# CHAPTER IX

## DETERMINATION OF SINGLE THREE CLASS ATTRIBUTES SAMPLING PLAN BASED UPON A LINEAR COST MODEL AND A PRIOR DISTRIBUTION

9.1 In this chapter we have given the techniques for the determination of a single three class attributes sampling plan based upon a linear cost model and a prior distribution. Three prior distributions are considered. The development of the techniques is similar to that given by Guenther [15] who considered the problem of the determination of a single two class attributes sampling plan.

9.2 The full significance of a sampling plan can only be developed on the basis of the prior distribution and the economic consequences of the decision of acceptance and rejection of a lot. In recent years a number of papers have appeared concerning two class attributes sampling inspection models which are constructed incorporating both costs and a prior distribution of the process fraction defective.

The costs associated with the decision of acceptance and rejection of a lot are more real to the firm and the corresponding choice is easier to make than the choice of risk points and risks. This is mainly because the various decision costs are closer to the type of data that firm can supply on a rational basis than are the various risks and risk points. Tippett [46, pp.146] has pointed out that, in practice the sampling plan based on even the rough estimates of costs can be found quite satisfactory. During fifties, based on various decision costs some valuable work on the method of determining inspection sampling plan on the economic basis was done. Some of them are by Anscombe [1], Hamaker [26], Weibull [47], Pandey [36] and many others cited by Pandey himself in [36].

To evaluate the minimum expected value of the associated cost function, it is required to consider some prior distribution of the quality of a lot. Barnard [2] pointed out the close correspondence between the theory of statistical decisions and the theory of sampling inspection. He established an important result which shows that, not only must we know the prior distribution in order to solve a decision problem, but we may have to know it in considerable detail. Assuming prior distribution for the lot quality several other papers on Bayesian sampling inspection such as Guthrie and Johns [18], Wetherill [48], Pfanzagl [39], Hald [19],[21],[22],[23],[24], Johansen [27], Hald and Thyregod [25], Thyregod [45], Guenther [15] have appeared. The sampling plans discussed in these papers are two class attributes sampling plans and are based on two decision criteria, i.e., they are either acceptance-rectification or acceptance-rejection sampling plans. Pandey [36] has discussed Bayesian single two class attributes sampling plan with three decision criteria for discrete prior distribution.

In the following sections of this chapter we have considered the problem for the determination of a single three class attributes sampling plan (with two decision criteria) based on a linear cost model and a prior distribution. Three bivariate distributions for a lot quality given in terms of $p_1$ and $p_2$ are considered as prior distributions. They are the bivariate degenerate, the bivariate two point, and the bivariate beta distributions. The linear cost model formulized by Hald [19], [20], [22], is modified for the single three class attributes sampling plan. The expressions for the expected value of the cost function based on this linear cost model and the above prior distributions are obtained. Determination of a sampling plan under two side conditions on AOC and under a prior bivariate degenerate distribution is illustrated numerically.

## 9.3 Terminology, Assumptions and Some Results :

### 9.3.1 Single Three Class Attributes Sampling Plan :

Single three class attributes sampling plan (curtailed as well as uncurtailed) is discussed in Chapter V. We rewrite here the statement of the uncurtailed single three class attributes sampling plan. Assume that units produced by a process are assembled at random into lots of size N. From each lot a random sample of size n is selected. During the inspection of a sample, each unit is classified as either bad, marginal, or good. Let $D_1$ be the number of marginal units and $D_2$ be the number of bad units observed during the inspection of a sample of size n. The decision rule is then to accept a lot when $d_1+d_2 \leq a_1$ and $d_2 \leq a_2$ and to reject a lot otherwise. Here $a_1$ and $a_2$ are called the acceptance numbers.

### 9.3.2 Assumptions and Some Results

Assume that, when lots are accepted, all marginal units and bad units observed during the sampling inspection are replaced by good units. Furthermore, rejected lots are inspected completely and marginal and bad units found during the screening are replaced by good units. This can be considered as an obvious extension of acceptance-rectification criterion commonly used in two-class attributes sampling plan.

Let

$p_1$ = proportion of marginal units in the production run.

$p_2$ = proportion of bad units in the production run.

$p_o$ = proportion of good units in the production run

$= 1-p_1-p_2$.

$Y_1$ = number of marginal units in a lot.

$Y_2$ = number of bad units in a lot.

We assume the following marginal and joint probability functions of $Y_1$ and $Y_2$.

$$P(Y_1=y_1) = \binom{N}{y_1} p_1^{y_1}(1-p_1)^{N-y_1} \qquad y_1=0,1,\ldots,N \qquad \ldots(9.3.1)$$

$$P(Y_2=y_2) = \binom{N}{y_2} p_2^{y_2}(1-p_2)^{N-y_2} \qquad y_2=0,1,\ldots,N \qquad \ldots(9.3.2)$$

and
$$P(Y_1=y_1, Y_2=y_2) = \frac{N!\, p_o^{N-y_1-y_2}\, p_1^{y_1}\, p_2^{y_2}}{y_1!\, y_2!\, (N-y_1-y_2)!} \qquad y_1=0,1,\ldots,N;\ y_2=0,1,\ldots,N;\ y_1+y_2 \le N. \qquad \ldots(9.3.3)$$

For a given lot $Y_1=m_1$ and $Y_2=m_2$ are fixed. It follows that the joint probability function of $D_1$ and $D_2$ for given $Y_1=m_1$.

and $Y_2=m_2$ is the bivariate hypergeometric distribution. Hence

$$P(D_1=d_1,\ D_2=d_2 / Y_1=m_1,\ Y_2=m_2)$$

$$= \frac{\binom{m_1}{d_1}\binom{m_2}{d_2}\binom{N-m_1-m_2}{n-d_1-d_2}}{\binom{N}{n}} \qquad \begin{array}{l} d_1 = 0,1,\ldots,n, \\ d_2 = 0,1,\ldots,n, \\ d_1+d_2 \le n. \end{array} \qquad \ldots(9.3.4)$$

Define

$U_1 = Y_1-D_1$

= the number of marginal units in the remaining $N-n$ units in a lot

$U_2 = Y_2-D_2$

= the number of bad units in the remaining $N-n$ units in a lot.

Consider the unconditional events $U=\{U_1=u_1, U_2=u_2\}$ and $D=\{D_1=d_1,\ D_2=d_2\}$ .

Result : The unconditional events U and D are independently distributed.

Proof : $P[(D_1=d_1, D_2=d_2),\ (U_1=u_1, U_2=u_2)]$

$= P[(D_1=d_1,\ D_2=d_2),\ (Y_1=u_1+d_1,\ Y_2=u_2+d_2)]$

$= P(Y_1=u_1+d_1,\ Y_2=u_2+d_2)\ P[(D_1=d_1, D_2=d_2) / (Y_1=y_1, Y_2=y_2)]$

Using (9.3.3) and (9.3.4), we get

$$P\left[(D_1=d_1, D_2=d_2), (U_1=u_1, U_2=u_2)\right] = P(D_1=d_1, D_2=d_2)\ P(U_1=u_1, U_2=u_2) \quad \ldots(9.3.5)$$

where $P(D_1=d_1,\ D_2=d_2) = \dfrac{n!\, p_0^{\,n-d_1-d_2}\ p_1^{\,d_1}\ p_2^{\,d_2}}{d_1!\, d_2!\, (n-d_1-d_2)!}$

and

$$P(U_1=u_1,\ U_2=u_2) = \frac{(N-n)!\ p_0^{\,N-n-u_1-u_2}\ p_1^{\,u_1}\ p_2^{\,u_2}}{u_1!\ u_2!\ (N-n-u_1-u_2)!}$$

Hence the result.

From this result it follows that

$D_1$ is binomial with parameters n and $p_1$,

$D_2$ is binomial with parametsrs n and $p_2$,

$U_1$ is binomial with parameters N-n and $p_1$,

$U_2$ is binomial with parameters N-n and $p_2$.

9.3.3 The Probability of Acceptance (Operating Characteristic) :

$$OC = \sum_{d_1=o}^{a_1-d_2} \sum_{d_2=o}^{a_2} \binom{m_1}{d_1}\binom{m_2}{d_2}\binom{N-m_1-m_2}{n-d_1-d_2}\Big/\binom{N}{n} \quad \ldots(9.3.6)$$

$$= \sum_{d_2=o}^{a_2} p(N,n,m_2,d_2)\ P(N-m_2, n-d_2, m_1, a_1-d_2) \quad \ldots(9.3.7)$$

where $p(N,n,M,x) = \binom{M}{x}\binom{N-M}{n-x}\Big/\binom{N}{n}, \quad a \le x \le b$ ...(9.3.8)

$a=\max\left[0,\ n-N+M\right]$ , $b=\min[M,n]$

and $P(N,n,M,r) = \sum_{x=a}^{r} p(N,n,M,x)$ ...(9.3.9)

Since (9.3.7) is a conditional probability depending on $m_1$ and $m_2$, we may be more interested in the average operating characteristic (AOC) which is the expected value of (9.3.7) taken over $Y_1$ and $Y_2$. The AOC is given by

$$\text{AOC} = \sum_{y_1=o}^{N-y_2} \sum_{y_2=o}^{N} \left[ \sum_{d_1=o}^{a_1-d_2} \sum_{d_2=o}^{a_2} \binom{y_1}{d_1}\binom{y_2}{d_2}\binom{N-y_1-y_2}{n-d_1-d_2} \Big/ \binom{N}{n} \right] \cdot \frac{N!\; p_o^{N-y_1-y_2}\; p_1^{y_1}\; p_2^{y_2}}{y_1!\; y_2!(N-y_1-y_2)!}$$

$$= \sum_{d_1=o}^{a_1-d_2} \sum_{d_2=o}^{a_2} \frac{n!\; p_o^{n-d_1-d_2}\; p_1^{d_1}\; p_2^{d_2}}{d_1!\; d_2!(n-d_1-d_2)!}$$

$$= \sum_{d_2=o}^{a_2} b(d_2;n,p_2)\; B(a_1-d_2;\; n-d_2,p_1') \qquad \ldots(93.10)$$

where $p_1' = p_1/(p_1+p_o)$

## 9.4. The Linear Cost Model :

Consider the following costs associated with a three class attributes sampling plan :

$S_o$ = Cost per unit of sampling and testing.

$S_1$ = Repair cost of a marginal unit found in sampling.

$S_2$ = Repair cost of a bad unit found in sampling.

$A_o$ = Cost per unit associated with handling N-n units not inspected in an accepted lot (frequently it is assumed to be zero).

$A_1$ = Cost associated with a marginal unit in N-n units not inspected in an accepted lot.

$A_2$ = Cost associated with a bad unit in N-n units not inspected in an accepted lot.

$R_o$ = Cost per unit of inspecting the remaining N-n units in a rejected lot.

$R_1$ = Repair cost associated with a marginal unit in the remaining N-n units of a rejected lot.

$R_2$ = Repair cost associated with a bad unit in the remaining N-n units of a rejected lot.

We assume that $S_o \geq R_o$, $S_1 \geq R_1$, $S_2 \geq R_2$, $S_1 \leq S_2$, and $R_1 \leq R_2$. It may be noted that equality sign holds frequently in all the cases. Furthermore, assume that $A_1$ and $A_2$ are very large quantities. This assumption implies that the occurrence of a bad or a marginal unit in the remaining units of an accepted lot is a very costly affair.

The linear cost function is then given as below ;

$$H(d_1, d_2, m_1, m_2, p_1, p_2;\ N, n, a_1, a_2)$$

$$= \begin{cases} nS_o + d_1S_1 + d_2S_2 + (N-n)A_o + (m_1 - d_1)A_1 + (m_2 - d_2)A_2 \\ \qquad\qquad \text{for } d_1 + d_2 \leq a_1;\ d_2 \leq a_2 \\ nS_o + d_1S_1 + d_2S_2 + (N-n)R_o + (m_1 - d_1)R_1 + (m_2 - d_2)R_2 \text{ otherwise.} \end{cases}$$

$$= \begin{cases} nS_o+d_1S_1+d_2S_2+(N-n)A_o+u_1A_1+u_2A_2 & \text{for } d_1+d_2 \leq a_1; d_2 \leq a_2 \\ nS_o+d_1S_1+d_2S_2+(N-n)R_o+u_1R_1+u_2R_2 & \text{otherwise} \end{cases} \quad \ldots(9.4.1)$$

Generally, the sampling plans are based upon the average cost per lot. For this purpose one should take the expected value of the random variables involved in (9.4.1). Thus the average cost is

$K(N,n,a_1,a_2,p_1,p_2)$

$$= \begin{cases} nS_o+S_1np_1+S_2np_2+(N-n)A_o+(N-n)p_1A_1+(N-n)p_2A_2 \\ \qquad\qquad \text{for } d_1+d_2 \leq a_1; \quad d_2 \leq a_2 \\ nS_o+S_1np_1+S_2np_2+(N-n)R_o+(N-n)p_1R_1+(N-n)p_2R_2 \quad \text{otherwise.} \end{cases}$$

$$= n\left[S_o+S_1p_1+S_2p_2\right]+(N-n)\left[A_o+p_1A_1+p_2A_2\right]P_a$$
$$+(N-n)\left[R_o+p_1R_1+p_2R_2\right]P_r$$

$$= nK_s(p_1,p_2)+(N-n)\left[K_a(p_1,p_2)P_a + K_r(p_1,p_2)\cdot P_r\right] \quad \ldots(9.4.2)$$

$$= nK_s(p_1,p_2)+(N-n)\left[K_a(p_1,p_2)+\{K_r(p_1,p_2)-K_a(p_1,p_2)\}P_r\right] \quad \ldots(9.4.3)$$

where $K_s(p_1,p_2) = S_o+S_1p_1+S_2p_2$

$K_a(p_1,p_2) = A_o+A_1p_1+A_2p_2$,

and $K_r(p_1,p_2) = R_o+R_1p_1+R_2p_2$

are nonnegative linear functions of $p_1$ and $p_2$. $P_a$ is given by either expression (9.3.7) or (9.3.10) and $P_r=1-P_a$. Since $K_S(p_1,p_2)$, $K_r(p_1,p_2)$, and $K_a(p_1,p_2)$ are nonnegative linear functions of $p_1$, and $p_2$, expression (9.4.3) will be minimum provided

$$K_r(p_1,p_2) - K_a(p_1,p_2) = 0 \qquad \text{...(9.4.4)}$$

i.e. $$(R_o-A_o)-(A_1-R_1)p_1-(A_2-R_2)p_2 = 0 \qquad \text{...(9.4.5)}$$

Let $(p_{1r}, p_{2r})$ be the solution of the equation (9.4.4) or (9.4.5), $0 < p_{1r}, p_{2r} < 1$. Then if

$$p_1 < p_{1r} \text{ and } p_2 < p_{2r}, \quad K_r(p_1,p_2) - K_a(p_1,p_2) > 0$$

and $$p_1 > p_{1r} \text{ and } p_2 > p_{2r}, \quad K_r(p_1,p_2)-K_a(p_1,p_2) < 0. \qquad \text{...(9.4.6)}$$

## 9.5 Bayesian Sampling Plan when a Prior Distribution of the Lot Quality is Bivariate Degenerate :

The problem is to determine a sampling plan which minimizes the expected value of the cost function given by either (9.4.2) or (9.4.3). The expected value of the cost function is obtained under the given prior distribution of the lot quality. We first consider the case in which the distribution of $(p_1,p_2)$ is degenerate. This means that the whole mass of the distribution is concentrated at a single point.

If we knew $p_1$ and $p_2$, say, $p_1=p_{10}$ and $p_2=p_{20}$ then we could always minimize (9.4.2). If $p_{10}< p_{1r}$ and $p_{20}<p_{2r}$, then we would always accept the lot without sampling. In this case $P_a=1$ and $n=0$, hence

$$K(N,n,a_1,a_2,\ p_{10},p_{20}) = NK_a(p_{10},\ p_{20}) \qquad \ldots(9.5.1)$$

If $p_{10}>p_{1r}$ and $p_{20}>p_{2r}$, then we would always reject the lot without sampling. In this case $P_r=1$ and $n=0$. Hence

$$K(N,n,a_1,a_2,\ p_{10},\ p_{20}) = NK_r(p_{10},p_{20}) \qquad \ldots(9.5.2)$$

In practice $p_1$ and $p_2$ are unknown. Then the problem is to minimize $K(N,n,a_1,a_2,\ \bar{p}_1,\bar{p}_2)$, where $\bar{p}_1$ and $\bar{p}_2$ are the guess values for the true values of $p_1$ and $p_2$ respectively, subject to one or two conditions on the OC (or AOC) curve given below :

(a) OC (or AOC)$\leq\beta$ , ...(9.5.3)

(b) OC (or AOC)$\geq 1-\alpha$ , ...(9.5.4)

(c) OC (or AOC)$\leq\beta$ and OC (or AOC)$\geq 1-\alpha$ ...(9.5.5)

where $\alpha$ and $\beta$ are producer's and consumer's risks respectively. Rewrite the expression (9.4.3) as given below :

$$K(N,n,a_1,a_2,p_1,p_2)=n\left[K_s(p_1,p_2)-K_r(p_1,p_2)\right]+(N-n)\left[K_a(p_1,p_2)-K_r(p_1,p_2)\right]\ P_a+NK_r(p_1,p_2) \quad \ldots(9.5.6)$$

Then the expected value of the cost function under given bivariate degenerate prior distribution is

$$K(N,n,a_1,a_2,\overline{p}_1,\overline{p}_2)$$
$$=n\left[K_s(\overline{p}_1,\overline{p}_2) - K_r(\overline{p}_1,\overline{p}_2)\right]+ (N-n)\left[K_a(\overline{p}_1,\overline{p}_2)-K_r(\overline{p}_1,\overline{p}_2)\right]$$
$$\cdot P_a(\overline{p}_1, \overline{p}_2) + NK_r(\overline{p}_1,\overline{p}_2) \quad \ldots(9.5.7)$$

where $P_a(\overline{p}_1,\overline{p}_2)$ is obtained from (9.3.10) by substituting for $p_1=\overline{p}_1$ and $p_2=\overline{p}_2$.

Consider the expected value of the cost function under the following two cases :

Case-I : $R_o=S_o$, $R_1=S_1$, $R_2=S_2$

Under this case $K_s(\overline{p}_1,\overline{p}_2) - K_r(\overline{p}_1,\overline{p}_2)=0$, hence the expected value of the cost function is

$$\therefore K(N,n,a_1,a_2,\overline{p}_1,\overline{p}_2) =(N-n)\left[K_a(\overline{p}_1,\overline{p}_2)-K_r(\overline{p}_1,\overline{p}_2)\right] P_a(\overline{p}_1,\overline{p}_2) + NK_r(\overline{p}_1,\overline{p}_2) \quad \ldots(9.5.8.)$$

$$\therefore R(N,n,a_1,a_2,\overline{p}_1,\overline{p}_2) = (N-n)P_a(\overline{p}_1,\overline{p}_2) \quad \ldots(9.5.9)$$

where $R(N,n,a_1,a_2,\overline{p}_1,\overline{p}_2) = \dfrac{K(N,n,a_1,a_2),\overline{p}_1,\overline{p}_2)-NK_r(\overline{p}_1,\overline{p}_2)}{K_a(\overline{p}_1,\overline{p}_2)-K_r(\overline{p}_1,\overline{p}_2)}$

To minimize (9.5.8) one has to maximize or minimize (9.5.9) depending on whether the quantity $K_a(\bar{p}_1,\bar{p}_2)-K_r(\bar{p}_1,\bar{p}_2)$ is negative or positive.

Case-II : $R_o < S_o$, $R_1 < S_1$, $R_2 < S_2$ (at least one of these inequalities should be true) :

In this case $K_s(\bar{p}_1,\bar{p}_2)-K_r(\bar{p}_1,\bar{p}_2)$ is always positive. Then, the function to be minimized is

$$R_1'(N,n,a_1,a_2,\bar{p}_1,\bar{p}_2) = n+(N-n)\gamma P_a(\bar{p}_1,\bar{p}_2) \qquad \ldots(9.5.10)$$

where $R_1'(N,n,a_1,a_2,\bar{p}_1,\bar{p}_2) = \dfrac{K(N,n,a_1,a_2,\bar{p}_1,\bar{p}_2)-NK_r(\bar{p}_1,\bar{p}_2)}{K_s(\bar{p}_1,\bar{p}_2) - K_r(\bar{p}_1,\bar{p}_2)}$

and $\gamma=[K_a(\bar{p}_1,\bar{p}_2)-K_r(\bar{p}_1,\bar{p}_2)]/[K_s(\bar{p}_1,\bar{p}_2)-K_r(\bar{p}_1,\bar{p}_2)]$

## 9.6 Expected Cost Function when a Prior Distribution of the Lot Quality is Bivariate Two Point Distribution :

### 9.6.1 The Two Point Prior Distribution :

The probability function of the bivariate two-point prior distribution for the lot quality is given below :

$$f(p_1,p_2) = \begin{cases} w_1 & (p_1,p_2) = (p_{10},p_{20}) \\ w_2 & (p_1,p_2) = (p_{11},p_{21}) \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(9.6.1)$$

where $w_1, w_2, (p_{10}, p_{20})$ and $(p_{11}, p_{21})$ are assumed to be known. The entire probability distribution is concentrated on two points $(p_{10}, p_{20})$ and $(p_{11}, p_{21})$. Alternatively the above probability distribution can be represented as follows :

| $p_2 \setminus p_1$ | $p_{10}$ | $p_{11}$ | Total |
|---|---|---|---|
| $p_{20}$ | $w_1$ | 0 | $w_1$ |
| $p_{21}$ | 0 | $w_2$ | $w_2$ |
| Total | $w_1$ | $w_2$ | 1 |

...(9.6.2)

From (9.6.2) it follows that the marginal distributions of $p_1$ and $p_2$ are as given below :

(i) Marginal Distribution of $p_1$ :

$$\begin{aligned} f_1(p_1) &= w_1 \quad & p_1 = p_{10} \\ &= w_2 & p_1 = p_{11} \\ &= 0 & \text{elsewhere} \end{aligned} \qquad \ldots(9.6.3)$$

$$\begin{aligned} f_2(p_2) &= w_1 \quad & p_2 = p_{20} \\ &= w_2 & p_2 = p_{21} \\ &= 0 & \text{elsewhere} \end{aligned} \qquad \ldots(9.6.4)$$

Hald's justification for the use of this type of model appears to be that those associated with quality control can frequently identify good quality (with quality level $(p_{10}, p_{20})$) and poor quality (with quality level $(p_{11}, p_{21})$) together with $w_1$ and $w_2$ even if they cannot furnish any additional information.

### 9.6.2 Expected Value of the Cost Function :

Consider the cost function given by (9.4.2) in the following form :

$$K(N,n,a_1,a_2,p_1,p_2) = nK_s(p_1,p_2)+(N-n)\left[\{K_a(p_1,p_2)-K_s(p_1,p_2)\}P_a + K_r(p_1,p_2)\right] \quad \ldots(9.6.5)$$

To find the expected value of (9.6.5) with respect to the given prior distribution (9.6.1), we require the following expectations :

$$E\left[K_s(p_1,p_2)\right] = w_1 \; K_s(p_{10},p_{20})+ w_2K_s(p_{11},p_{21}) = K_s \quad \ldots(9.6.6)$$

$$E\left[K_r(p_1,p_2)\right] = w_1K_r(p_{10},p_{20})+ w_2K_r(p_{11},p_{21}) \quad \ldots(9.6.7)$$

$$E(P_a)= w_1P_a(p_{10},p_{20})+ w_2P_a(p_{11},p_{21}) \quad \ldots (9.6.8)$$

$$E\left[p_1 \; P_a\right] = w_1p_{10}P_a(p_{10},p_{20})+ w_2p_{11}P_a(p_{11},p_{21}) \quad \ldots(9.6.9)$$

$$E\left[p_2P_a\right]=w_1p_{20}P_a(p_{10},p_{20}) +w_2p_{21}P_a(p_{11},p_{21}) \quad \ldots(9.6.10)$$

where $P_a = P_a(p_1, p_2)$

$= $ Probability of acceptance at point $(p_1, p_2)$ and is given by the expression (9.3.10)

$K_s(p_{10}, p_{20})$ = the value of $K_s(p_1, p_2)$ at point $(p_{10}, p_{20})$

$K_s(p_{11}, p_{21})$ = the value of $K_s(p_1, p_2)$ at point $(p_{11}, p_{21})$

$K_{\ell}(p_{10}, p_{20})$ = the value of $K_r(p_1, p_2)$ at point $(p_{10}, p_{20})$.

$K_r(p_{11}, p_{21})$ = the value of $K_r(p_1, p_2)$ at point $(p_{11}, p_{21})$.

Using these expectations the expected value of the cost function which is to be minimized is

$$\begin{aligned} K(N, n, a_1, a_2) = nK_s + (N-n) \{ & w_1K_r(p_{10}, p_{20}) - w_1K_a(p_{10}, p_{20}) \\ & - w_1K_r(p_{10}, p_{20})P_a(p_{10}, p_{20}) + w_1K_a(p_{10}, p_{20}) \\ & \cdot P_a(p_{10}, p_{20}) + w_2K_a(p_{11}, p_{21})P_a(p_{11}, p_{21}) \\ & - w_2K_r(p_{11}, p_{21})\, P_a(p_{11}, p_{21}) + K_m \} \end{aligned} \qquad \ldots(9.6.11)$$

where $K_m = w_1K_a(p_{10}, p_{20}) + w_2K_r(p_{11}, p_{21})$.

$K_a(p_{10},p_{20})$ = the value of $K_a(p_1,p_2)$ at point $(p_{10},p_{20})$.

$K_a(p_{11},p_{21})$ = the value of $K_a(p_1,p_2)$ at point $(p_{11},p_{21})$

Alternative form of the function which is to be minimized is given below :

$$R_2'(N,n,a_1,a_2) = n+(N-n)\left\{\gamma_1 P_r(p_{10},p_{20})+\gamma_2 P_a(p_{11},p_{21})\right\} \quad \ldots(9.6.12)$$

where $\gamma_1 = w_1\left[K_r(p_{10},p_{20}) - K_a(p_{10},p_{20})\right]/(K_s-K_m)$.

$\gamma_2 = w_2\left[K_a(p_{11},p_{21}) - K_r(p_{11},p_{21})\right]/(K_s-K_m)$.

$P_r(p_1,p_2) = 1-P_a(p_1,p_2)$

= Probability of rejection at point $(p_1,p_2)$

## 9.7 Expected Cost Function when a Prior Distribution of the Lot Quality is Bivariate Beta Distribution :

### 9.7.1 The Bivariate Beta Distribution :

Consider the following distribution as a prior distribution for the lot quality :

$$g(p_1,p_2) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma\alpha_1\,\Gamma\alpha_2\,\Gamma\alpha_3}\; p_1^{\alpha_1-1}\; p_2^{\alpha_2-1}\; p_3^{\alpha_3-1} \quad \ldots(9.7.1)$$

where (i) $0 \leq p_i \leq 1, \quad \sum_{i=1}^{3} p_i=1.$

(ii) $p_3 = p_o$ defined in Section 9.3.2.

(iii) $\alpha_i$ are positive and real.

The distribution is known as bivariate beta distribution (or Dirichlet distribution). The means, variances and covariances of the distribution are given by the following expressions :

$$E(p_i) = \alpha_i/\alpha_o \quad \text{for } i=1,2,3 \qquad \ldots(9.7.2)$$

$$\text{var}\ (p_i) = \alpha_i(\alpha_o-\alpha_i)/[\alpha_o^2\ (\alpha_o+1)] \quad \text{for } i=1,2,3 \qquad \ldots(9.7.3)$$

and $$\text{cov}\ (p_i,p_j) = \alpha_i\,\alpha_j/[\alpha_o^2\ (\alpha_o+1)] \quad \text{for } i\neq j=1,2,3 \qquad \ldots(9.7.4)$$

where $$\alpha_o=\alpha_1+\alpha_2+\alpha_3 \qquad \ldots(9.7.5)$$

For the further developments of the results in the following section we assume that $\alpha_i(i=1,2,3)$ are integers.

9.7.2 Expected Value of the Cost Function :

Consider the linear cost function in the form given by (9.6.5). To find the expected value of the linear cost function with respect to the prior distribution given by (9.7.1) following expectations are found useful :

$$E\left[K_s(p_1,p_2)\right] = (S_o\alpha_o + S_1\alpha_1 + S_2\alpha_2)/\alpha_o \qquad \ldots(9.7.6)$$

$$E\left[K_r(p_1,p_2)\right] = (R_o\alpha_o + R_1\alpha_1 + R_2\alpha_2)/\alpha_o \qquad \ldots(9.7.7)$$

$$E\left[P_a(p_1,p_2)\right] = \int_0^{1-p_1}\int_0^1 P_a(p_1,p_2)\, g(p_1,p_1)\, dp_1 dp_2$$

$$= \sum_{d_2=o}^{a_2}\sum_{d_1=o}^{a_1-d_2} \frac{\binom{\alpha_1+d_1-1}{\alpha_1-1}\binom{\alpha_2+d_2-1}{\alpha_2-1}\binom{n+\alpha_3-d_1-d_2-1}{\alpha_3-1}}{\binom{n+\alpha_o-1}{\alpha_o-1}} \qquad \ldots(9.7.8)$$

Consider the following result which gives the relation between the hypergeometric distribution and the inverse hypergeometric distribution.

$$P^*(N,M,k,r) = 1-P(N,r,M,k-1) \qquad \ldots(9.7.9)$$

where (i) $P^*(N,M,k,r) = \sum_{x=k}^{r} \binom{x-1}{k-1}\binom{N-x}{M-k}/\binom{N}{M}$

(ii) $P(N,r,M,k-1) = \sum_{x=o}^{k-1} \binom{M}{x}\binom{N-M}{r-x}/\binom{N}{r}$

Making use of this result, (9.7.8) can be expressed as

$$E\left[P_a(p_1,p_2)\right] = \sum_{d_2=o}^{a_2}\left\{P(n+\alpha_o-1,\ \alpha_2+d_2-1,\ \alpha_o-1,\alpha_2-1)-P(n+\alpha_o-1,\alpha_2+d_2,\ \alpha_o-1,\alpha_2-1)\right\}\left\{1-P(n+\alpha_1+\alpha_3-d_2-1,\alpha_1+a_1-d_2,\alpha_1+\alpha_3-1,\alpha_1-1)\right\}$$

$$\ldots(9.7.10)$$

$$E\left[p_1 P_a(p_1,p_2)\right] = \int_0^{1-p_1} \int_0^1 p_1 P_a(p_1,p_2)\, g(p_1,p_1)\, dp_1\, dp_2$$

$$= \frac{\alpha_1}{\alpha_0} \sum_{d_2=o}^{a_2} \sum_{d_1=o}^{a_1-d_2} \frac{\binom{\alpha_1+d_1}{\alpha_1}\binom{\alpha_2+d_2-1}{\alpha_2-1}\binom{n+\alpha_3-d_1-d_2-1}{\alpha_3-1}}{\binom{n+\alpha_0}{\alpha_0}} \quad \ldots(9.7.11)$$

Again making use of the relation between hypergeometric and inverse hypergeometric given by (9.7.9) we get

$$E\left[p_1 P_a(p_1,p_2)\right]$$

$$= \frac{\alpha_1}{\alpha_0} \sum_{d_2=o}^{a_2} \left\{P(n+\alpha_o,\alpha_2+d_2-1,\alpha_o,\alpha_2-1)-P(n+\alpha_o,\alpha_2+d_2,\alpha_o,\alpha_2-1)\right\}\left\{1-P(n+\alpha_3+\alpha_1-d_2,\alpha_1+1+a_1-d_2,\alpha_1+\alpha_3,\alpha_1)\right\} \quad \ldots(9.7.12)$$

$$E\left[p_2 P_a(p_1,p_2)\right]$$

$$= \int_0^{1-p_1} \int_0^1 p_2 P_a(p_1,p_2)\, g(p_1,p_2)\, dp_1\, dp_2$$

$$= \frac{\alpha_2}{\alpha_0} \sum_{d_2=o}^{a_2} \sum_{d_1=o}^{a_1-d_2} \frac{\binom{\alpha_1+d_1-1}{\alpha_1-1}\binom{\alpha_2+d_2}{\alpha_2}\binom{n+\alpha_3-d_1-d_2-1}{\alpha_3-1}}{\binom{n+\alpha_0}{\alpha_0}} \quad \ldots(9.7.13)$$

Making use of result (9.7.9) we get

$$E\left[p_2 P_a(p_1,p_2)\right]$$

$$= \frac{\alpha_2}{\alpha_0} \sum_{d_2=o}^{a_2} \left\{P(n+\alpha_o,\alpha_2+d_2,\alpha_o,\alpha_2)-P(n+\alpha_o,\alpha_2+d_2+1,\alpha_o,\alpha_2)\right\} \cdot \left\{1-P(n+\alpha_1+\alpha_3-d_2-1,\alpha_1+a_1-d_2,\alpha_1+\alpha_3-1,\alpha_1-1)\right\} \quad \ldots(9.7.14)$$

Expressions (9.7.10), (9.7.12), and (9.7.14) can be evaluated using hypergeometric table given by Lieberman and Owen [30] .

Now define

$$R'(N,n,a_1,a_2) = K(N,n,a_1,a_2) - NK_m/(K_s-K_m) \quad \ldots(9.7.15)$$

where $K_s = (\alpha_o S_o+\alpha_1 S_1+\alpha_2 S_2)/\alpha_o$

$K_m = (\alpha_o A_o+\alpha_1 R_1+\alpha_2 R_2)/\alpha_o$ .

Then using the results of expectations, namely, (9.7.6), (9.7.7), (9.7.10), (9.7.12), (9.7.14), we get

$$R'(N,n,a_1,a_2) = n+(N-n)\ (\gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3) \quad \ldots(9.7.16)$$

where $\gamma_1 = (R_o-A_o)/[\alpha_o(K_s-K_m)]$

$\gamma_2 = [(A_1-R_1)\alpha_1] / [\alpha_o(K_s-K_m)]$

$\gamma_3 = [(A_2-R_2)\alpha_2] / [\alpha_o(K_s-K_m)]$

$$E_1 = \Big[1-\alpha_o \sum_{d_2=o}^{a_2} \{ P(n+\alpha_o-1,\alpha_2+d_2-1,\alpha_o-1,\alpha_2-1)-P(n+\alpha_o-1, \alpha_2+d_2, \alpha_o-1,\alpha_2-1)\} \{ 1-P(n+\alpha_1+\alpha_3-d_2-1,\alpha_1+a_1-d_2, \alpha_1+\alpha_3-1, \alpha_1-1)\}\Big]$$

$$E_2 = \sum_{d_2=o}^{a_2} \{ P(n+\alpha_o,\alpha_2+d_2-1,\alpha_o,\alpha_2-1)-P(n+\alpha_o,\alpha_2+d_2,\alpha_o,\alpha_2-1)\} \cdot\{1-P(n+\alpha_3+\alpha_1-d_2,\alpha_1+1+a_1-d_2,\alpha_1+\alpha_3,\alpha_1)\}$$

$$E_3 = \sum_{d_2=o}^{a_2} \{P(n+\alpha_o, \alpha_2+d_2, \alpha_o, \alpha_2)-P(n+\alpha_o, \alpha_2+d_2+1, \alpha_o, \alpha_2)\}$$
$$\cdot \{1-P(n+\alpha_1+\alpha_3-d_2-1, \alpha_1+a_1-d_2, \alpha_1+\alpha_3-1, \alpha_1-1)\}$$

## 9.8 Determination of the Sampling Plan when Prior Distribution is Bivariate Degenerate :

In Sections 9.5, 9.6, 9.7 we have obtained the expressions for the expected cost functions under three prior distributions of the lot quality. Here, the problem is to determine a sampling plan which minimizes the expected cost when a particular prior distribution is given. If a triplet $(n, a_1, a_2)$ is determined such that the expected value of the cost function is minimum, the sampling plan $(n, a_1, a_2)$ is called the unrestricted Bayesian sampling plan. When a triplet $(n, a_1, a_2)$ is determined such that the expected value of the cost function is minimized subject to the side conditions on either OC or AOC curve, then the sampling plan $(n, a_1, a_2)$ is called a restricted Bayesian sampling plan. Generally, three side conditions are used: (i) $P_a(p_{10}, p_{20}) \geq 1-\alpha$, (ii) $P_a(p_{11}, p_{21}) \leq \beta$, (iii) $P_a(p_{10}, p_{20}) \geq 1-\alpha$, and $P_a(p_{11}, p_{21}) \leq \beta$, where $(p_{10}, p_{20})$ is the good quality level and $(p_{11}, p_{21})$ is the bad quality level and $\alpha$ and $\beta$ are the producer's and consumer's risks respectively. In the following sections we consider the determination of the restricted Bayesian sampling plan when prior distribution is bivariate degenerate. The side conditions under which the sampling plan is determined are (i) $P_a(p_{10}, p_{20}) \geq 1-\alpha$ and (ii) $P_a(p_{11}, p_{21}) \leq \beta$ .

9.8.1 A Numerical Example :

To illustrate the determination of the restricted Bayesian sampling plan when the prior distribution is bivariate degenerate we consider the following example :

Lot size : N=100

Different costs associated with the linear cost model :

$A_o = 0.00$, $A_1 = 4.00$, $A_2 = 4.00$

$R_o = 0.10$, $R_1 = 2.00$, $R_2 = 2.00$

$S_o = 0.20$, $S_1 = 2.00$, $S_2 = 2.00$

Good quality level : $p_{10} = 0.15$, $p_{20} = 0.05$

Bad quality level : $p_{11} = 0.30$, $p_{21} = 0.10$

Guess Value : $\overline{p}_1 = 0.06$, $\overline{p}_2 = 0.02$

Consumer's risk : $\beta = 0.10$

Producer's risk : $\alpha = 0.05$

It follows that $K_s(\overline{p}_1, \overline{p}_2) \neq K_r(\overline{p}_1, \overline{p}_2)$. This leads to the Cases-II described in Section 9.5. Hence, the problem is to determine a sampling plan which satisfies the given side condition and minimizes the expression (9.5.10). We have determined the sampling plans under two side conditions (i) $P_a(p_{10},p_{20}) \geq 1-\alpha$ and (ii) $P_a(p_{11},p_{21}) \leq \beta$ .

<u>Sampling Plan under the Side Condition $P_a(p_{10},p_{20}) \geq 1-\alpha$</u> :

It may be noted that for fixed n and increasing $a_1$ or $a_2$ or both $(a_2 < a_1 < n) P_a$ will increase. This fact leads to the following practical procedure for the determination of the sampling plan under the side condition $P_a(p_{10},p_{20}) \geq 1-\alpha$:

For any fixed n, one may start with $a_2=0$, $a_1=1$ and continues to increase either $a_1$ or $a_2$ or both till the condition $P_a(p_{10},p_{20}) \geq 1-\alpha$ is satisfied. Once such pair is determined, [redacted] any further increase in either $a_1$ or $a_2$ or both will also satisfy the given side condition. Thus there will be number of pairs $(a_1,a_2)$ satisfying the side condition $P_a(p_{10},p_{20}) \geq 1-\alpha$ for fixed n. For all such pairs one finds the value of $R_1'(N,n,a_1,a_2,\bar{p}_1,\bar{p}_2)$ given by the expression (9.5.10) and selects the pair for which the value is minimum.

There are different minimum values for the expression (9.5.10) for different n. One should select the smallest value from all such minimum values. This smallest minimum value will determine the desired plan.

For the given example it is observed that for any fixed n, the pair $(a_1, a_2)$ which satisfies the given side condition for the first time (starting from $a_2=0$, $a_1=1$) will give the minimum value for $R_1'(N, n, a_1, a_2, \overline{p}_1, \overline{p}_2)$ given by (9.5.10). Table 9.1 gives an idea how the minimum values of (9.5.10) behave for some initial values of n :

Table 9.1

| n | $a_1$ | $a_2$ | Minimum Value of (9.5.10) |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 1 | 2 | 61.105957 |
| 4 | 1 | 2 | 61.372101 |
| 5 | 1 | 3 | 61.772949 |
| 6 | 1 | 3 | 62.056946 |
| 7 | 1 | 4 | 62.359512 |
| 8 | 2 | 4 | 63.169708 |
| 9 | 2 | 4 | 63.551193 |
| 10 | 2 | 4 | 63.925156 |

It is observed that the minimum value of (9.5.10) has an increasing pattern as n goes on increasing. Hence the plan which satisfies the given side condition $P_a(p_{10}, p_{20}) \geq 1-\alpha$ and minimizes the expected cost function is n=3, $a_1=2$, $a_2=1$.

The minimum value of the expected cost function, given by the expression (9.5.7), is 32.11057.

<u>Sampling Plan Under the Side Condition $P_a(p_{11},p_{21}) \le \beta$</u> :

It may be noted that for fixed $(a_1,a_2)$ and increasing $n(a_2 < a_1 < n)$ $P_a$ will decrease. This fact would lead to the following practical procedure for the determination of the sampling plan under the side condition $P_a(p_{11},p_{21}) \le \beta$ :

For any fixed $(a_1,a_2)$ (one may start with $a_2=0$, $a_1=1$) one continues to increase n till the condition $P_a(p_{11},p_{21}) \le \beta$ is satisfied.Once such n is determined, then for any further increment in n one will observe that $P_a(p_{11},p_{21})$ reduces. Therefore in this case also the condition $P_a(p_{11},p_{21}) \le \beta$ is satisfied. Thus there are different values of n satisfying the side condition $P_a(p_{11},p_{21}) \le \beta$ for fixed $(a_1,a_2)$. For all such values of n, one finds the value of $R_1'(N,n,a_1,a_2,\overline{p}_1,\overline{p}_2)$ given by the expression (9.5.10) and selects the n along with $(a_1,a_2)$ for which the value of $R_1'(N,n,a_1,a_1,\overline{p}_1,\overline{p}_2)$ is minimum.

Next one considers the variation in $a_1$ for fixed $a_2$ and then variation in $a_2$. These variations will give different minimum values for the expression (9.5.10). The smallest minimum value among all such minimum values will determine the desired plan.

Table 9.2 represents few triplets to give an idea about the behaviour of the minimum values of $R_1'(N,n,a_1,\bar{p}_1,\bar{p}_2)$

Table 9.2

| n | $a_1$ | $a_2$ | Minimum Value of (9.5.10) |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 26 | 1 | 0 | 39.694000 |
| 29 | 2 | 0 | 46.527679 |
| 28 | 3 | 0 | 50.350800 |
| ... | ... | ... | ... |
| 35 | 2 | 1 | 52.405609 |
| 39 | 3 | 1 | 59.842270 |
| 17 | 4 | 1 | 64.305847 |
| ... | ... | ... | ... |
| 41 | 3 | 2 | 61.484390 |
| 18 | 4 | 2 | 66.440491 |
| 21 | 5 | 2 | 67.823166 |
| ... | ... | ... | ... |
| 18 | 4 | 3 | 66.622391 |
| 21 | 5 | 3 | 68.140762 |
| 24 | 6 | 3 | 69.453888 |
| ... | ... | ... | ... |
| 21 | 5 | 4 | 68.163956 |
| 24 | 6 | 4 | 69.499420 |
| 27 | 7 | 4 | 70.752777 |

The explanation for the first row of Table 9.2 is as follows :

For fixed pair $a_2=0$, $a_1=1$, and for different values of n from 8 to 50 the side condition $P_a(p_{11},p_{21}) \leq \beta$ is satisfied. Among all these triplets, the expression (9.5.10) is minimum at $a_2=0$, $a_1=1$, n=26.

It is then observed that for fixed $a_2$ and changing $a_1$ the minimum value of the expression(9.5.10) increases. Similarly, for fixed $a_1$ and changing $a_2$ the minimum value of the expression (9.5.10) also increases. Thus, the sampling plan which satisfies the given side condition $P_a(p_{11},p_{21}) \leq \beta$ and minimizes the expected value of the cost function is n=26, $a_1=1$, and $a_2=0$. The expected value of the cost function, given by the expression (9.5.7), is 29.96938.

Evaluation of different triplets satisfying the given side condition and calculation of the value of the function (9.5.10) for such triplets under two side conditions was dome with the help of computer. We have used EC 1030 computer at Operations Research Group, Baroda. It may be noted that the maximum value of n attained in the program used was 50 (a 50% value of the lot size).

Thus, for the numeric values stated at the beginning of this section the Bayesian Sampling Plan satisfying the side condition $P_a(p_{10}, p_{20}) \geq 1-\alpha$ is $n=3$, $a_1=2$, $a_2=0$ and the one satisfying the side condition $P_a(p_{11}, p_{21}) \leq \beta$ is $n=26$, $a_1=1$, $a_2=0$.