CHAPTER VIII

MISCELLANEOUS PROBELMS

8.1 In this chapter two problems are studied. One is about the investigation of the bounds on the variance of the MVUE of the fraction defective under fully-curtailed single two class attributes sampling plan. The other problem is regarding the determination of the MVUE of the proportions of good, marginal and bad units under single three class attributes sampling plan, curtailed as well as uncurtailed.

8.2 Investigation of the Bounds on the Variance of the MVUE in Case of Single Two Class Attributes Sampling Plan :

Girshick, Mosteller, and Savage [12] have studied the problem of the unique unbiased estimator of the fraction defective p in a sequence of binomial trials under various stopping rules. Some of the stopping rules described in their paper [12] resembled to the curtailed single and double two class attributes sampling plans discussed in the earlier chapters(II and IV). Let the unique unbiased estimators of p for uncurtailed, semi-curtailed, and fully-curtailed single sampling plans for the inspection results of one lot on hand be denoted as $\overline{p}$(uncu), $\overline{p}$(semi), $\overline{p}$(fully) respectively. The variance of $\overline{p}$(uncu) is well known and it is equal to pq/n. Nelson, Williams, and Fletcher [35] have given the approximate expression for the variance of $\overline{p}$(semi). In the following sections we have studied the problem of the variance of $\overline{p}$(fully). In this case the exact evaluation of the variance is possible but not simple. Hence we have studied the bounds for the variance of $\overline{p}$(fully).

8.2.1 Probability Function and MVUE :

We recall the statement of the fully-curtailed single two class attributes sampling plan from Section 4.3.2 of Chapter IV. The probability function associated with this sampling plan under binomial probability law can be stated as given below :

$$P(Y=y,\ T=t) = \begin{cases} \binom{y-1}{g-1}\, p^{y-g} q^{g} & y=g,g+1,\ldots,n;\ t=1 \\ \binom{y-1}{k-1}\, p^{k} q^{y-k} & y=k,k+1,\ldots,n;\ t=2 \end{cases} \qquad \ldots(8.2.1)$$

where $0 \leq p \leq 1$ and $q = 1-p$

n = size of the sample

k = rejection number

$g = n-k+1$

Y = actual number of units inspected when sampling is stopped due to observing either k defective units or g nondefective units.

The MVUE $\overline{p}$(fully) in this case is as given below :

$$\overline{p}(\text{fully}) = \begin{cases} \frac{y-g}{y-1} & y=g,g+1,\ldots,n \\ \frac{k-1}{y-1} & y=k,k+1,\ldots,n \end{cases} \qquad \ldots(8.2.2)$$

It may be noted that the first part of the expressions (8.2.1) and (8.2.2) is associated with the acceptance of a lot and the second part of the same is associated with the rejection of a lot.

8.2.2 Variance of $\overline{p}$(fully) :

The variance of $\overline{p}$(fully) is given by

$$\begin{aligned} \text{Var}\left[\overline{p}(\text{fully})\right] &= E\left[\overline{p}(\text{fully}) - E\left\{\overline{p}(\text{fully})\right\}\right]^2 \\ &= E\left[\overline{p}(\text{fully})\right]^2 - p^2 \qquad \ldots(8.2.3) \end{aligned}$$

Now consider the $E\left[\overline{p}(\text{fully})\right]^2$. From definition of expectation we have

$$E\left[\overline{p}(\text{fully})\right]^2 = \sum_{y=g}^{n} \left(\frac{y-g}{y-1}\right)^2 \binom{y-1}{g-1} p^{y-g} q^{g} + \sum_{y=k}^{n} \left(\frac{k-1}{y-1}\right)^2 \cdot \binom{y-1}{k-1} p^k q^{y-k}$$

$$= pB(k-2;n-1,p)-qB(k-1;n-1,p) + \sum_{y=g}^{n} \left(\frac{g-1}{y-1}\right)^2 \binom{y-1}{g-1} p^{y-g} q^g$$

$$+ \sum_{y=k}^{n} \left(\frac{k-1}{y-1}\right)^2 \binom{y-1}{k-1} p^k q^{y-k} \qquad \ldots(8.2.4)$$

where $B(r;n,p) = \sum_{x=0}^{r} \binom{n}{x} p^x q^{n-x}$. To find the variance one has merely to subtract p^2 from the evaluation of (8.2.4). It may be noted that the exact evaluation of the expression (8.2.4) is possible but it is not simple because of the summation terms involved (particularly for large n). Hence in the following section we try to get the bounds for the variance of $\overline{p}$(fully).

8.2.3 Bounds for the Variance of $\overline{p}$(fully) :

Define a random variable X with the probability function given as

$$h(x) = \frac{\binom{x-1}{t-1} p^{x-t} q^t}{\sum_{x=t}^{n} \binom{x-1}{t-1} p^{x-t} q^t} \qquad x=t,\ t+1, \ldots, n$$

$$= 0 \qquad \text{elsewhere} \qquad \ldots(8.2.5)$$

Using the general result

$$E(Z^2) \geq E^2(Z) \qquad \ldots(8.2.6)$$

and taking $Z = \frac{1}{X-1}$, where X has probability function (8.2.5), we get the following inequality :

$$\sum_{x=t}^{n} \left(\frac{t-1}{x-1}\right)^2 \binom{x-1}{t-1} p^{x-t} q^t \geq \frac{q^2 \{B(n-t;n-1,p)\}^2}{B(n-t;n,p)} \qquad \ldots(8.2.7)$$

By interchanging p and q in (8.2.5) and using the result

(8.2.6) again we get the following inequality,

$$\sum_{x=t}^{n}\left(\frac{t-1}{x-1}\right)^2\binom{x-1}{t-1} p^t q^{x-t} \geq \frac{p^2\{1-B(t-2;n-1,p)\}^2}{\{1-B(t-1;n,p)\}} \qquad \ldots(8.2.8)$$

Furthermore, define a random variable U with the following probability function :

$$f(u) = \frac{\binom{u-2}{r-2} p^{u-r} q^r}{\sum_{u=r}^{n}\binom{u-2}{r-2} p^{u-r} q^r} \qquad u=r,\ r+1,\ldots,n$$

$$= 0 \qquad \text{elsewhere} \qquad \ldots(8.2.9)$$

Using the following result

$$E(Z) \leq 1-\frac{1}{E\left(\frac{1}{1-Z}\right)} \qquad \ldots(8.2.10)$$

and taking $Z = \frac{1}{U-1}$, where U has probability function (8.2.9), we get the following inequality :

$$\sum_{u=r}^{n}\left(\frac{r-1}{u-1}\right)\binom{u-2}{r-2} p^{u-r} q^r \leq \frac{(r-1)q^2 B(n-r;n-2,p)B(n-r;n-1,p)}{(r-2)B(n-r;n-1,p)+qB(n-r;n-2,p)}$$

$$\ldots(8.2.11)$$

If we interchange p and q in (8.2.9) and use the relation (8.2.10) again we get the inequality as

$$\sum_{u=r}^{n}\left(\frac{r-1}{u-1}\right)\binom{u-2}{r-2} p^r q^{u-r} \leq \frac{(r-1)p^2\{1-B(r-3;n-2,p)\}\{1-B(r-2;n-1,p)\}}{(r-2)\{1-B(r-2;n-1,p)\}+p\{1-B(r-3;n-2,p)\}}$$

$$\ldots(8.2.12)$$

Applying the inequalities (8.2.7), (8.2.8), (8.2.11), and (8.2.12) on the third term and fourth term of (8.2.4) we get the following bounds for the variance of unbiased estimator.

$$\text{var}\left[\overline{p}(\text{fully})\right] \leq pB(k-2;n-1,p) - qB(k-1;n-1,p) + \frac{(g-1)q^2 B(k-1;n-2,p)\; B(k-1;n-1,p)}{(g-2)B(k-1;n-1,p)+qB(k-1;n-2,p)} + \frac{(k-1)p^2\,[1-B(k-3;n-2,p)]\,[1-B(k-2;n-1,p)]}{(k-2)\,[1-B(k-2;n-1,p)]+p\,[1-B(k-3;n-2,p)]} - p^2 \quad \ldots(8.2.13)$$

and

$$\text{var}\left[\overline{p}(\text{fully})\right] \geq pB(k-2;n-1,p) - qB(k-1;n-1,p) + \frac{q^2\{B(k-1;n-1,p)\}^2}{B(k-1;n,p)} + \frac{p^2\{1-B(k-2;n-1,p)\}^2}{\{1-B(k-1;n,p)\}} - p^2 \quad \ldots(8.2.14)$$

Expression (8.2.13) gives an upper bound and (8.2.14) gives a lower bound.

The Cramer-Rao lower bound (C-R bound) in this case is given as

$$\text{Var}\left[\overline{p}(\text{fully})\right] \geq \frac{pq}{ASN} \quad \ldots(8.2.15)$$

where $ASN = \frac{g}{q}\,B(k-1;n+1,p) + \frac{k}{p}\,[1-B(k;n+1,p)]$

It may be emphasized that the expressions (8.2.13) through (8.2.15) can be evaluated using the usual Binomial Probability Tables such as[34],[43] etc. and desk calculators.

8.2.4 A Numerical Example :

In this section we have illustrated the evaluation of the bounds and the exact variance discussed in the previous sections considering the following single sampling plan :

$n=25$, $k=3$ (and $g=23$).

Table 8.1 gives the bounds and the exact variance for different values of p.

Table 8.1

Plan : $n=25$, $k=3$ $(g=23)$

p	Lower bound	Upper bound	C-R bound	$V[\bar{p}(\text{fully})]$
1	2	3	4	5
0.04	0.00067551	0.00209541	0.00164348	0.00194467
0.05	0.00101344	0.00282383	0.00204930	0.00257758
0.06	0.00135667	0.00366709	0.00246527	0.00328810
0.07	0.00168146	0.00463130	0.00289651	0.00407786
0.08	0.00197106	0.00571843	0.00334746	0.00496044
0.09	0.00221481	0.00692713	0.00382194	0.00590889
0.10	0.00240668	0.00825399	0.00432318	0.00690384

8.2.5 Remarks

(i) Lower bound given by (8.2.14) is not better than the C-R bound. This could be observed by comparing columns (2) and (4) of Table 8.1.

(ii) Exact variance has a tendency to be nearer to the upperbound as compared to the C-R bound. This is revealed from the comparison of column (5) with columns (3) and (4) of Table 8.1 . Hence if one is interested in substituting the exact variance by any of the bounds (C-R bound or upper bound) from the view point of easy calculations, the upper bound will serve as a substitute for the variance.

(iii) The motivation to use the inequalities (8.2.6) and (8.2.10) as the basis for obtaining the bounds was from Sathe's paper [42] in which he uses these inequalities to get the bounds for the variance of the MVUE of the parameter θ of the usual negative binomial distribution. Somehow the same technique in his case gives sharper bounds and that too the lower bound better than the usual C-R bound. So is not the case in the problem we studied.

8.3 Unique Unbiased Estimators in Case of Single Three Class Attributes Sampling Plan :

We have defined the uncurtailed, semi-curtailed and fully-curtailed single three class attributes plans in Chapter V. In the following sections we give the unique unbiased estimators for the proportions of good, marginal and bad units.

8.3.1 Method for Unique Unbiased Estimator :

In case of a trinomial distribution a sequential method of estimation of p_0, p_1, and p_2, developed by Muhamedhanova and Suleimanova [33], is described in Johnson and Kotz [28, pp.289]. The method is a natural generalization of that of Girshick, Mosteller, and Savage [12]. The method is based on the enumeration of the paths followed by the poimt (d_0, d_1, d_2), as y increases, which conclude at an observed termination point. Here $y(d_0+d_1+d_2)$ represents the number of trials. The termination points are points on the boundary of the continuation region, and this boundary defines the sampling procedure. For a given sampling procedure boundary points are classified as the boundary points of the acceptance region and the boundary points of the rejection region. If the termination point is (d_0, d_1, d_2) and $K(d_0, d_1, d_2)$ be the number of possible paths from (0,0,0) ending at (d_0,d_1,d_2), while $K_j(d_0,d_1,d_2)$ is the number of such paths starting from (1,0,0) for j=0, (0,1,0) for j=1, or (0,0,1) for j=2, then

$$K_j(d_0,d_1,d_2)/K(d_0,d_1,d_2) \qquad \ldots(8.3.1)$$

is the unique unbiased estimator of p_j.

Using the method described above and evaluating

$K_j(d_o,d_1,d_2)$ and $K(d_o,d_1,d_2)$ in case of uncurtailed, semi--curtailed, and fully-curtailed three class attributes sampling plans we have obtained the unique unbiased estimators of p_o, p_1 and p_2 under each sampling procedure in the following sections. Of course, these estimators are based on the inspection results of one lot on hand. To evaluate $K_j(d_o,d_1,d_2)$ and $K(d_o,d_1,d_2)$, define a quantity $N(a_1,a_2,a_3)$ as given below :

$$N(a_1,a_2,a_3) = \frac{(a_1+a_2+a_3)!}{a_1!a_2!a_3!} \qquad \ldots(8.3.2)$$

Evaluation of $K(\cdot)$ and $K_j(\cdot)$ in terms of $N(\cdot)$ is done in the succeeding sections.

8.3.2 Probability Functions :

Recollect the probability functions of a semi-curtailed and a fully-curtailed single three class attributes sampling plans from sections (5.4.1) and (5.5.1) of Chapter V, respectively. We have to reproduce them into some convenient forms for the purpose of evaluation of the number of paths needed for the calculation of unbiased estimators. The probability functions of uncurtailed, semi-curtailed, and fully-curtailed single three class attributes sampling plans in convenient forms are as given below :

(i) Uncurtailed sampling plan :

$P(Y=n,\ D_1=d_1,\ D_2=d_2,\ I=i)$

$$
= \begin{cases}
f_1(n,d_1,d_2;p_1,p_2) & d_1=0,1,\ldots,k_1-d_2-1 \\
 & d_2=0,1,\ldots,k_2-1 \\
 & i=1 \\
f_2(n,d_1,d_2;p_1,p_2) & d_1=0,1,\ldots,k_1-d_2-1 \\
 & d_2=k_2,\ldots,n \\
 & i=2 \\
f_3(n,d_1,d_2;p_1,p_2) & d_1=k_1-d_2,\ldots,n \\
 & d_2=0,1,\ldots,k_2-1 \\
 & i=3 \\
f_4(n,d_1,d_2;p_1,p_2) & d_1=k_1-d_2,\ldots,n \\
 & d_2=k_2,\ldots,n \\
 & i=4
\end{cases} \qquad \ldots(8.3.3)
$$

where $f_i(n,d_1,d_2;p_1,p_2) = \dfrac{n!\,p_o^{\,n-d_1-d_2}\,p_1^{\,d_1}\,p_2^{\,d_2}}{d_1!\ d_2!\,(n-d_1-d_2)!}$ for i=1,2,3,4.

For i=1 the lot is accepted and for i=2,3,4 the lot is rejected.

(ii) Semi-curtailed Sampling Plan :

$P(Y=y,\ D_1=d_1,\ D_2=d_2,\ I=i)$

$$
= \begin{cases}
h_1(y,d_1,d_2;p_1,p_2) & \begin{array}{l} y=n \\ d_1=0,1,\ldots,k_1-d_1-1 \\ d_2=0,1,\ldots,k_2-1 \\ i=1 \end{array} \\
h_2(y,d_1,d_2;p_1,p_2) & \begin{array}{l} y=k_2,k_2+1,\ldots,n \\ d_1=0,1,\ldots,k_1-k_2 \\ d_2=k_2 \\ i=2 \end{array} \\
h_3(y,d_1,d_2;p_1,p_2) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_1=k_1-d_2 \\ d_2=1,2,\ldots,k_2-1 \\ i=3 \end{array} \\
h_4(y,d_1,d_2;p_1,p_2) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_1=k_1-d_2 \\ d_2=0,1,\ldots,k_2-1 \\ i=4 \end{array}
\end{cases} \qquad \ldots(8.3.4)
$$

where
$$h_1(y,d_1,d_2;p_1,p_2) = \frac{n!\,p_o^{\,n-d_1-d_2}\,p_1^{\,d_1}\,p_2^{\,d_2}}{d_1!\,d_2!\,(n-d_1-d_2)!}$$

$$h_2(y,d_1,d_2;p_1,p_2) = \frac{(y-1)!\,p_o^{\,y-k_2-d_1}\,p_1^{\,d_1}\,p_2^{\,k_2}}{d_1!\,(k_2-1)!\,(y-k_2-d_1)!}$$

$$h_3(y,d_1,d_2;p_1,p_2) = \frac{(y-1)!\,p_o^{\,y-k_1}\,p_1^{\,k_1-d_2}\,p_2^{\,d_2}}{(k_1-d_2)!\,(d_2-1)!\,(y-k_1)!}$$

$$h_4(y,d_1,d_2;p_1,p_2) = \frac{(y-1)!\ p_o^{y-k_1}\ p_1^{k_1-d_2}\ p_2^{d_2}}{(k_1-d_2-1)!\ d_2!(y-k_1)!}$$

For i=1, the lot is accepted and for i=2,3,4 the lot is rejected.

(iii) Fully-Curtailed Plan :

$P(Y=y,\ D_o=d_o,\ D_1=d_1,\ D_2=d_2,\ I=i)$

$$= \begin{cases} u_1(y,d_o,d_1;p_1,p_2) & y=g_1,\ldots,n \\ & d_o=g_o,\ldots,g_1 \\ & d_1=g_1-d_o \\ & d_2=y-g_1 \\ & i=1 \\ u_2(y,d_o,d_1;p_1,p_2) & y=g_1,\ldots,n \\ & d_o=g_o,\ldots,g_1-1 \\ & d_1=g_1-d_o-1 \\ & d_2=y-g_1 \\ & i=2 \\ u_3(y,d_o,d_1;p_1,p_2) & y=g_1,\ldots,n \\ & d_o=g_o-1 \\ & d_1=k_1-k_2+1,\ldots,n \\ & d_2=y-g_o-d_1 \\ & i=3 \end{cases}$$

cont...

$$
\begin{cases}
u_4(y,d_1,d_2;p_1,p_2) & \begin{array}{l} y=k_2,k_2+1,\ldots,n \\ d_o=y-k_2-d_1 \\ d_1=0,1,\ldots,k_1-k_2 \\ d_2=k_2 \\ i=4 \end{array} \\
u_5(y,d_1,d_2;p_1,p_2) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=1,2,\ldots,k_2-1 \\ i=5 \end{array} \\
u_6(y,d_1,d_2;p_1,p_2) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=0,1,\ldots,k_2-1 \\ i=6 \end{array}
\end{cases}
\qquad \ldots(8.3.5)
$$

where $u_1(y,d_o,d_1;p_1,p_2) = \dfrac{(y-1)!\, p_o^{d_o}\, p_1^{g_1-d_o}\, p_2^{y-g_1}}{(g_1-d_o)!(y-g_1)!(d_o-1)!}$

$$u_2(y,d_o,d_1;p_1,p_2) = \frac{(y-1)!\, p_o^{d_o}\, p_1^{g_1-d_o}\, p_2^{y-g_1}}{(g_1-d_o-1)!\,(y-g_1)!\,d_o!}$$

$$u_3(y,d_o,d_1;p_1,p_2) = \frac{(y-1)!\, p_o^{g_o}\, p_1^{d_1}\, p_2^{y-g_o-d_1}}{d_1!(y-g_o-d_1)!(g_o-1)!}$$

$$u_4(y,d_1,d_2;p_1,p_2) = h_2(y,d_1,d_2;p_1,p_2)$$

$$u_5(y,d_1,d_2;p_1,p_2) = h_3(y,d_1,d_2;p_1,p_2)$$

$$u_6(y,d_1,d_2;p_1,p_2) = h_4(y,d_1,d_2;p_1,p_2)$$

where $g_0=n-k_1+1$ and $g_1=n-k_2+1$.

For $i=1,2,3$ the lot is accepted and for $i=4,5,6$ the lot is rejected.

8.3.3 Unbiased Estimator Under Uncurtailed Sampling Plan :

In case of uncurtailed sampling plan the number of units inspected (number of trials) is equal to n. The number of paths under uncurtailed sampling plan are as given below :

Number of Paths from (0,0,0) :

$$K(d_0,d_1,d_2)=N(n-d_1-d_2,\ d_1,d_2) \quad \text{for } i=1,2,3,4$$

Number of paths from (1,0,0) :

$$K_0(d_0,d_1,d_2) = N(n-d_1-d_2-1,d_1,d_2) \quad \text{for } i=1,2,3,4$$

Number of paths from (0,1,0) :

$$K_1(d_0,d_1,d_2) = N(n-d_1-d_2,d_1-1,d_2) \quad \text{for } i=1,2,3,4$$

Number of paths from (0,0,1) :

$$K_2(d_0,d_1,d_2) = N(n-d_1-d_2,d_1,d_2-1) \quad \text{for } i=1,2,3,4$$

Using the definition given in (8.3.1) and denoting the unbiased estimators as $\overline{p}_j(\text{uncu})$ $j=0,1,2$, we have

$$\overline{p}_0(\text{uncu}) = (n-d_1-d_2)/n \qquad \text{for } i = 1,2,3,4$$

$$\overline{p}_1(\text{uncu}) = d_1/n \qquad \text{for } i = 1,2,3,4$$

$$\overline{p}_2(\text{uncu}) = d_2/n \qquad \text{for } i= 1,2,3,4.$$

8.3.4 Unbiased Estimators under Semi-Curtailed Sampling Plan :

The number of paths under semi-curtailed sampling plan are given below :

Number of Paths from (0,0,0) :

$$K(d_o,d_1,d_2)=\begin{cases} N(n-d_1-d_2,d_1,d_2) & y=n \\ & d_1=0,1,\ldots,k_1-d_2-1 \\ & d_2=0,1,\ldots,k_2-1 \\ & i=1 \\ N(y-k_2-d_1,d_1,k_2-1) & y=k_2,k_2+1,\ldots,n \\ & d_1=0,1,\ldots,k_1-k_2 \\ & d_2=k_2 \\ & i=2 \\ N(y-k_1,\cancel{k_1},k_1-d_2,d_2-1) & y=k_1,k_1+1,\ldots,n \\ & d_1=k_1-d_2 \\ & d_2=1,2,\ldots,k_2-1 \\ & i=3 \end{cases}$$

cont...

$$\begin{cases} N(y-k_1,k_1-d_2-1,d_2) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_1=k_1-d_2 \\ d_2=0,1,\ldots,k_2-1 \\ i=4 \end{array} \end{cases} \quad \ldots(8.3.6)$$

Number of Paths from (1,0,0) :

$$K_o(d_o,d_1,d_2)=\begin{cases} N(n-d_1-d_2-1,d_1,d_2) & \begin{array}{l} y=n \\ d_1=0,1,\ldots,k_1-d_2-1 \\ d_2=0,1,\ldots,k_2-1 \\ i=1 \end{array} \\ N(y-k_2-d_1-1,d_1,k_2-1) & \begin{array}{l} y=k_2,k_2+1,\ldots,n \\ d_1=0,1,\ldots,k_1-k_2 \\ d_2=k_2 \\ i=2 \end{array} \\ N(y-k_1-1,k_1-d_2,d_2-1) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_1=k_1-d_2 \\ d_2=1,2,\ldots,k_2-1 \\ i=3 \end{array} \\ N(y-k_1-1,k_1-d_2-1,d_2) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_1=k_1-d_2 \\ d_2=0,1,\ldots,k_2-1 \\ i=4 \end{array} \end{cases} \quad \ldots(8.3.7)$$

Number of Paths from (0,1,0) :

$$K_1(d_o,d_1,d_2)=\begin{cases} N(n-d_1-d_2,d_1-1,d_2) & y=n \\ & d_1=0,1,\ldots,k_1-d_2-1 \\ & d_2=0,1,\ldots,k_2-1 \\ & i=1 \\ N(y-k_2-d_1,d_1-1,k_2-1) & y=k_2,k_2+1,\ldots,n \\ & d_1=0,1,\ldots,k_1-k_2 \\ & d_2=k_2 \\ & i=2 \\ N(y-k_1,k_1-d_2-1,d_2-1) & y=k_1,k_1+1,\ldots,n \\ & d_1=k_1-d_2 \\ & d_2=1,2,\ldots,k_2-1 \\ & i=3 \\ N(y-k_1,k_1-d_2-2,d_2) & y=k_1,k_1+1,\ldots,n \\ & d_1=k_1-d_2 \\ & d_2=0,1,\ldots,k_2-1 \\ & i=4 \end{cases} \qquad \ldots(8.3.8)$$

Number of Paths from(0,0,1):

$$K_2(d_o,d_1,d_2)=\begin{cases} N(n-d_1-d_2,d_1,d_2-1) & y=n \\ & d_1=0,1,\ldots,k_1-d_2-1 \\ & d_2=0,1,\ldots,k_2-1 \\ & i=1 \\ N(y-k_2-d_1,d_1,k_2-2) & y=k_2,k_2+1,\ldots,n \\ & d_1=0,1,\ldots,k_1-k_2 \\ & d_2=k_2 \\ & i=2 \\ N(y-k_1,k_1-d_2,d_2-2) & y=k_1,k_1+1,\ldots,n \\ & d_1=k_1-d_2 \\ & d_2=1,2,\ldots,k_2-1 \\ & i=3 \\ N(y-k_1,k_1-d_2-1,d_2-1) & y=k_1,k_1+1,\ldots,n \\ & d_1=k_1-d_2 \\ & d_2=0,1,\ldots,k_2-1 \\ & i=4 \end{cases} \qquad \ldots(8.3.9)$$

Substituting for $K(d_o,d_1,d_2)$ and $K_j(d_o,d_1,d_2)$, $j=0,1,2$ from (8.3.6) through (8.3.9) in the definition given by (8.3.1) we get the unbiased estimators $\overline{p}_j(\text{semi})$, $j=0,1,2$ as given below :

$$\overline{p}_o(\text{semi})=\begin{cases}(n-d_1-d_2)/n & d_1=0,1,\ldots,k_1-d_2-1 \\ & d_2=0,1,\ldots,k_2-1 \\ (y-k_2-d_1)/(y-1) & y=k_2,k_2+1,\ldots,n \\ & d_1=0,1,\ldots,k_1-k_2 \\ (y-k_1)/(y-1) & y=k_1,k_1+1,\ldots,n \\ & d_2=1,2,\ldots,k_2-1 \\ (y-k_1)/(y-1) & y=k_1,k_1+1,\ldots,n \\ & d_2=0,1,\ldots,k_2-1\end{cases}$$

$$\overline{p}_1(\text{semi})=\begin{cases}d_1/n & d_1=0,1,\ldots,k_1-d_2-1 \\ & d_2=0,1,\ldots,k_2-1 \\ d_1/(y-1) & y=k_2,k_2+1,\ldots,n \\ & d_1=0,1,\ldots,k_1-k_2 \\ (k_1-d_2)/(y-1) & y=k_1,k_1+1,\ldots,n \\ & d_2=1,2,\ldots,k_2-1 \\ (k_1-d_2-1)/(y-1) & y=k_1,k_1+1,\ldots,n \\ & d_2=0,1,\ldots,k_2-1\end{cases}$$

$$\overline{p}_2(\text{semi})=\begin{cases} d_2/n & d_1=0,1,\ldots,k_1-d_2-1 \\ & d_2=0,1,\ldots,k_2-1 \\ (k_2-1)/(y-1) & y=k_2,k_2+1,\ldots,n \\ & d_1=0,1,\ldots,k_1-k_2 \\ (d_2-1)/(y-1) & y=k_1,k_1+1,\ldots,n \\ & d_2=1,2,\ldots,k_2-1 \\ d_2/(y-1) & y=k_1,k_1+1,\ldots,n \\ & d_2=0,1,\ldots,k_2-1 \end{cases}$$

8.3.5 Unbiased Estimators under Fully-Curtailed Sampling Plan :

Number of paths under fully-curtailed sampling plan are given below :

Number of Paths from (0,0,0) :

$$
K(d_o,d_1,d_2)=\begin{cases}
N(d_o-1,g_1-d_o,y-g_1) & \begin{array}{l} y=g_1,\ldots,n \\ d_o=g_o,\ldots,g_1 \\ d_1=g_1-d_o \\ d_2=y-g_1 \\ i=1 \end{array} \\[2ex]
N(d_o,g_1-d_o-1,y-g_1) & \begin{array}{l} y=g_1,\ldots,n \\ d_o=g_o,\ldots,g_1-1 \\ d_1=g_1-d_o-1 \\ d_2=y-g_1 \\ i=2 \end{array} \\[2ex]
N(g_o-1,d_1,y-g_o-d_1) & \begin{array}{l} y=g_1,\ldots,n \\ d_o=g_o-1 \\ d_1=k_1-k_2+1,\ldots,n \\ d_2=y-g_o-d_1 \\ i=3 \end{array} \\[2ex]
N(y-k_2-d_1,d_1,k_2-1) & \begin{array}{l} y=k_2,k_2+1,\ldots,n \\ d_o=y-k_2-d_1 \\ d_1=0,1,\ldots,k_1-k_2 \\ d_2=k_2 \\ i=4 \end{array} \\[2ex]
N(y-k_1,k_1-d_2,d_2-1) & \begin{array}{l} y=k_1,\ldots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=1,2,\ldots,k_2-1 \\ i=5 \end{array} \\[2ex]
N(y-k_1,k_1-d_2-1,d_2) & \begin{array}{l} y=k_1,\ldots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=0,1,\ldots,k_2-1 \\ i=6 \end{array}
\end{cases}
\qquad \ldots(8.3.10)
$$

Number of Paths from (1,0,0) :

$$K_o(d_o,d_1,d_2)=\begin{cases}
N(d_o-2,g_1-d_o,y-g_1) & \begin{array}{l} y=g_1,\ldots,n \\ d_o=g_o,\ldots,g_1 \\ d_1=g_1-d_o \\ d_2=y-g_1 \\ i=1 \end{array} \\
N(d_o-1,g_1-d_o-1,y-g_1) & \begin{array}{l} y=g_1,\ldots,n \\ d_o=g_o,\ldots,g_1-1 \\ d_1=g_1-d_o-1 \\ d_2=y-g_1 \\ i=2 \end{array} \\
N(g_o-2,d_1,y-g_o-d_1) & \begin{array}{l} y=g_1,\ldots,n \\ d_o=g_o-1 \\ d_1=k_1-k_2+1,\ldots,n \\ d_2=y-g_o-d_1 \\ i=3 \end{array} \\
N(y-k_2-d_1-1,d_1,k_2-1) & \begin{array}{l} y=k_2,k_2+1,\ldots,n \\ d_o=y-k_2-d_1 \\ d_1=0,1,\ldots,k_1-k_2 \\ d_2=k_2 \\ i=4 \end{array} \\
N(y-k_1-1,k_1-d_2,d_2-1) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=1,2,\ldots,k_2-1 \\ i=5 \end{array} \\
N(y-k_1-1,k_1-d_2-1,d_2) & \begin{array}{l} y=k_1,k_1+1,\ldots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=0,1,\ldots,k_2-1 \\ i=6 \end{array}
\end{cases} \qquad \ldots(8.3.11)$$

Number of Paths from (0,1,0) :

$$
K_1(d_o,d_1,d_2)=\begin{cases}
N(d_o-1,g_1-d_o-1,y-g_1) & \begin{array}{l} y=g_1,\dots,n \\ d_o=g_o,\dots,g_1 \\ d_1=g_1-d_o \\ d_2=y-g_1 \\ i=1 \end{array} \\[2ex]
N(d_o,g_1-d_o-2,y-g_1) & \begin{array}{l} y=g_1,\dots,n \\ d_o=g_o,\dots,g_1-1 \\ d_1=g_1-d_o-1 \\ d_2=y-g_1 \\ i=2 \end{array} \\[2ex]
N(g_o-1,d_1-1,y-g_o-d_1) & \begin{array}{l} y=g_1,\dots,n \\ d_o=g_o-1 \\ d_1=k_1-k_2+1,\dots,n \\ d_2=y-g_o-d_1 \\ i=3 \end{array} \\[2ex]
N(y-k_2-d_1,d_1-1,k_2-1) & \begin{array}{l} y=k_2,k_2+1,\dots,n \\ d_o=y-k_2-d_1 \\ d_1=0,1,\dots,k_1-k_2 \\ d_2=k_2 \\ i=4 \end{array} \\[2ex]
N(y-k_1,k_1-d_2-1,d_2-1) & \begin{array}{l} y=k_1,k_1+1,\dots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=1,2,\dots,k_2-1 \\ i=5 \end{array} \\[2ex]
N(y-k_1,k_1-d_2-2,d_2) & \begin{array}{l} y=k_1,k_1+1,\dots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=0,1,\dots,k_2-1 \\ i=6 \end{array}
\end{cases}
\qquad \dots(8.3.12)
$$

Number of Paths from (0,0,1) :

$$
K_2(d_o,d_1,d_2)=\begin{cases}
N(d_o-1,g_1-d_o,y-g_1-1) & \begin{array}{l} y=g_1,\dots,n \\ d_o=g_o,\dots,g_1 \\ d_1=g_1-d_o \\ d_2=y-g_1 \\ i=1 \end{array} \\[2ex]
N(d_o,g_1-d_o-1,y-g_1-1) & \begin{array}{l} y=g_1,\dots,n \\ d_o=g_o,\dots,g_1-1 \\ d_1=g_1-d_o-1 \\ d_2=y-g_1 \\ i=2 \end{array} \\[2ex]
N(g_o-1,d_1,y-g_o-d_1-1) & \begin{array}{l} y=g_1,\dots,n \\ d_o=g_o-1 \\ d_1=k_1-k_2+1,\dots,n \\ d_2=y-g_o-d_1 \\ i=3 \end{array} \\[2ex]
N(y-k_2-d_1,d_1,k_2-2) & \begin{array}{l} y=k_2,k_2+1,\dots,n \\ d_o=y-k_2-d_1 \\ d_1=0,1,\dots,k_1-k_2 \\ d_2=k_2 \\ i=4 \end{array} \\[2ex]
N(y-k_1,k_1-d_2,d_2-2) & \begin{array}{l} y=k_1,k_1+1,\dots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=1,2,\dots,k_2-1 \\ i=5 \end{array} \\[2ex]
N(y-k_1,k_1-d_2-1,d_2-1) & \begin{array}{l} y=k_1,k_1+1,\dots,n \\ d_o=y-k_1 \\ d_1=k_1-d_2 \\ d_2=0,1,\dots,k_2-1 \\ i=6 \end{array}
\end{cases}
\qquad \ldots(8.3.13)
$$

Substituting for $K(d_o,d_1,d_2)$ and $K_j(d_o,d_1,d_2)$ from (8.3.10) through (8.3.13) in the definition given by (8.3.1) we get the unbiased estimators $\overline{p}_j$(fully), j=0,1,2 as given below :

$$\overline{p}_o(\text{fully})=\begin{cases}(d_o-1)/(y-1) & y=g_1,\ldots,n \\ & d_o=g_o,\ldots,g_1 \\ d_o/(y-1) & y=g_1,\ldots,n \\ & d_o=g_o,\ldots,g_1-1 \\ (g_o-1)/(y-1) & y=g_1,\ldots,n \\ & d_1=k_1-k_2+1,\ldots,n \\ (y-k_2-d_1)/(y-1) & y=k_2,k_2+1,\ldots,n \\ & d_1=0,1,\ldots,k_1-k_2 \\ (y-k_1)/(y-1) & y=k_1,\ldots,n \\ & d_2=1,2,\ldots,k_2-1 \\ (y-k_1)/(y-1) & y=k_1,\ldots,n \\ & d_2=0,1,\ldots,k_2-1\end{cases}$$

$$
\overline{p}_1(\text{fully}) = \begin{cases}
(g_1-d_o)/(y-1) & y=g_1,\ldots,n \\
 & d_o=g_o,\ldots,g_1 \\
(g_1-d_o-1)/(y-1) & y=g_1,\ldots,n \\
 & d_o=g_o,\ldots,g_1-1 \\
d_1/(y-1) & y=g_1,\ldots,n \\
 & d_1=k_1-k_2+1,\ldots,n \\
d_1/(y-1) & y=k_2,\ldots,n \\
 & d_1=0,1,\ldots,k_1-k_2 \\
(k_1-d_2)/(y-1) & y=k_1,\ldots,n \\
 & d_2=1,2,\ldots,k_2-1 \\
(k_1-d_2-1)/(y-1) & y=k_1,\ldots,n \\
 & d_2=0,1,\ldots,k_2-1
\end{cases}
$$

$$
\overline{p}_2(\text{fully}) = \begin{cases}
(y-g_1)/(y-1) & y=g_1,\ldots,n \\
 & d_o=g_o,\ldots,g_1 \\
(y-g_1)/(y-1) & y=g_1,\ldots,n \\
 & d_o=g_o,\ldots,g_1-1 \\
(y-g_o-d_1)/(y-1) & y=g_1,\ldots,n \\
 & d_1=k_1-k_2+1,\ldots,n \\
(k_2-1)/(y-1) & y=k_2,\ldots,n \\
 & d_1=0,1,\ldots,k_1-k_2 \\
(d_2-1)/(y-1) & y=k_1,\ldots,n \\
 & d_2=1,2,\ldots,k_2-1 \\
d_2/(y-1) & y=k_1,\ldots,n \\
 & d_2=0,1,\ldots,k_2-1
\end{cases}
$$

From the formulas given above it is observed that the estimate of the proportion of bad units, for instance, in the case of fully-curtailed single sampling plan takes one of the following forms

$$\frac{\text{(Number of bad units observed)}}{\text{(number of units inspected - 1)}}$$

or

$$\frac{\text{(Number of bad units observed - 1)}}{\text{(Number of units inspected - 1)}} .$$