Chapter 3

Quark Masses And Mixings

As we have seen in the last chapter, the standard model does not provide one with any guideline as to what the fermion masses and the mixings should be, the only criterion for determining these being experimental consistency. But this situation is aesthetically not a very pleasing one and there have been many efforts to formulate models that remove the arbitrariness to some degree. The methodology is quite simple. One imposes certain symmetries on the quark mass matrices to relate at least some of the ten parameters in this sector. Mainly motivated by phenomenological considerations, some of these models can no doubt be looked upon as having arisen from theories with higher gauge symmetries.

In this chapter we start with a brief discussion of some of these models and the motivation behind each. Once the predictions due to each are identified, the next logical step is obviously to check their validity in the light of the current experimental results. Finally we end with a model independent study of the three generation quark mass matrices and identify the various models as special cases.

3.1 Models for Quark Masses and Mixings

3.1.1 Stech Model:

This model [29] was motivated by grand unified theories where the gauge group has a $SU(5)$ subgroup and all the fermions of a generation are contained in an irreducible representation. The fermion masses arise from non-zero vacuum expectation value of Higgs fields transforming under different representations. The assumption was that the mass contributions due to the symmetric Higgs representations dominate and that the antisymmetric representations do not contribute to the up-sector. Such a scenario was to be ensured by suitable discrete symmetries and a proper choice of the higgs couplings. A further choice of hermiticity of the mass matrices restrict their form to

$$M_{u(S)} = \widehat{M_u} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \tag{3.1.1}$$

and

$$M_{d(S)} = \alpha \widehat{M_u} + A, \tag{3.1.2}$$

where α is a constant and A is an antisymmetric matrix.

$M_{d(S)}$ can be brought into a diagonal form by an orthogonal transformation. In this basis A is still hermitian and antisymmetric. It should however be noticed that choosing a particular basis for the CKM matrix would necessitate a unitary transformation by a phase matrix. While this would leave M invariant, A would lose its antisymmetry and would be a hermitian matrix with all diagonal elements zero. In fact the basis independent statement is that $det(A) = 0$.

As shall be shown in section 3.3.4 , this model is characterised by seven parameters and hence we expect three relations between the quark sector parameters. These can be read off from the matrix equation

$$K^{\dagger} M_{d(S)} K = \widehat{M_d} = diag(m_d, m_s, m_b) \tag{3.1.3}$$

The analysis is exactly similar to that employed for the model independent case [section 3.3] and shall not be presented here. The relations one is looking for are

$$\begin{aligned} s_{12}^2 &\approx \left(\frac{m_d}{m_s} - \frac{m_u}{m_c}\right)\left(1 - \frac{m_u m_d}{m_s m_c}\right)^{-1}, \\ s_{23}^2 &\approx \left(\frac{m_s}{m_b} - \frac{m_c}{m_t}\right)\left(1 - \frac{m_s m_c}{m_b m_t}\right)^{-1}, \\ q\cos\delta &\approx -\frac{m_s}{m_b} s_{12}. \end{aligned} \tag{3.1.4}$$

There is indeed a fourth relation claimed by Stech:

$$s_{13}^2 = \frac{m_u}{m_c} s_{23}^2, \tag{3.1.5}$$

but in section 3.3.2 it shall be shown to be not a consequence of the model but to have arisen from a flaw in the analysis.

Though the ansatz looks simple enough, it is very difficult to ensure such a form in viable models. The first such scheme was presented in the context of left–right models [30], but in these the tree level derivations of the Stech model were somewhat spoiled by infinite corrections at higher loops. An alternative model [31] based on a supersymmetric $SO(10)$ theory with softly broken supersymmetry is probably the best candidate available in the literature. The symmetric parts of the mass matrices are unaltered at the one–loop level and the antisymmetric part for the down quarks arises as a one–loop correction and is hence smaller than the tree level terms. The relevant diagrams involve charged color triplet scalars and because of the choice for their quantum numbers give a net antisymmetric contribution. The cornerstone of the model is the proportionality of such corrections with the Majorana mass of the ν_R (for a definition of Majorana neutrinos, see Chapter 4) and hence there are no corresponding diagrams for the up–sector.

3.1.2 The Fritzsch Model:

The Fritzsch model [32] envisages a scenario where, to begin with, only the heaviest quarks in either sector are massive and all others gain mass successively through charged current mixings with the next higher generation. This model was first obtained [32] for a field theory with $SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ as the electroweak gauge group and two Higgs fields, on imposition of a certain discrete symmetry. The simplest such construction is given by a scenario where one considers only the matter fields q_{iL} $(2, 1, \frac{1}{6})$, q_{iR} $(1, 2, \frac{1}{6})$ (i being the generation index) and two Higgs fields $\phi_{1,2}(2, 2, 0)$. The number in the parentheses here represent the transformation properties of the field under the gauge group. The most general Yukawa term then reads

$$\mathcal{L}_{Yuk} = \overline{q_{iL}} q_{jR} \left(h_{ij} \phi_1 + g_{ij} \phi_2 \right) + H.c. \qquad (3.1.6)$$

Parity invariance obviously requires that the matrices h_{ij} and g_{ij} be hermitian. If we further demand that the Lagrangian should respect the discrete symmetry

$$\begin{array}{lllllll} q_{1L} & \rightarrow & -iq_{1L} & \qquad & q_{1R} & \rightarrow & iq_{1R} \\ q_{2L} & \rightarrow & iq_{2L} & & q_{2R} & \rightarrow & -iq_{2R} \\ q_{3L} & \rightarrow & q_{3L} & & q_{3R} & \rightarrow & q_{3R} \\ \phi_1 & \rightarrow & \phi_1 & & \phi_2 & \rightarrow & i\phi_2, \end{array} \tag{3.1.7}$$

the only non-zero Yukawa couplings would be

$$h_{21} = h_{12}^*, \quad g_{33}, \quad h_{31} = h_{13}^* \quad \text{and} \quad h_{32} = h_{23}^*,$$

and one obtains the desired form for the mass matrices.

In a basis where the up-quark mass matrix is real, one then has

$$M_{u(F)} = \begin{pmatrix} 0 & a_u & 0 \\ a_u & 0 & b_u \\ 0 & b_u & c_u \end{pmatrix} \tag{3.1.8}$$

and

$$M_{d(F)} = \begin{pmatrix} 0 & a_d e^{i\varphi_1} & 0 \\ a_d e^{-i\varphi_1} & 0 & b_d e^{i\varphi_2} \\ 0 & b_d e^{-i\varphi_2} & c_d \end{pmatrix}. \tag{3.1.9}$$

The quark sector is thus characterized by eight parameters and hence two relations between the masses and the CKM matrix parameters are predicted. The form of the mass matrices also imply that the middle (in magnitude) eigenvalue of both $M_{u(F)}$ and $M_{d(F)}$ would have a sign opposite to the other two.

$M_{d(F)}$ can be brought into real form by performing a phase rotation on both the left– and the right–handed down quark fields

$$M_{d(F)} = P^\dagger M'_{d(F)} P, \tag{3.1.10}$$

where

$$P = diag\left(1, e^{i\varphi_1}, e^{i(\varphi_1+\varphi_2)}\right) \tag{3.1.11}$$

and

$$M'_{d(F)} = \begin{pmatrix} 0 & a_d & 0 \\ a_d & 0 & b_d \\ 0 & b_d & c_d \end{pmatrix}. \tag{3.1.12}$$

$M_{u(F)}$ and $M'_{d(F)}$ being real symmetric matrices, can now be diagonalized by orthogonal transformations. For example

$$O_u^T M_{u(F)} O_u = \widehat{M_u} \equiv diag\,(m_u, m_c, m_t) \tag{3.1.13}$$

with

$$O_u = \begin{pmatrix} \frac{1}{N_1} & \frac{1}{N_2} & \frac{1}{N_3} \\ \frac{m_u}{N_1 a_u} & \frac{m_c}{N_2 a_u} & \frac{m_t}{N_3 a_u} \\ \frac{-m_u b_u}{N_1 a_u(m_c+m_t)} & \frac{-m_c b_u}{N_2 a_u(m_u+m_t)} & \frac{-m_t b_u}{N_3 a_u(m_u+m_c)} \end{pmatrix}. \tag{3.1.14}$$

The eigenvalues m_i can be obtained by inverting the relations

$$\begin{aligned} a_u &= (-m_u m_c m_t/c_u)^{1/2} \\ b_u &= [-(m_u+m_c)(m_u+m_t)(m_c+m_t)/c_u]^{1/2} \\ c_u &= (m_u+m_c+m_t), \end{aligned} \tag{3.1.15}$$

and N_i are the normalizations for the eigenvectors of $M_{u(F)}$:

$$\begin{aligned} N_1^2 &= \frac{m_c - m_u}{m_c}\left[1+\frac{(m_c+m_u)m_u}{(m_t+m_c)m_c}\right] \\ N_2^2 &= \frac{m_u - m_c}{m_u}\left[1+\frac{(m_u-m_c)m_c}{(m_t+m_u)m_t}\right] \\ N_3^2 &= \frac{m_t^3}{m_u m_c(m_c+m_u)}\left[1-\frac{m_c^2+m_c m_u+m_u^2}{m_t^2}+\frac{m_u m_c(m_c+m_u)}{m_t^3}\right]. \end{aligned} \tag{3.1.16}$$

The weak mixing matrix being given by

$$K = O_u^T P^\dagger O_d, \tag{3.1.17}$$

we have

$$\begin{aligned} s_{12} &\approx \left|\sqrt{-\frac{m_d}{m_s}} - e^{-i\varphi_1}\sqrt{-\frac{m_u}{m_c}}\right|, \\ s_{23} &\approx \left|\sqrt{-\frac{m_s}{m_b}} - e^{-i\varphi_2}\sqrt{-\frac{m_c}{m_t}}\right|, \\ s_{13} &\approx \left|\frac{m_s}{m_b}\sqrt{\frac{m_d}{m_b}} - e^{-i\varphi_1}\left(\sqrt{-\frac{m_s}{m_b}} - e^{-i\varphi_2}\sqrt{-\frac{m_c}{m_t}}\right)\right|, \\ \frac{\sin\delta}{\frac{s_{12}s_{23}}{s_{13}} - \cos\delta} &\approx \frac{\sin\varphi_1}{\cos\varphi_1 - \sqrt{\frac{m_d m_c}{m_s m_u}}}. \end{aligned} \tag{3.1.18}$$

The first two of these equations can be used to determine the phases φ_1 and φ_2 and then the last two represent the predictions of the model.

3.1.3 Fritzsch-Shin Model:

The two phases $\varphi_{1,2}$ in the Fritzsch mass matrix are not determined by the imposed discrete symmetry and hence are quite arbitrary. If these could be fixed by some means, the arbitrariness could be reduced somewhat and further relations between the parameters would be predicted. To this end, Shin [33] made a choice for the two phases namely $\varphi_1 = 90^\circ$ and $\varphi_2 = 0$, the hope being that this could be achieved on imposition of further discrete symmetries. Thus the model is now characterised by six parameters and this results in two further constraints on the system over and above those obtained for the general Fritzsch case. For example, all of equations (3.1.18) are now predictions of the model.

3.1.4 Fritzsch-Stech model:

That the Stech and the Fritzsch ansätze are not inconsistent with each other is easy to see. Exploiting this freedom, Gronau, Johnson and Schechter [34] proposed a scenario in which both these sets of assumptions are incorporated. (However no realistic model to achieve this has been constructed.) In a suitable basis the mass matrices are thus given by

$$M_{u(FS)} = \begin{pmatrix} 0 & a_u & 0 \\ a_u & 0 & b_u \\ 0 & b_u & c_u \end{pmatrix} \tag{3.1.19}$$

and

$$M_{d(FS)} = \alpha M_{u(FS)} + \begin{pmatrix} 0 & ia & 0 \\ -ia & 0 & ib \\ 0 & -ib & 0 \end{pmatrix} . \tag{3.1.20}$$

Like the Fritzsch–Shin model, this also results in a six parameter family and the predictions over and above either the Stech or the Fritzsch scheme can easily be obtained. (For a detailed account of the same, see Section 3.3.4.)

3.2 Validity of Models

In the last section we had a brief overview of the more popular models to explain the quark masses and mixings. All these ansätze predict some relations between the otherwise free parameters in this sector and hence the most obvious check for the validity of such models would be the comparison of their respective predictions with experimental data. A detailed analysis was performed by Harari & Nir and by Nir [36] where they compared the relations with the restrictions imposed by the CP–violation parameter ϵ_K and the B_d^0–$\overline{B_d^0}$ mixing extent r_d to conclude that while the Stech ansatz was ruled out, the Fritzsch scheme barely survived. However it was pointed out in the literature [12,37] that these authors had put unnecessarily severe constraints on some ill–determined parameters involved in the calculations. In the light of this, we redid the exercise to obtain results quite different from that in ref.[36]. For example, the Stech ansatz was not yet ruled out and the Fritzsch model had much more freedom than claimed [35]. However the newer result on the $\Delta S = 1$ CP–violating parameter ϵ'_K proved to be a very useful constraint to check models by. In the rest of this section [1] we shall describe the method of comparison adopted and the results thereof.[2]

3.2.1 The Stech Model : Consistency Check

As we have seen earlier, the Stech ansatz results in three predictions *viz.* eqns.(3.1.4). The first of the three relations is obviously consistent with the experimental values, while the second, on scanning through the entire range allowed to the masses gives $m_t(1\ GeV) \leq 82.4\ GeV$ thus implying that we need to examine only the range $45\ GeV \leq m_t^{\rm phys} \leq 51.5\ GeV$. An examination of the overlap of ϵ_K and x_d bounds in the q–δ plane (where $q \equiv s_{13}/s_{23}$) for various choices of $B_B f_B^2$ and B_K indicates $q \sim 0.1$ thus requiring δ to be nearly 90° for the Stech scheme to be valid. This indicates a near 'maximal' CP violation in the neutral kaon system as expected from the choice for M_u and M_d. A thorough examination shows that the Stech ansatz agrees with the ϵ_K– and x_d–values only for $m_t^{\rm phys} \sim 51.5\ GeV$, $s_{23} \sim 0.07$, $B_K \sim 0.33$ and $B_B f_B^2 \sim 0.04$. This overlap was absent in the analyses in refs.[36] as they had limited s_{23} to be below 0.05. But even this tenuous agreement is destroyed by the ϵ'_K observation. Noting that $m_t(1\ GeV) \sim 82\ GeV$ and $s_{23} \sim 0.07$ implies $m_s(1\ GeV) \sim 120 MeV$, a substitution of all relevant variables in equation(2.3.33) predicts $|\epsilon'_K/\epsilon_K| \gtrsim 9 \times 10^{-3}$ which is considerably higher than the experimental upper limit of 4.4×10^{-3}.

[1] This section is based on the work in ref. [35]

[2] It must be noted that independent of the contents of this section, the recent improved bounds [17] on the top quark mass ($m_t \gtrsim 89\ GeV$) effectively rules out all the models under discussion

3.2.2 The Fritzsch Model: Consistency Check

As far as this model goes, equation (3.1.18) gives $m_t \leq m_c/(\sqrt{m_s/m_b} - s_{23})^2$ and using the entire range for the other parameters, one gets $m_t(1\ GeV) \leq 223\ GeV$ thus requiring us to look only at the interval $45\ GeV \leq m_t^{\text{phys}} \leq 127\ GeV$. To check the validity of the model, we select a combination of m_t, m_d, s_{23}, B_K and $B_B f_B^2$ and look for any overlap in the q–δ plane of the x_d– and ϵ_K– bands and the region allowed by the model. Note that equation (3.1.18) gives rise to two bands (corresponding to the two different relative signs between φ_1 and φ_2, as yet undetermined) independent of δ, whose widths are determined by the error bars on m_i and s_{12} and which may, in some cases, coalesce into one. Furthermore these selections are to be checked for consistency with the ϵ'_K/ϵ_K results.

Our analysis shows that unlike in ref. [36] one does obtain a large number of solutions for this ansatz. Though most of the solutions obtained are for $s_{23} \sim 0.07$, a significant number do exist for $s_{23} \sim 0.059$, an upper bound many authors have quoted. More important the extremal conditions required in the earlier analyses [36] are released. We divide the solution into three broad categories

Large m_t:

For $95\ GeV \leq m_t^{\text{phys}} \leq 127\ GeV$, the requirement of $B_B f_B^2$ and x_d being respectively at the top and the bottom of their individual given ranges is relaxed with their ratio being allowed to take central values. But $B_K \leq 0.6$ and $s_{23} \geq 0.06$ are slowly pushed to their respective minimum and maximum as m_t increases. Also m_u, m_d and m_s need to assume almost the lowest values allowed. Small $q(\sim 0.035$–$0.06)$ is favoured while δ is allowed over a considerable range ($40°$–$120°$) with progressively higher values for lower m_t.

But this solution is in contradiction with $q \geq 0.07$, a limit imposed by observed levels of charmless b-decay. So for this range the Fritzsch scheme is effectively ruled out.

Low m_t:

For 70 $GeV \geq m_t^{\rm phys} \geq 45$ GeV, on the other hand $x_d/B_B f_B^2$ needs to be constrained near the lowest value. While B_K, m_d and consequently m_u and m_s have the freedom to assume values close to or slightly below the centre of the range, s_{23} again needs to be larger and larger as one progresses to lower m_t's. q takes on a typically a larger value (0.08 – 0.1) than allowed for large m_t and δ is constrained between 110° and 130°.

Middle m_t:

In this range (70 $GeV \leq m_t^{\rm phys} \leq 95$ GeV) the results are similar to those in ref. [36] and though many more solutions are obtained, we are not detailing them here.

It is to be noted that ϵ'_K results hardly constrain the Fritzsch model solution domains. One of the very few examples where this result did rule out this model is

$$\begin{array}{rclrcll} B_K & = & 0.85 & B_B f_B^2 & = & 0.02 & s_{23} = 0.06 \\ m_d(1\;GeV) & = & 6.3MeV & m_t^{\rm phys} & = & 90\;GeV & \end{array}$$

The other contraints agreed for $\delta \sim 113°$–$123°$, and $q \sim 0.067$ which would have required $|\epsilon'_K/\epsilon_K| \leq 2.09 \times 10^{-3}$.

3.2.3 Fritzsch–Shin Model: Consistency Check

Since this ansatz is but a special case of the Fritzsch scheme, it stands to reason that the agreement would be narrower. Indeed our check shows that of the three zones we demarcate, the Shin choice is invalid in both the high m_t and the low m_t regions. Even for the middle m_t region, the agreement is very marginal as in ref.[36] and not much improved by relaxing the upper bound on s_{23}. It is however noted that $\varphi_2 = 0$ alone has much better agreement than the Shin ansatz.

3.2.4 Fritzsch–Stech model: Consistency Check

That the experimental agreement of such models would be narrower than that of either the Fritzsch or the Stech model is obvious on account of its incorporating the assumptions of both the latter ones. Of particular relevance is the Stech lineage. Hence even without bothering to examine we could safely conclude that this ansatz is phenomenologically inconsistent.

3.2.5 Conclusions

Our analysis has shown that if one takes into consideration the entire range allowed [12] experimentally to K_{cb}, a much wider range of solutions is allowed to the Fritzsch ansatz predictions than claimed hitherto. The necessity of a heap of theoretically and experimentally ill-determined parameters assuming extreme values allowed is removed. But the model fails to take advantage of one concession that a higher value of s_{23} gives it, *i.e.* agreeing for high m_t. As we have seen earlier, $m_t^{\rm phys} > 95$ GeV is ruled out as it entails a value of q smaller than the experimentally allowed minimum. If the limit $m_t^{\rm phys} \leq 55$ GeV [37] is taken seriously, the Fritzsch model would still be in the running. But in that case the Shin modification is totally ruled out. On the other hand, the Stech scheme which was allowed a marginal agreement with the earlier data for $s_{23} \sim 0.07$, is totally ruled out by the $|\epsilon'_K/\epsilon_K|$ results. As a corollary, other models incorporating the Stech ansatz like the Fritzsch–Stech model of Gronau, Johnson and Schechter are automatically invalidated.

3.3 A Model Independent Analysis

The results of the last section demonstrate that none of the current models for quark mass matrices do the job efficiently. This naturally prompts a model independent study [3] of the problem in the hope that such an activity would help us in gaining some insight into the matter at hand and possibly indicate fertile but as yet untapped territory for future model building.

To begin with, we start with the most general mass matrices for three generations. The

[3] This section is based on the work in ref. [38]

arguments of section 2.1 show that by making a phase transformation on the right handed quark fields alone we can make these matrices into hermitian ones with all their eigenvalues to be positive. However such a choice for the basis would, in general, not be consistent with the particular form of the CKM matrix that we have chosen to work with. Hence we shall only demand hermiticity and allow the eigenvalues to take either sign. This need not cause any alarm as in the standard model the sign of the fermion mass has no significance and can be changed by a chiral transformation.

In the basis in which M_u is diagonal, we then have for the most general case

$$M_d = \alpha\widehat{M_u} + A \tag{3.3.1}$$

where

$$A = \begin{pmatrix} 0 & R_1 e^{i\rho_1} & R_2 e^{i\rho_2} \\ R_1 e^{-i\rho_1} & f & R_3 e^{i\rho_3} \\ R_2 e^{-i\rho_2} & R_3 e^{-i\rho_3} & d \end{pmatrix}. \tag{3.3.2}$$

Thus the mass matrices are a ten parameter family determined by the values of m_u, m_c, m_t, α, f, d, $R_{1,2,3}$ and the invariant phase $(\rho_1+\rho_3-\rho_2)$. That the other phase combinations are unphysical can easily be seen by making the most general phase redefinitions of the quark wavefunctions. This then leads to $M_d \longrightarrow P_1^\dagger M_d P_2$ where $P_{1,2}$ are some arbitrary phase matrices. While the magnitudes of the individual elements are invariant under this change of basis, their phases are not. The simplest nontrivial combinations resisting change are given by [39] the "cycles" $arg\left[(M_d)_{ij}(M_d^\dagger)_{jk}(M_d)_{kl}(M_d^\dagger)_{li}\right]$ (no summation) and this in the present case simplifies to the expression given earlier.

Though on the face of it this parametrization has no predictive power as we are using ten parameters to relate ten others, in our analysis we would not be using all of them and most of our conclusions would be drawn by considering only the diagonal elements.

On diagonalizing M_d we have

$$K\widehat{M_d}K^\dagger = M_d = \alpha M_u + A$$

where

$$\widehat{M_d} = diag(m_d, m_s, m_b).$$

The diagonal elements of the matrix equation give three relations, of which one is the trace condition

$$\alpha = \frac{m_d + m_s + m_b - f - d}{m_u + m_c + m_t} \tag{3.3.3}$$

and the others are

$$\begin{aligned} \alpha m_u &= m_d + c_{13}^2 s_{12}^2 (m_s - m_d) + s_{13}^2 (m_b - m_d) \\ \text{and} \quad \alpha m_c + f &= m_d + \left| c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} \right|^2 + c_{13}^2 s_{23}^2 (m_b - m_d). \end{aligned} \tag{3.3.4}$$

3.3.1 The two generation limit:

As a first approximation we assume that the third generation essentially decouples from the first two – a not too strong assumption as experimentally both s_{23} and s_{13} are small compared to s_{12}. In this limit (3.3.4) reduces to

$$\begin{aligned} \alpha m_u &= m_d + s_{12}^2 (m_s - m_d) \\ \alpha m_c + f &= m_d + c_{12}^2 (m_s - m_d). \end{aligned} \tag{3.3.5}$$

Eliminating α from above, we obtain

$$s_{12}^2 = \frac{\left(1 - \frac{f}{m_s}\right) \frac{m_u}{m_c} - \frac{m_d}{m_s}}{\left(1 + \frac{m_u}{m_c}\right)\left(1 - \frac{m_d}{m_s}\right)} \tag{3.3.6}$$

Using eqns.(2.1.18) and (2.2.12) in (3.3.6) gives an allowed range for f for a given m_d which has been plotted in Figure 3.1. It is seen that for $m_d/m_s < 0$, f assumes small values irrespective of the sign of m_u/m_c, and is consistent with zero. While for $m_d/m_s > 0$, f is comparatively larger and its sign is opposite to that of m_u/m_c.

3.3.2 Back to three generations:

Assuming the two generation limit for s_{12} and using it as an input in eqn. (3.3.4), we have (for $c_{13} \approx 1$),

$$s_{23}^2 = \frac{m_c}{m_t + m_c + m_u} \frac{(m_b - d)(m_c + m_u) + m_t(f - m_s - m_d)}{m_b(m_c + m_u) - (m_d m_c + m_s m_c + f m_u)} \tag{3.3.7}$$

i.e.

$$\begin{aligned} &\frac{d}{m_b}\left[s_{23}^2\left\{\frac{m_b}{m_d}\left(1 + \frac{m_u}{m_c}\right) - \left(1 + \frac{m_s}{m_d} + \frac{f m_u}{m_d m_c}\right)\right\} + \frac{m_s}{m_d}\left(1 - \frac{f}{m_s}\right) + 1\right] \\ &\quad = \left(1 - \frac{m_t}{m_c}\right)\left(1 + \frac{m_u}{m_c}\right)\frac{m_b}{m_d} \\ &\quad -s_{23}^2\left(1 + \frac{m_u}{m_c}\right)\left\{\frac{m_b}{m_d}\left(1 + \frac{m_u}{m_c}\right) - \left(1 + \frac{m_s}{m_d} + \frac{f m_u}{m_d m_c}\right)\right\} \end{aligned}$$

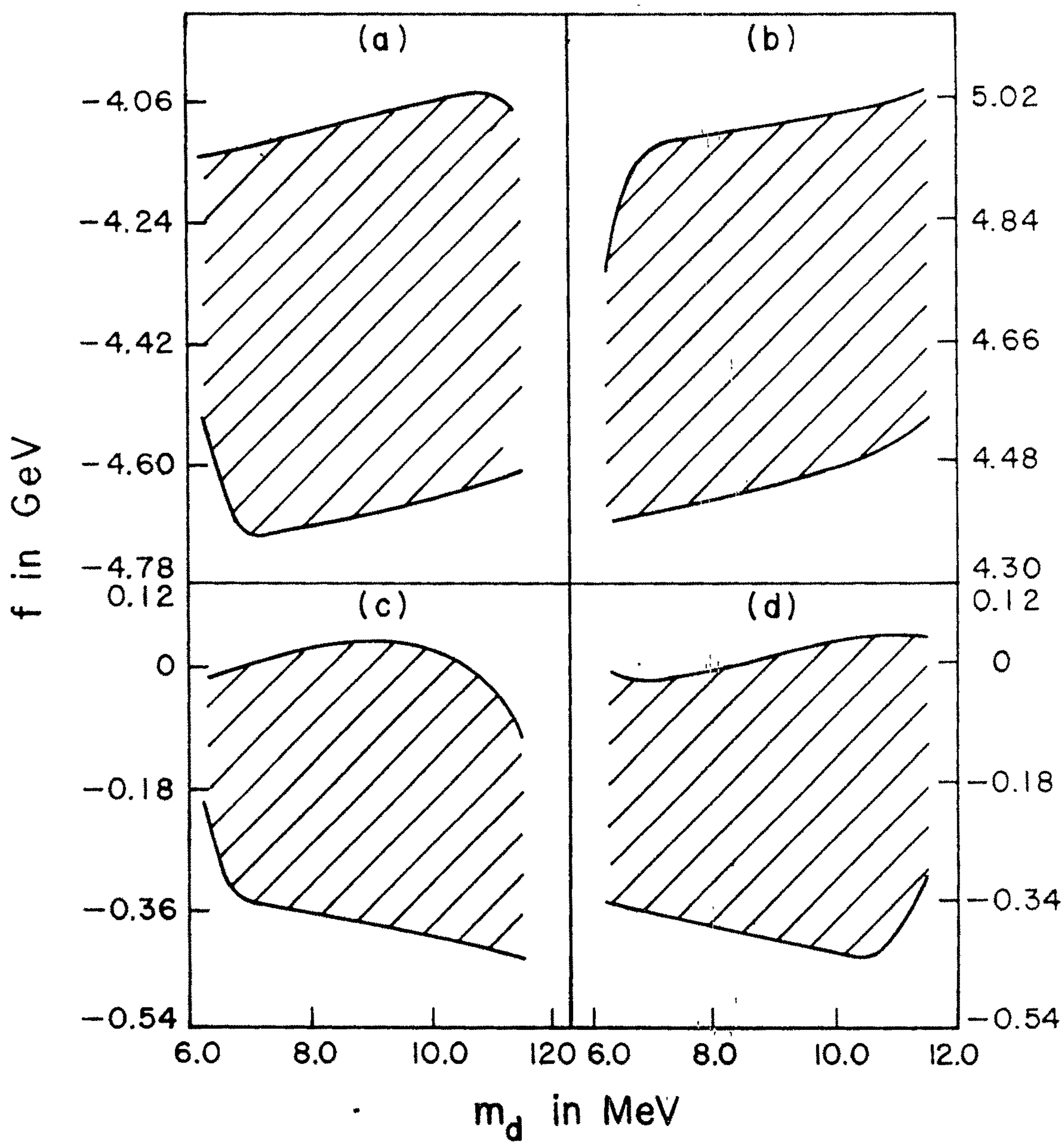

Figure 3.1: The allowed region for f (shaded region) as a function of m_d (see eqn 3.3 6)

$$a)\ \frac{m_s}{m_d} > 0, \quad \frac{m_u}{m_c} > 0 \qquad b)\ \frac{m_s}{m_d} > 0, \quad \frac{m_u}{m_c} < 0$$

$$c)\ \frac{m_s}{m_d} < 0, \quad \frac{m_u}{m_c} > 0 \qquad d)\ \frac{m_s}{m_d} < 0, \quad \frac{m_u}{m_c} < 0$$

All values are calculated at $\mu = 1\ GeV$. m_d has been assumed to be positive. For $m_d < 0$, $f \to -f$.

Thus for a given s_{23} we have a linear relation between d and m_t with the slope and intercept depending on the signs of the various mass ratios. In Stech model, for example, d was required to be zero, thus fixing m_t upto error bars due to the experimental uncertainties. A non-zero value of d would unfreeze this restriction and allow for better agreements with the experiments. The allowed regions for d for a fixed m_t has been given in Table 3.1. Similar to the case for f, d takes 'small' values about zero for $m_d/m_s < 0$ while for a positive value of this ratio it is considerably larger and a vanishing value is not consistent with the observations.

$Sign(\frac{m_b}{m_d}, \frac{m_b}{m_d}, m_c)$	Limits on $d(1\ GeV)$ (in $GeVs$)
$(+,+,+)$	$-3.43m_t + 5.19 < d < -3.27m_t + 5.39$
$(+,+,-)$	$-3.47m_t + 5.19 < d < -3.29m_t + 5.39$
$(+,-,+)$	$-0.15m_t + 5.19 < d < +0.19m_t + 5.19$
$(+,-,-)$	$-0.18m_t + 5.19 < d < +0.15m_t + 5.19$
$(-,+,+)$	$-3.46m_t + 5.21 < d < -3.35m_t + 5.21$
$(-,+,-)$	$-3.44m_t + 5.21 < d < -3.26m_t + 5.41$
$(-,-,+)$	$-0.18m_t + 5.12 < d < +0.16m_t + 5.21$
$(-,-,-)$	$-0.16m_t + 5.21 < d < +0.18m_t + 5.21$

Table 3.1: Limits on $d(1\ GeV)$ in terms of $m_t(1\ GeV)$ as imposed by eqn. (3.3.7). The limits are calculated for positive m_u, m_t, and m_d. For $m_d < 0$, $d \to -d$.

Taking the two generation result to be exact and substituting in eqn.(3.3.4) one obtains

$$\begin{aligned} s_{13}^2 &= \frac{m_u}{m_t + m_c + m_u} \frac{(m_c + m_u)(m_b - d) - m_t(m_s + m_d - f)}{(m_c + m_u)m_b - m_u(m_s + m_d - f)} \\ &\approx \frac{m_u}{m_c}\left(1 - \frac{m_s}{m_b}\right) s_{23}^2 \end{aligned} \tag{3.3.8}$$

This implies that $m_u/m_c > 0$. The analysis and the result are similar to Stech's (Section 3.1.1). An attempt to obtain a better approximation by an iterative procedure (*i.e.* substituting the current expressions for s_{23} and s_{13} in eqns. (3.3.5) and (3.3.6), instead of taking them to be zero and then redoing the same analysis) yields an extra term much smaller in magnitude.

But this result is in direct contradiction to the Fritzsch model (Section 3.1.2) wherein alternate generations have masses of opposite signs. Indeed, if equations (3.1.18) are squared, one obtains an equation similar to (3.3.8), but with an extra term typically larger than the right hand side.

The inconsistency lies in the analysis where one is aiming to solve for three angles from two equations. The relation (3.3.8) is thus shown not to be an outcome of the Stech ansatz but rather arising from an overkill of the equations (3.3.5) and (3.3.6). The best one can achieve without using the off-diagonal terms is an expression for s_{13} in terms of three unknowns m_t, f and d, the measured parameters s_{12} and the other five quark masses:

$$s_{13}^2 = \frac{\alpha m_u - \left[m_d + (m_s - m_d)s_{13}^2\right]}{m_b - \left[m_d + (m_s - m_d)s_{13}^2\right]}. \tag{3.3.9}$$

3.3.3 The off-diagonal terms:

Till now we have used only the diagonal terms of the matrix equation (3.3.1), ignoring the off diagonal terms, inclusion of which would give exact but contentless results. We continue in the same vein but would nevertheless like to look at these relations so as to get an idea of the relative magnitudes of these terms. We have

$$\begin{aligned}
R_1 e^{i\rho_1} &= c_{12}c_{23}c_{13}s_{12}(m_s - m_d) + c_{13}s_{13}s_{23}(m_b - c_{12}^2 c_{23} m_d - s_{12}^2 m_s)e^{-i\delta} \\
R_2 e^{i\rho_2} &= c_{23}c_{13}s_{13}(m_b - s_{12}^2 m_s - c_{12}^2 m_d) + c_{12}c_{13}s_{12}s_{23}(m_d - m_s)e^{i\delta} \\
R_3 e^{i\rho_3} &= c_{12}s_{12}s_{13}(m_d - m_s)\left(c_{23}^2 - s_{23}^2 e^{2i\delta}\right) \\
&\quad + c_{23}s_{23}e^{i\delta}\left[c_{13}^2 m_b + (c_{12}^2 s_{13}^2 - s_{12}^2)m_d - (s_{12}^2 s_{13}^2 - c_{12}^2)m_s\right]
\end{aligned} \tag{3.3.10}$$

The complex phases ρ_1 and ρ_2 are relatively small and lie in the same quadrant as can be seen from the fact that $\tan\rho_1 \tan\rho_2 \approx s_{23}^2$. While $\sin\rho_1$ attains its maximum of 0.12 when m_s and s_{12} assume the lowest allowed values and s_{13}, s_{23}, m_b the highest and $\delta \approx 83°$, $\sin\rho_2$ is maximized to 0.15 by giving $\left|\frac{K_{ub}}{K_{cb}}\right|$ and m_b their lowest values, m_s, s_{12} their highest and putting $\delta \approx 81.5°$. On the other hand $\rho_3 \approx \delta$. Hence this dominates the invariant phase $\rho_1 - \rho_2 + \rho_3$ and most of the CP–violating contribution comes from this term.

3.3.4 Models as special cases of the general form:

In this section we revert to a discussion of the models mentioned earlier. We demonstrate how these models could be obtained from the general mass–matrix on imposing suitable constraints. This would exhibit the restrictions one is pre–imposing on the various parameters and hopefully afford a better understanding of the implications of an ansatz.

Stech ansatz

A straightforward comparison of equations (3.1.2) and (3.3.2) gives the constraints to be

$$\begin{aligned} f &= 0 \\ d &= 0 \\ \rho_1 - \rho_2 + \rho_3 &= 90^\circ \end{aligned} \tag{3.3.11}$$

Using (3.3.11) in (3.3.10) one gets

$$\frac{s_{13}}{s_{23}} \cos\delta \approx 0 \tag{3.3.12}$$

This could be directly seen in the light of the discussion following equation (3.3.10). The Stech ansatz thus restricts the mass matrices to a seven parameter family which predicts near 'maximal' CP violation. One of the three promised predictions is then equation (3.3.12) or equivalently the last of of equations (3.1.4) and the others can be obtained from equations (3.3.6) and (3.3.7) by substituting $f = 0$ and $d = 0$ in them respectively.

The first two conditions obviously restrict the mass matrices to the negative m_s/m_d sector. Also a lower limit on the mass of the d-quark is set:

$$\begin{aligned} m_d(1GeV) &> 7MeV \quad \text{for } m_u/m_c > 0 \\ m_d(1GeV) &> 8.5MeV \quad \text{for } m_u/m_c < 0 \end{aligned} \tag{3.3.13}$$

There exist in the literature certain modifications of the Stech scheme as for example a non-zero d or an invariant phase $\rho_1 - \rho_2 + \rho_3$ different from 90°. Such models have reasonable agreement with experiments at the cost of loss of predictive power.

Fritzsch model

The simplest way to find the constraints to be put on the general form to obtain the Fritzsch form is to rotate M_d with O_u and compare the resultant with $M_{d(F)}$.

$$M_{d(F)} = O_u M_d O_u^T$$

gives the two required constraints:

$$\begin{aligned}
0 &= \frac{\alpha m_u}{N_1^2} + \frac{\alpha m_c + f}{N_2^2} + \frac{\alpha m_t + d}{N_3^2} + \frac{2R_1 \cos\rho_1}{N_1 N_2} + \frac{2R_2 \cos\rho_2}{N_1 N_3} + \frac{2R_3 \cos\rho_3}{N_2 N_3} \\
0 &= \frac{\alpha m_u}{N_1^2} m_u^2 + \frac{\alpha m_c + f}{N_2^2} m_c^2 + \frac{\alpha m_t + d}{N_3^2} m_t^2 \\
&\quad + \frac{2R_1 \cos\rho_1}{N_1 N_2} m_u m_c + \frac{2R_2 \cos\rho_2}{N_1 N_3} m_u m_t + \frac{2R_3 \cos\rho_3}{N_2 N_3} m_c m_t
\end{aligned} \tag{3.3.14}$$

Using equations (3.1.16) and (3.3.10) in the above, any two of the ten parameters can be eliminated. For example if f and d are evaluated in terms of the masses and the CKM parameters, then substituting the expressions for them in (3.3.6) and (3.3.7) would give us, say s_{13} and m_t in terms of the others and these would be the predictions of the model.

Fritsch-Shin scheme

Shin's choice for the phases in the Fritzsch mass matrix reduces the parameters by a further two and now we have four predictions. The choice is equivalent to imposing two additional constraints on the general form over and above eqns. (3.3.14):

$$\begin{aligned}
0 &= (m_c^2 - m_u^2)\frac{m_u m_c}{N_1 N_2} R_1 \sin\rho_1 + (m_t^2 - m_u^2)\frac{m_u m_t}{N_1 N_3} R_2 \sin\rho_2 \\
&\quad + (m_t^2 - m_c^2)\frac{m_c m_t}{N_2 N_3} R_3 \sin\rho_3 \\
0 &= \frac{\alpha m_u}{N_1^2} m_u + \frac{\alpha m_c + f}{N_2^2} m_c + \frac{\alpha m_t + d}{N_3^2} m_t + \frac{2R_1 \cos\rho_1}{N_1 N_2}(m_u + m_c) \\
&\quad + \frac{2R_2 \cos\rho_2}{N_1 N_3}(m_u + m_t) + \frac{2R_3 \cos\rho_3}{N_2 N_3}(m_c + m_t)
\end{aligned} \tag{3.3.15}$$

Proceeding in a manner similar to that for the general Fritzsch form, eqns.(3.3.15) give two more relations between the masses, the weak mixing angles and the CP–violating phase.

Fritzsch-Stech matrix

The simplest way to write the two additional constraints that take the general Fritzsch form to the one of interest is to demand that

$$\begin{aligned} Re(a_d) &= \frac{c_d}{c_u} a_u \\ Re(b_d) &= \frac{c_d}{c_u} b_u \end{aligned} \tag{3.3.16}$$

In our language this would look

$$\begin{aligned}
&\frac{(m_u+m_c)(m_u+m_t)(m_c+m_t)}{(m_u+m_c+m_t)^2} \Bigg[\frac{\alpha m_u^3}{N_1^2(m_c+m_t)^2} + \frac{2m_u m_c R_1 \cos\rho_1}{N_1 N_2 (m_u+m_c)(m_c+m_t)} \\
&\qquad + \frac{(\alpha m_c + f)m_c^2}{N_2^2(m_u+m_t)^2} + \frac{2m_u m_t R_2 \cos\rho_2}{N_1 N_3 (m_u+m_c)(m_c+m_t)} \\
&\qquad + \frac{(\alpha m_t + d)m_t^2}{N_2^2(m_u+m_c)^2} + \frac{2m_c m_t R_3 \cos\rho_3}{N_2 N_3 (m_u+m_c)(m_u+m_t)}\Bigg] \\
&= -\Bigg[\frac{\alpha m_u}{N_1^2} m_u + \frac{\alpha m_c + f}{N_2^2} m_c + \frac{\alpha m_t + d}{N_3^2} m_t + \frac{2R_1\cos\rho_1}{N_1 N_2}(m_u+m_c) \\
&\qquad + \frac{2R_2\cos\rho_2}{N_1 N_3}(m_u+m_t) + \frac{2R_3\cos\rho_3}{N_2 N_3}(m_c+m_t)\Bigg] \\
&= \frac{\alpha m_u^3}{N_1^2(m_c+m_t)} + \frac{(\alpha m_c+f)m_c^2}{N_2^2(m_u+m_t)} + \frac{(\alpha m_t+d)m_t^2}{N_2^2(m_u+m_c)} + \frac{m_u m_c(m_u+m_c+2m_t)}{(m_u+m_t)(m_c+m_t)}\frac{R_1\cos\rho_1}{N_1 N_2} \\
&\qquad + \frac{m_c m_t(2m_u+m_c+m_t)}{(m_u+m_c)(m_u+m_t)}\frac{R_2\cos\rho_2}{N_2 N_3} + \frac{m_u m_t(m_u+2m_c+m_t)}{(m_u+m_c)(m_c+m_t)}\frac{R_3\cos\rho_3}{N_1 N_3}
\end{aligned} \tag{3.3.17}$$

3.3.5 Conclusions

Our analysis has shown that the parameters involved in the general three-generation quark mass matrix are not fixed by the current experimental data but are allowed a continuous range. However this range is limited to different sectors depending on the relative signs of the mass terms. Most of the large width of these sectors arise due to the large indeterminacy in the masses of the lighter quarks and only to a lesser degree from the inaccuracy of the knowledge of the c–b mixing strength.

From the expressions in Section 3.3.3 we see that the off-diagonal terms in M_d are relatively small. In fact $\left|\frac{R_{1,2}}{m_s}\right| < O(0.1)$ and $\left|\frac{R_3}{m_s}\right| < O(1)$. In the light of this, if one demands that $\left|\frac{f}{m_s}\right|$ and $\left|\frac{d}{m_b}\right|$ not be too large either, then from Fig. 3.1 and Table 3.1 we are limited to the $\frac{m_d}{m_s} < 0$ sector. All the specific models that we have encountered so far lie in this category.

This implies that future model building could take two different courses. The more conservative course, given the moderate success of the current models would be to reexamine the present constraints and offer slight modifications that would alter or extend the models to a degree without drastically changing the basic structure. All these would be expected to lie in the $\frac{m_d}{m_s} < 0$ sector. The other and more radical approach would be to consider an entirely different class of models. This would entail f and d assuming much larger values compared to the other parameters in M_d and would demand a theoretical justification for such a behaviour.