrated here by means of the choices $a_i=1$, $b_i= -q^{p-i+1}$ and, by writing m for λ in theorem-7A.

Since in this case,

$$\prod_{i=1}^{sk+1} (a_i+q^{mn}\, b_i) = [q^{p+mn-sk}]_{sk+1} = \frac{[q]_{p+mn}}{[q]_{p+mn-sk-1}} ,$$

and

$$\prod_{i=1}^{sn} (a_i+q^{mk}\, b_i) = [q^{p+mk-sn+1}]_{sk} = \frac{[q]_{p+mk}}{[q]_{p+mk-sn}} ,$$

therefore theorem-7A gets reduced to the form :

$$(5.7.4)\begin{cases} C_n = \sum_{k=0}^{[n/s]} q^{msk(sk-1)/2}\, \dfrac{1-q^{p+msk-sk}}{1-q^{p+mn-sk}}\, \dfrac{[q]_{n-sk}}{(q^m;q^m)_{n-sk}} \cdot \begin{bmatrix} p+mn-sk \\ n-sk \end{bmatrix} D_k \\ D_n = \sum_{k=0}^{sn} (-1)^{sn-k}\, q^{mk(k-2sn+1)/2} \begin{bmatrix} p+mk-k \\ sn-k \end{bmatrix} \dfrac{[q]_{sn-k}}{(q^m;q^m)_{sn-k}}\, C_k ; \end{cases}$$

which provides an extension of the basic Gould class (1) of Table-15 (Ch.3). This pair also provides a basic analogue of the first pair of Table-22.

In a similar manner, by specializing a_i and b_i appropriately, one obtains the extensions of the other basic classes from theorem 7 or 7A. The following table encompasses all the extended basic classes.

Table-23 ***Extensions of basic analogues of Riordan's inverse pairs***

$$D_n = \sum_{k=0}^{sn} q^{\lambda k(k-2sn+1)/2} A_{n,k} C_k \;;\; C_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} q^{\lambda sk(sk-1)/2} B_{n,k} D_k$$

Theorem-No. with $a_i=1$	b_i	$A_{n,k}$	$B_{n,k}$	Extension of class (No.) in Table-No.
Theorem-7A with $\lambda = m$	$-q^{p-i+1}$	$\frac{[q]_{p+mk-k}}{[q]_{p+mk-sn}\,(q^m;q^m)_{sn-k}}$	$\frac{(1-q^{p+msk-sk})\,[q]_{p+mn-sk-1}}{[q]_{p+mn-n}\,(q^m;q^m)_{n-sk}}$	Gould class (1) Table-2
Theorem-7A* with $\lambda = m$	$-q^{p-i+2}$	$\frac{[q]_{p+mk-k}\,(1-q^{p+msn-sn+1})}{[q]_{p+mk-sn+1}\,(q^m;q^m)_{sn-k}}$	$\frac{[q]_{p+mn-sk}}{[q]_{p+mn-n}\,(q^m;q^m)_{n-sk}}$	Gould class (2) Table-2
Theorem-7A with $\lambda = 1$	$-q^{p+i}$	$\frac{[q]_{p+sn+k}}{[q]_{p+2k}\;[q]_{sn-k}}$	$\frac{(1-q^{p+2sk+1})\,[q]_{p+2n}}{[q]_{p+n+sk+1}\,[q]_{n-sk}}$	simpler Legendre Class (1), Table-5
Theorem-7A* with $\lambda = 1$	$-q^{p+i-1}$	$\frac{[q]_{p+sn+k-1}\,(1-q^{p+2sn})}{[q]_{p+2k}\;[q]_{sn-k}}$	$\frac{[q]_{p+2n}}{[q]_{p+n+sk}\,[q]_{n-sk}}$	simpler Legendre Class (2), Table-5
Theorem-7A* with $\lambda = c-1$	$-q^{p+i-1}$	$\frac{[q]_{p+ck-k+sn-1}\,(1-q^{p+scn})}{[q]_{p+ck}\,(q^{c-1};q^{c-1})_{sn-k}}$	$\frac{[q]_{p+cn}}{[q]_{p+cn-n+sk}\,(q^{c-1};q^{c-1})_{n-sk}}$	Legendre-Chebyshev Class (1), Table-6
Theorem-7A with $\lambda = c+1$	$-q^{p-i+1}$	$\frac{[q]_{p+ck}}{[q]_{p+ck+k-sn}\,(q^{c+1};q^{c+1})_{sn-k}}$	$\frac{(1-q^{p+sck})\,[q]_{p+cn+n-sk-1}}{[q]_{p+cn}\,(q^{c+1};q^{c+1})_{n-sk}}$	Legendre-Chebyshev Class (3), Table-6
Theorem-7A with $\lambda = c-1$	$-q^{p+i}$	$\frac{[q]_{p+ck-k+sn}}{[q]_{p+ck}\,(q^{c-1};q^{c-1})_{sn-k}}$	$\frac{(1-q^{p+sck+1})\,[q]_{p+cn}}{[q]_{p+cn-n+sk+1}\,(q^{c-1};q^{c-1})_{n-sk}}$	Legendre-Chebyshev Class (5), Table-6
Theorem-7A* with $\lambda = c+1$	$-q^{p-i+2}$	$\frac{[q]_{p+ck}\,(1-q^{p+scn+1})}{[q]_{p+ck+k-sn+1}\,(q^{c+1};q^{c+1})_{sn-k}}$	$\frac{[q]_{p+cn+n-sk}}{[q]_{p+cn}\,(q^{c+1};q^{c+1})_{n-sk}}$	Legendre-Chebyshev Class (7), Table-6

(* indicates that in theorem-7A, D_k should be replaced by $D_k/(a_{sk+1} + q^{\lambda sk} b_{sk+1})$ first, and then a_i and b_i should be specialized).