CHAPTER II

THE MAXIMUM LIKELIHOOD ESTIMATION OF THE FRACTION DEFECTIVE UNDER CURTAILED MULTIPLE TWO CLASS ATTRIBUTES SAMPLING PLAN

2.1 In this chapter we define curtailed multiple sampling plan by attributes. A particular case of a curtailed multiple sampling plan, namely, the curtailed double sampling plan is studied extensively under two different situations, Situation-A and Situation-B. Situation-A takes into consideration the reporting of complete information of the records of sampling inspection whereas Situation-B occurs when censored information of Type-I on inspection records, as defined by Gupta [17] is reported. The maximum likelihood estimator (MLE) of the fraction defective and the asymptotic variance of the MLE are given under both the situations, Situation-A and Situation-B. In Situation-B the MLE is not available in the explicit form. Hence in the Appendix of this chapter we have given a SUBROUTINE which will evaluate the MLE by an iterative method. The SUBROUTINE also evaluates the asymptotic variance of the MLE.

2.2 Curtailed Multiple Sampling Plan :

2.2.1 In the multiple sampling plan (MSP) by attributes a sequence of k samples of size n_i (i=1,2,...,k) is taken from a lot of size N. The design of the sampling plan specifies 2k numbers a_i and r_i (i=1,2,...,k). If the accumulated number of defectives, d_i, is equal to or less than a_i, the lot is accepted. If d_i is equal to or greater than r_i, the lot is rejected. If d_i falls between a_i and r_i, the decision of acceptance or rejection of the lot is differed until the next sample of size n_{i+1} is inspected. The constants a_i and r_i, known as acceptance and rejection numbers, are predetermined numbers satisfying the following conditions :

(i) $0 \leq a_1 \leq a_2 \leq \cdots \leq a_{k-1} < a_k$,

(ii) $r_1 \leq r_2 \leq \cdots \leq r_k$,

(iii) $a_i + 1 < r_i$ i =1,2,...,k-1,

(iv) $a_k + 1 = r_k$,

(v) $a_i < \sum_{j=1}^{i} n_j$ i = 1,2,..., k ,

(vi) $r_k \leq \sum_{i=1}^{k} n_i$.

The condition (iv) ensures that not more than k samples are required for inspection. It also implies that only 2k-1 numbers are to be specified.

2.2.2 If the inspection has no other purpose than to determine which ~~inspection~~ lots to accept and which to reject, it would be obvious to stop the inspection as soon as the decision of acceptance or rejection is reached. This leads to the curtailment in the inspection. Two forms of the curtailed inspection can be distinguished. The sampling plan which considers the curtailment in the inspection arising due to observing enough defectives to reject a lot is termed here as a semi-curtailed sampling plan, following the terminology of the Statistical Research Group [44]. Similarly the sampling plan which considers the curtailment in the inspection arising due to observing either enough defectives to reject a lot or enough nondefectives to accept a lot is called a fully-curtailed sampling plan.

2.2.3 Statement of a Fully-Curtailed MSP :

Consider an attributes sampling plan in which individual units randomly selected from a lot of size N are inspected one at a time till one of the following 2k events occurs :

(α_i) r_i defectives are observed and the number of units inspected is greater than $\sum_{j=1}^{i} n_{j-1}$ and less than or equal to $\sum_{j=1}^{i} n_j$,

(β_i) g_i nondefectives are observed and the number of units inspected is greater than $\sum_{j=1}^{i} n_{j-1}$ and less than or equal to $\sum_{j=1}^{i} n_j$,

for $i=1,2,\ldots,k$

Here n_0 is assigned a value zero.

Then, the decision rule is to reject the lot if one of the k events of the set α_i occurs and to accept the lot if one of the k events of the set β_i occurs.

The relations between the consants given in the above statement and those of the uncurtailed MSP are

$$n_i=n_i,\ r_i=r_i,\ g_i = \sum_{j=1}^{i} n_j-a_i.$$

2.3 Curtailed Double Sampling Plan :

2.3.1 A double sampling plan (DSP) is a particular case of MSP for k=2 given in Section 2.2.1. The design of the DSP specifies four numbers a_i and r_i (i=1,2.). The relation $a_2+1=r_2$ ensures that not more than two samples are required to inspect. In the usual text books [3],[11],[13] and other literature [8] , in the definition of DSP, the common practice is to take $r_1=r_2=r$ and hence $a_2+1=r$. Due to this practice the design of the sampling plan specifies only two numbers a_1 and a_2. We call this latter sampling plan, usual DSP (UDSP).

2.3.2 Statement of Fully-Curtailed Sampling Plan :

Statement of a fully-curtailed DSP can be easily obtained from the statement of the fully-curtailed MSP, given in Section 2.2.3, by considering k=2. The four events $\alpha_1, \alpha_2, \beta_1$, and β_2 of fully-curtailed DSP are designated here as $E_i (i=1,2,3,4)$ and are as given below :

(E_1) r_1 defectives are observed and the number of units inspected is less than or equal to n_1,

(E_2) r_2 defectives are observed and the number of units inspected is greater than n_1 but less than or equal to n_1+n_2,

(E_3) g_1 nondefectives are observed and the number of units inspected is less than or equal to n_1,

(E_4) g_2 nondefectives are observed and the number of units inspected is greater than n_1 but less than or equal to n_1+n_2.

The decision rule is then to reject the lot if one of the events E_1 and E_2 occurs and to accept the lot if one of the events E_3 and E_4 occurs. The constants r_1, r_2, g_1 and g_2 are the predetermined numbers such that

$$n_1-g_1+1 < r_1 \leq r_2 \leq n_1+n_2 \qquad \ldots(2.3.1)$$

$$0 < g_1 < g_2 \qquad \ldots(2.3.2)$$

$$g_1 \leq n_1 \qquad \ldots(2.3.3)$$

$$g_2=n_1+n_2-r_2+1 \qquad \ldots(2.3.4)$$

It may be noted that the constants of a fully-curtailed DSP are related to the constants of the corresponding uncurtailed DSP of Section 2.3.1 as given below :

$$n_1=n_1,\ n_2=n_2,\ r_1=r_1,\ r_2=r_2$$

$$g_1=n_1-a_1,\quad g_2=n_1+n_2-a_2 \qquad \ldots(2.3.5)$$

It is then clear that the events E_1 and E_3 of a fully--curtailed DSP lead respectively to rejection and acceptance of a lot on the basis of enough information of the defectives and nondefectives observed during the inspection of the first sample. Similarly the events E_2 and E_4 lead to rejection and acceptance of a lot respectively on the basis of enough information of the accumulated total of defectives and nondefectives observed during the inspection of the second sample.

2.3.3 A Remark on Statement of [12].

Statement of a fully-curtailed DSP is also given by Girschick, Mosteller and Savage [12]. This statement is somewhat confusing. For ready reference we reproduce the same here :

"A sample of size n_1 is drawn and items are inspected until (i) r_1 $(1 < r_1 \leq n_1)$ defectives are found, or (ii) n_1-a+1 $(a \geq 0)$ nondefectives are found or (iii) the sample is exhausted with neither of these events occurring. If case (iii) arises, a second sample of size n_2 is drawn and inspection proceeds until a grand total of $r_2 (r_1 \leq r_2 \leq n_1+n_2)$ defectives are found or $n_1+n_2-r_2+1$ nondefectives are found. In this scheme we call r_1 and r_2 rejection numbers and a an acceptance number".

Following remarks will reveal the confusion involved in the above statement :

(i) They aught to have called a-1 as an acceptance number.

(ii) a=0 is meaningless, for a=0 implies finding n_1+1 nondefectives in inspection of n_1 items.

(iii) Non-existence of a condition of the type $n_1-g_1+1 < r_1$ as given by (2.3.1) allows in their plan to have

$a=r_1$. Then for instance, $a = r_1 = 3$ and $n_1 = 6$ will not allow us to take a second sample at all.

2.4 Fully-Curtailed DSP under Situation-A :

As stated earlier, Situation-A takes into consideration the reporting of complete information of the records of sampling inspection. Now in case of fully-curtailed DSP the complete information of the sampling inspection means the information on (i) the number of units inspected (or the number of defectives found) when the inspection is stopped by finding sufficient number of nondefectives and (ii) the number of units inspected (or the number of nondefectives observed) when the inspection is stopped by finding sufficient number of defectives. A lot is accepted when (i) occurs and is rejected when (ii) occur. In the succeeding sections of this section we study fully-curtailed DSP in detail (particularly its probability function, the maximum likelihood estimate of the fraction defective, asymptotic variance of the maximum likelihood estimate etc.) under this situation. At the end of this section the results of our study are generalized to fully-curtailed MSP.

2.4.1 Probability Function :

Let the process average proportion of defectives be p and for sufficiently large lots it can be considered as the probability of selecting a defective in a single trial. Furthermore, let the probability p remain constant from trial to trial and the trials be stochastically independent. This applies to the type B situation of Dodge and Romig [10], hence, the lot size N does not subsequently appear.

Let Y denote the number of units inspected when the inspection is stopped due to the occurrence of the event $E_i (i=1,2,3,4)$. Let $A_i (i=1,2,3,4)$ be the set of possible values attained by Y. Then

$$A_1 = \{r_1, r_1+1, \ldots, n_1\},$$

$$A_2 = \{r_2-r_1+n_1+1,\ r_2-r_1+n_1+2,\ \ldots,\ n_1+n_2\},$$

$$A_3 = \{g_1,\ g_1+1,\ \ldots\ n_1\},$$

$$A_4 = \{g_2-g_1+n_1+1,\ g_2-g_1+n_1+2,\ \ldots,\ n_1+n_2\}.$$

Further define a random variable T as follows :

$$T = i \text{ if } E_i \text{ occurs, } i=1,2,3,4.$$

Then the joint probability function of the random variables Y and T can be expressed as

$$P\,(Y=y,\ T=i) = \begin{cases} f_i\,(y;p) & y \in A_i,\ i = 1,2,3,4 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(2.4.1)$$

where

$$f_1(y;p) = \binom{y-1}{r_1-1} p^{r_1} q^{y-r_1} \qquad \ldots(2.4.2)$$

$$f_2(y;p) = \sum_{u=1}^{b_1} \binom{n_1}{g_1-u} \binom{y-n_1-1}{b_2-u} p^{r_2} q^{y-r_2} \qquad \ldots(2.4.3)$$

$$f_3(y;p) = \binom{y-1}{g_1-1} p^{y-g_1} q^{g_1} \qquad \ldots(2.4.4)$$

$$f_4(y;p) = \sum_{u=1}^{b_1} \binom{n_1}{g_1-u} \binom{y-n_1-1}{g_2-g_1+u-1} p^{y-g_2} q^{g_2} \qquad \ldots(2.4.5)$$

and $q = 1-p$, $u=d_1-a_1$, $b_1=g_1+r_1-n_1-1$, $b_2=g_1+r_2-n_1-1$.

While calculating the various terms of the summation involved in $f_2(y;p)$ and $f_4(y;p)$, $\binom{n}{x}$ is regarded as zero whenever x exceeds n or whenever x is negative.

The probability function of the number of units inspected, Y, which is the marginal probability function of (2.4.1) can be expressed as

$$P\,(Y=y) = \sum_{i=1}^{4} f_i(y;p) \qquad \ldots(2.4.6)$$

Similarly the probability of occurrence of the events $E_i\,(i = 1,2,3,4)$ is given by $\pi_i\,(\,i=1,2,3,4\,)$

where

$$\pi_i = P(T=i)$$

$$= \sum_{y \in A_i} f_i(y;p) \qquad i = 1,2,3,4 \qquad \ldots(2.4.7)$$

Then the probability that a lot is rejected is

$$\pi_1 + \pi_2 \qquad \ldots(2.4.8)$$

and that it is accepted is

$$\pi_3 + \pi_4 \qquad \ldots(2.4.9)$$

2.4.2 The Maximum Likelihood Estimate :

In this section we derive the maximum likelihood estimate (MLE) of the fraction defective, p, when m lots are inspected in accordance with the fully-curtailed DSP. Suppose for the inspection of every lot, the information about the number of units inspected and about the fact that the event E_i has occurred is supplied. This information could be concisely expressed by the following pairs.

$$(y_{ij}, T=i) \qquad j = 1,2,\ldots,m_i; \quad i = 1,2,3,4 \qquad \ldots(2.4.10)$$

where $y_{ij} \in A_i$, $j = 1,2,\ldots, m_i$ for fixed i and $\sum_{i=1}^{4} m_i = m$.

The m pairs given by (2.4.10) can be considered as a random sample of size m from a bivariate distribution whose probability function is given by (2.4.1). The likelihood function, L, based on this sample can be expressed as

$$L = \prod_{i=1}^{4} \prod_{j=1}^{m_i} f(y_{ij};p)$$

$$= (\text{const.}) \prod_{j=1}^{m_1} (p^{r_1} q^{y_{1j}-r_1}) \prod_{j=1}^{m_2} (p^{r_2} q^{y_{2j}-r_2})$$

$$\prod_{j=1}^{m_3} (p^{y_{3j}-g_1} q^{g_1}) \prod_{j=1}^{m_4} (p^{y_{4j}-g_2} q^{g_2}) \qquad \ldots(2.4.11)$$

where we use (2.4.2) through (2.4.5) to obtain (2.4.11)

On taking logarithms of (2.4.11), differentiating partially with respect to p, equating the partial derivative to zero, and solving for p we obtain the MLE of p, $\hat{p}$, as

$$\hat{p} = \frac{(TD)}{(TU)} \qquad \ldots(2.4.12)$$

where (TD) = Total number of defectives observed

$$= m_1 r_1 + m_2 r_2 + \sum_{j=1}^{m_3} (y_{3j} - g_1) + \sum_{j=1}^{m_4} (y_{4j} - g_2) \qquad \ldots(2.4.13)$$

(TU) = Total number of units inspected

$$= \sum_{j=1}^{m_1} y_{1j} + \sum_{j=1}^{m_2} y_{2j} + \sum_{j=1}^{m_3} y_{3j} + \sum_{j=1}^{m_4} y_{4j} \qquad \ldots(2.4.14)$$

This feature was also observed by Phatak and Bhatt [40] when the maximum likelihood estimators of the fraction defective under semi-curtailed and fully-curtailed single sampling plans were obtained.

2.4.3 The Asymptotic Variance of the MLE :

Differentiating partially the logarithm of the likelihood function, L, given by (2.4.11) twice it is found that

$$\frac{\partial^2 \log L}{\partial p^2} = \frac{q-p}{p^2 q^2}\,(TD) + \frac{1}{q^2}\,(TU) \qquad \ldots(2.4.15)$$

Noting that

(i) $E\,(TD) = p\,E\,(TU)$

$= p\,m\,(ASN)$

(ii) The expression for ASN is

$$ASN = \frac{r_1}{p}\left[1-B\,(r_1;\, n_1+1,\ p)\right] + \frac{g_1}{q}\,B\,(n_1-g_1;\ n_1+1, p)$$
$$+ \sum_{t=1}^{b_1} b\,(n_1-g_1+t;\ n_1, p)\left[\frac{b_2+1-t}{p}\right.$$
$$\left.\cdot\left\{1-B(b_2+1-t; n_2+1, p)\right\} + n_1 + \frac{g_2-g_1+t}{q}\left\{B(b_2-t; n_2+1, p)\right\}\right] \qquad \ldots(2.4.16)$$

where $b(x;n,p) = \binom{n}{x} p^x q^{n-x}$ and $B(r;n,p) = \sum_{x=0}^{r} b(x;n,p)$.

The detail about ASN is given in Chapter IV of the thesis.

(iii) The asymptotic variance of MLE of p is given by

$$V\,(\hat{p}) = -\frac{1}{E\left(\frac{\partial^2 \log L}{\partial p^2}\right)}$$

In case-I, when inspection is terminated, inspector reports the information on (i) acceptance or rejection of a lot and (ii) the number of defectives found. Information on either number of nondefectives found or number of units inspected is not reported.

In case-II, when inspection terminates, inspector reports the information on (i) acceptance or rejection of a lot and (ii) the number of nondefectives found. Information on either number of defectives found or number of units inspected is not reported.

In the succeeding sections of 2.5 we study the various aspects (such as the MLE of the fraction defective, asymptotic variance of the MLE etc.) of fully-curtailed DSP under both the cases of Situation-B. The cases given above are mutually exclusive. Furthermore, it is observed that the evaluation of the MLE in Situation-B is not as simple as that in Situation-A. To get the MLE in both the cases of Situation-B we have to follow an iterative procedure. We use the method of scoring for parameters given on page 49 of the Advanced Theory of Statistics vol.2 [29]. A SUBROUTINE is given for both the cases to evaluate the MLE by this method and the asymptotic variance of the MLE. Numerical examples are worked out, using this SUBROUTINE, on EC 1030 computer at Operation Research Group, Baroda.

2.5.1 Probability Function under Case-I :

Let V be the number of defectives reported by the inspector along with the information about the acceptance or rejection of a lot. Recall the events $E_i (i=1,2,3,4)$ defined in Section 2.4.1. These four events are modified to suit the Case-I of Situation-B and are given below :

(F_1) r_1 defectives are observed and it is reported that the lot is rejected,

(F_2) r_2 defectives are observed and it is reported that the lot is rejected,

(F_3) V $(0 \leq V \leq n_1 - g_1)$ defectives are observed and it is reported that the lot is accepted,

(F_4) V $(n_1 - g_1 + 1 \leq V \leq n_1 + n_2 - g_2)$ defectives are observed and it is reported that the lot is rejected.

Let $B_i (i=1,2,3,4)$ be the set of possible values attained by V. Then

$$B_1 = \{r_1\}$$
$$B_2 = \{r_2\}$$
$$B_3 = \{0, 1, \ldots n_1 - g_1\}$$
$$B_4 = \{n_1 - g_1 + 1, \ldots n_1 + n_2 - g_2\}$$

Further define a random variable I as follows :

$I = i$ if F_i occurs $i = 1,2,3,4$.

Then the joint probability function of the random variables V and I can be expressed as

$$P(V=v,\ I=i) = g_i(v;p) \quad v \in B_i,\ i=1,2,3,4$$
$$= 0 \qquad \text{elsewhere} \qquad \ldots(2.5.1)$$

where

$$g_1(v;p) = \sum_{y=r_1}^{n_1} \binom{y-1}{r_1-1} p^{r_1} q^{y-r_1} \qquad \ldots(2.5.2)$$

$$g_2(v;p) = \sum_{u=1}^{b_1} \binom{n_1}{g_1-u} \sum_{y=r_2-r_1+n_1+1}^{n_1+n_2} \binom{y-n_1-1}{b_2-u} p^{r_2} q^{y-r_2} \qquad \ldots(2.5.3)$$

$$g_3(v;p) = \binom{v+g_1-1}{g_1-1} p^v q^{g_1} \qquad \ldots(2.5.4)$$

$$g_4(v;p) = \sum_{u=1}^{b_1} \binom{n_1}{g_1-u} \binom{v+g_2-n_1-1}{g_2-g_1+u-1} p^v q^{g_2} \qquad \ldots(2.5.5)$$

and b_1 and b_2 are defined in Section 2.4.1.

It may be noted that while calculating different terms $\binom{n}{x}$ is regarded as zero whenever x exceeds n or whenever x is negative.

2.5.2 The Maximum Likelihood Estimator under Case-I :

Let m be the number of lots which are inspected according to the fully-curtailed DSP under Case-I. Let the event $F_i (i=1,2,3,4)$ has occurred $m_i (i=1,2,3,4)$ times. For the inspection of every lot, the information about the number of defectives observed and the occurrence of the event F_i has supplied. This information could be expressed by the following pairs

$$(v_{ij}, i) \quad j=1,2,\ldots,m_i, \quad i=1,2,3,4. \qquad \ldots(2.5.6)$$

where $v_{ij} \in B_i$, $j=1,2,\ldots,m_i$ for fixed i and $\sum_{i=1}^{4} m_i = m$.

Considering these m pairs as a random sample of size m from a bivariate distribution with probability function given by (2.5.1), the likelihood function, L, based on this sample can be expressed as

$$L = \prod_{i=1}^{4} \prod_{j=1}^{m_i} g_i\,(v_{ij};\,p)$$

$$= (\text{const.}) \left[\sum_{y=r_1}^{n_1} \binom{y-1}{r_1-1} p^{r_1} q^{y-r_1} \right]^{m_1}$$

$$\cdot \left[\sum_{u=1}^{b_1} \binom{n_1}{g_1-u} \sum_{y=r_2-r_1+n_1+1}^{n_1+n_2} \binom{y-n_1-1}{b_2-u} p^{r_2} q^{y-r_2} \right]^{m_2}$$

$$\prod_{j=1}^{m_3} (p^{v_{3j}} q^{g_1}) \prod_{j=1}^{m_4} (p^{v_{4j}} q^{g_2}) \qquad \ldots(2.5.7)$$

Taking logarithm of (2.5.7), differentiating partially with respect to p, and equating the partial derivative to zero we get the likelihood equation as given below :

$$m_1(n_1-r_1+1)\, p\, \phi_1 + m_2 p \phi_2 + (1-p)\left(\sum_{j=1}^{m_3} v_{3j} + \sum_{j=1}^{m_4} v_{4j}\right) - (g_1 m_3 + g_2 m_4)p = 0$$

$$\therefore \hat{p} = \frac{\sum_{j=1}^{m_3} v_{3j} + \sum_{j=1}^{m_4} v_{4j}}{(g_1 m_3 + g_2 m_4) + \left(\sum_{j=1}^{m_3} v_{3j} + \sum_{j=1}^{m_4} v_{4j}\right) - m_1(n_1-r_1+1)\phi_1 - m_2\phi_2} \qquad \ldots(2.5.8)$$

where

$$\phi_1 = b(r_1-1;n_1,p)/\Delta_1,$$

$$\phi_2 = \frac{1}{\Delta_2}\Big[\sum_{u=1}^{b_1} (g_2-g_1+u)\, b(n_1-g_1+u;n_1,p)\, b(r_2-(n_1-g_1+u)-1;n_2,p) + n_1 \sum_{u=1}^{b_1} b(n_1-g_1+u-1;n_1-1,p)\, B'(r_2-(n_1-g_1+u);n_2,p) - n_1\Delta_2\Big],$$

$$\Delta_1 = B'(r_1;n_1,p),$$

$$\Delta_2 = \sum_{u=1}^{B_1} b(n_1-g_1+u;n_1,p)\, B'(r_2-(n_1-g_1+u);\, n_2,p),$$

and $b(x;n,p) = \binom{n}{x} p^x q^{n-x}$,

$$B(r;n,p) = \sum_{x=0}^{r} b(x;n,p),$$

$$B'(r;n,p) = \sum_{x=r}^{n} b(x;n,p).$$

2.5.3 <u>Asymptotic Variance of the MLE</u> :

Differentiating partially the logarithm of the likelihood function, L, given by (2.5.7) twice it is found that

$$\frac{\partial^2 \log L}{\partial p^2} = \frac{m_1(n_1-r_1+1)p\,\emptyset_1}{p^2\,q^2}\Big[(r_1-1)q + p\left\{1-(n_1-r_1+1)(1+\emptyset_1)\right\}\Big]$$

$$+ \frac{m_2 p}{p^2 q^2}\Big[\sum_{u=1}^{b_1}(g_2-g_1+u)\left\{B_1((r_2-1)\,q-g_2p) + n_1 pB_1'\right\}$$

$$+n_1\sum_{u=1}^{b_1} B_2\left\{(n_1-g_1+u-1)\,q-(g_1-u)p\right\} - p\emptyset_2^2 - p\,\emptyset_2(n_1-1)\Big]$$

$$-\frac{1}{p^2q^2}\Big[q^2\Big(\sum_{j=1}^{m_3} v_{3j} + \sum_{j=1}^{m_4} v_{4j}\Big) + p^2(g_1m_3+g_2m_4)\Big].$$

where

$$B_1 = b(n_1-g_1+u;n_1,p)\; b(r_2-(n_1-g_1+u)-1;n_2,p)/\Delta_2,$$

$$B_1' = b(n_1-g_1+u-1;n_1-1,p)\; b(r_2-(n_1-g_1+u)-1;n_2,p)/\Delta_2,$$

$$B_2 = b(n_1-g_1+u-1;n_1-1,p)\; B'(r_2-(n_1-g_1+u);n_2,p)/\Delta_2.$$

Noting that

i) $E(m_1) = mB'\,(r_1;n_1,p)$

ii) $E(m_2) = m\sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)\;B'(r_2-(n_1-g_1+u);n_2,p)$

iii) $E(m_3) = mB(n_1-g_1;n_1,p)$

iv) $E(m_4) = m\sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)\;B(n_2-g_2+g_1-u;n_2,p)$

v) $E\Big(\sum_{j=1}^{m_3} v_{3j}\Big) = \frac{mg_1p}{q}\;B(n_1-g_1-1;n_1,p)$

$$\text{vi) } E\left(\sum_{j=1}^{m_4} v_{4j}\right) = m \sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)$$

$$\cdot \left\{\frac{p(g_2-g_1+u)B(n_2-g_2+g_1-u-1;n_2,p)}{q} + (n_1-g_1+u)\, B(n_2-g_2+g_1-u;n_2,p)\right\}$$

and

(vii) the asymptotic variance of the MLE of p is given by

$$V(\hat{p}) = -\frac{1}{E(\partial^2 \log L/\partial p^2)}$$

one has

$$V(\hat{p}) = \frac{p^2q^2}{mH_1} \qquad \ldots(2.5.9)$$

where

$$\begin{aligned} H_1 = {} & p\,q\,g_1\, B(n_1-g_1;n_1,p) \\ & + pq \sum_{u=1}^{b_1} (g_2-g_1+u)\, b(n_1-g_1+u;n_1,p)\, B(n_2-(g_2-g_1+u-1);n_2,p) \\ & + q^2 \sum_{u=1}^{b_1} (n_1-g_1+u)\, b(n_1-g_1+u;n_1,p)\, B(n_2-(g_2-g_1+u);n_2,p) \\ & + p^2\, g_1\, B(n_1-g_1;n_1,p) \\ & + p^2 g_2 \sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)\, B(n_2-(g_2-g_1+u);n_2,p) \\ & -(n_1-r_1+1)\, p\, \phi_1 \left[(r_1-1)\, q + p\left\{1-(n_1-r_1+1)(1+\phi_1)\right\}\right] \\ & \cdot B'(r_1;n_1,p) \\ & -p\left\{(r_1-1)q-g_2\, p\right\} \sum_{u=1}^{b_1} (g_2-g_1+u)\, b(n_1-g_1+u;n_1,p) \\ & \quad \cdot b(r_2-(n_1-g_1+u)-1;n_2,p) \end{aligned}$$

cont...

$$-p^2 n_1 \sum_{u=1}^{b_1} (g_2-g_1+u)b(n_1-g_1+u-1;n_1-1,p)b(r_2-(n_1-g_1+u)-1;n_2,p)$$

$$-n_1 p \sum_{u=1}^{b_1} \{(n_1-g_1+u-1)q-(g_1-u)p\} b(n_1-g_1+u-1;n_1-1,p) \cdot B'(r_2-(n_1-g_1+u);n_2,p)$$

$$+ p^2 \phi_2 \Delta_2 \{\phi_2+(n_1-1)\}.$$

2.5.4 Probability Function under Case-II:

Let W be the number of nondefectives reported by the inspector along with the information of acceptance or rejection of a lot. Under this case the four events, given in Section 2.4.1, are modified as given below :

(G_1) $W(0 \leq W \leq n_1-r_1)$ nondefectives are observed and it is reported that the lot is rejected.

(G_2) $W(n_1-r_1+1 \leq W \leq n_1+n_2-r_2)$ nondefectives are observed and it is reported that the lot is rejected.

(G_3) g_1 nondefectives are observed and it is reported that the lot is accepted.

(G_4) g_2 nondefectives are observed and it is reported that the lot is accepted.

Let $D_i(i=1,2,3,4)$ be the set of possible values attained by W when event G_i occurs.

Then

$$D_1 = \{0,1,\ldots,n_1-r_1\}$$

$$D_2 = \{n_1-r_1+1, n_1-r_1+2, \ldots, n_1+n_2-r_2\}$$

$$D_3 = \{g_1\}$$

$$D_4 = \{g_2\}$$

Further, define a random variable I=i if G_i occurs i=1,2,3,4. Then the joint probability function of the random variable W and I can be given as :

$$P(W=w, I=i) = h_i(w,p) \qquad w \in D_i,\ i=1,2,3,4$$
$$= 0 \qquad \text{elsewhere} \qquad \ldots(2.5.10)$$

where

$$h_1(w;p) = \binom{w+r_1-1}{r_1-1} p^{r_1} q^{w} \qquad \ldots(2.5.11)$$

$$h_2(w;p) = \sum_{u=1}^{b_1} \binom{n_1}{g_1-u}\binom{w+r_2-n_1-1}{b_2-u} p^{r_2} q^{w} \qquad \ldots(2.5.12)$$

$$h_3(w;p) = \sum_{y=g_1}^{n_1} \binom{y-1}{g_1-1} p^{y-g_1} q^{g_1} \qquad \ldots(2.5.13)$$

$$h_4(w;p) = \sum_{u=1}^{b_1} \binom{n_1}{g_1-u} \sum_{y=g_2-g_1+n_1+1}^{n_1+n_2} \binom{y-n_1-1}{g_2-g_1+u-1} p^{y-g_2} q^{g_2} \qquad \ldots(2.5.14)$$

and b_1 and b_2 are as defined in Section 2.4.1. It may be noted that while calculating different terms $\binom{n}{x}$ is regarded as zero whenever x exceeds n or x is negative.

2.5.5 The MLE under Case-II :

Let m be the number of lots which are inspected according to the fully-curtailed DSP under Case-II. Let the event G_i (i=1,2,3,4) has occurred m_i(i=1,2,3,4) times. For the inspection of every lot, the information on the number of nondefectives observed and the occurrence of the event G_i has supplied. This information could be expressed by the following pairs :

$$(w_{ij}, i) \qquad j=1,2,\ldots,m_i, \quad i=1,2,3,4 \qquad \ldots(2.5.15)$$

where $w_{ij} \in D_i$, $j=1,2,\ldots,m_i$ for fixed i and $\sum_{i=1}^{4} m_i = m$.
Considering these m pairs as a random sample of size m from a bivariate distribution with probability function given by (2.5.10), the likelihood function, L, based on this sample can be expressed as

$$L = \prod_{i=1}^{4} \prod_{j=1}^{m_1} h_i(w_{ij};p)$$

$$= (\text{const.}) \prod_{j=1}^{m_1} (p^{r_1} q^{w_{1j}}) \prod_{j=1}^{m_2} (p^{r_2} q^{w_{2j}})$$

$$\cdot\left[\sum_{y=g_1}^{n_1} \binom{y-1}{g_1-1} p^{y-g_1} q^{g_1}\right]^{m_3}$$

$$\cdot\left[\sum_{u=1}^{b_1} \binom{n_1}{g_1-u} \sum_{y=g_2-g_1+n_1+1}^{n_1+n_2} \binom{y-n_1-1}{g_2-g_1+u-1} p^{y-g_2} q^{g_2}\right]^{m_4} \qquad \ldots(2.5.16)$$

Taking the logarithm of (2.5.16), differentiating partially with respect to p, and equating the partial derivative to zero we get the likelihood equation as given below :

$$m_1 r_1 + m_2 r_2 - p(m_1 r_1 + m_2 r_2) - p\left(\sum_{j=1}^{m_1} w_{1j} + \sum_{j=1}^{m_2} w_{2j} \right)$$

$$- m_3 g_1 p \Psi_1 - m_4 p \Psi_2 = 0$$

$$\therefore \; \hat{p} = \frac{m_1 r_1 + m_2 r_2}{(m_1 r_1 + m_2 r_2) + \left(\sum_{j=1}^{m_1} w_{1j} + \sum_{j=1}^{m_2} w_{2j} \right) + m_3 g_1 \Psi_1 + m_4 \Psi_2}$$

...(2.5.17)

where

$$\Psi_1 = b(n_1 - g_1; n_1, p) / \delta_1,$$

$$\Psi_2 = \frac{1}{\delta_2}\Big[\sum_{u=1}^{b_1} (g_2 - g_1 + u)\, b(n_1 - g_1 + u; n_1, p)\, b(n_2 - (g_2 - g_1 + u); n_2, p)$$

$$- n_1 \sum_{u=1}^{b_1} b(n_1 - g_1 + u - 1; n_1 - 1, p)\, B(n_2 - (g_2 - g_1 + u); n_2, p) + n_1 \delta_2 \Big],$$

$$\delta_1 = B(n_1 - g_1; n_1, p)$$

$$\delta_2 = \sum_{u=1}^{b_1} b(n_1 - g_1 + u; n_1, p)\, B(n_2 - (g_2 - g_1 + u); n_2, p)$$

and $b(x;n,p)$, $B(r;n,p)$, $B'(r;n,p)$ are as defined in Section 2.5.2.

2.5.6 Asymptotic Variance of the MLE under Case-II :

Differentiating partially the logarithm of the likelihood function, L, given by (2.5.16) twice it is found that

$$\frac{\partial^2 \log L}{\partial p^2} = - \frac{q^2(m_1 r_1 + m_2 r_2)}{p^2 q^2} - \frac{p^2(\sum_{j=1}^{m_1} w_{1j} + \sum_{j=1}^{m_2} w_{2j}}{p^2 q^2}$$

$$- \frac{m_3 g_1 p}{p^2 q^2}\left[\Psi_1\{(n_1-g_1)q + p(1-g_1+\Psi_1)\}\right]$$

$$- \frac{m_4 p}{p^2 q^2}\Big[\sum_{u=1}^{b_1} (g_2-g_1+u)\, L_1 \{(r_2-1)q - g_2 p + p\Psi_2\}$$

$$-n_1 \sum_{u=1}^{b_1}\{-pL_1'(g_2-g_1+u) + L_2((n_1-g_1+u-1)q-(g_1-u)p+p\Psi_2)\}$$

$$+p\Psi_2\Big]$$

where

$$L_1 = b(n_1-g_1+u;n_1,p)\ b(n_2-(g_2-g_1+u);n_2,p)/\delta_2,$$

$$L_1' = b(n_1-g_1+u-1;n_1-1,p)\ b(n_2-(g_2-g_1+u);n_2,p)/\delta_2,$$

$$L_2 = b(n_1-g_1+u-1;n_1-1,p)\ B(n_2-(g_2-g_1+u);n_2,p)/\delta_2.$$

Noting that

(i) $E(m_1) = m\ B'(r_1;n_1,p)$

(ii) $E(m_2) = m \sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)\ B'(r_2-(n_1-g_1+u);n_2,p)$

(iii) $E(m_3) = mB(n_1-g_1;n_1,p)$

iv) $E(m_4) = m \sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)\ B(n_2-(g_2-g_1+u);n_2,p)$

v) $E(\sum_{j=1}^{m_1} w_{1j}) = \frac{mr_1q}{p} B'(r_1+1;n_1,p)$

vi) $E(\sum_{j=1}^{m_2} w_{2j}) = m \sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)\ \Big[\frac{(r_2-(n_1-g_1+u))q}{p}$

$B'(r_2-(n_1-g_1+u)+1;n_2,p)+(g_1-u)B'(r_2-(n_1-g_1+u);n_2,p)\Big]$

and

vii) the asymptotic variance of the MLE of p is given by

$$v(\hat{p}) = -\frac{1}{E(\partial^2 \log L/\partial p^2)}$$

one has

$$V(\hat{p}) = p^2q^2/mH_2 \qquad \ldots(2.5.18)$$

where

$$H_2 = q^2r_1\ B'(r_1;n_1,p)+q^2r_2 \sum_{u=1}^{b_1} b(n_1-g_1+u;n_1,p)$$

$$.B'(r_2-(n_1-g_1+u);n_2p)$$

$$+ pqr_1\ B'(r_1+1;n_1,p)$$

$$+pq \sum_{u=1}^{b_1} (r_2-(n_1-g_1+u))b(n_1-g_1+u;n_1,p)B'(r_2-(n_1-g_1+u)+1;n_2,p)$$

$$+p^2 \sum_{u=1}^{b_1} (g_1-u)b(n_1-g_1+u;n_1,p)B'(r_2-(n_1-g_1+u);n_2,p)$$

$$+ \delta_1 pg_1 \{\psi_1((n_1-g_1)q + (1-g_1+\psi_1)p)\}$$

cont...

$$+ p \sum_{u=1}^{b_1} (g_2-g_1+u)\ (r_1-1)\ (q-g_2p)\ \delta_2\ L_1$$

$$+ n_1p^2 \sum_{u=1}^{b_1} (g_2-g_1+u)\ \delta_2 L_1' - n_1 p \sum_{u=1}^{b_1} \{(n_1-g_1+u-1)q-(g_1-u)p\} \cdot \delta_2\ L_2$$

$$+ p^2 \delta_2 \psi_2 \{\psi_2 - (n_1-1)\}\ .$$

2.5.7 Evaluation of the MLE and its Asymptotic Variance :

We observe that the expressions (2.5.8) and (2.5.17) are not in the explicit forms. Hence the actual evaluation of the MLE is not as simple as it is under the Situation-A. It is required to use an iterative procedure for the actual evaluation of the MLE. We have used an iterative procedure which is known as "The Method of Scoring for Parameters" due to Fisher (1925) explainad in the usual text books such as [29]. According to this method the expression for the MLE, $\hat{p}$, is

$$\hat{p} = t + \left(\frac{\partial \log L}{\partial p}\right)_t [V(\hat{p})]_t \qquad \ldots(2.5.19)$$

where t = initial value of $\hat{p}$ and $v(\hat{p})$ is the asymptotic variance at $\hat{p}$ = t. The method is as given below :

"Find $(\partial \log L/\partial p)$ and $v(\hat{p})$ for initial value of t, and hence find $\hat{p}$ using (2.5.19). Initial value of t is the first approximation of $\hat{p}$, denote it by $\hat{p}_0$(the procedure for the determination of the initial value is further explained in the next Section 2.5.8). If $|\hat{p} - \hat{p}_0|$ is negligible stop the iteration and the MLE of p is $\hat{p}$. If the absolute difference is not negligible take $t = \hat{p}$, where $\hat{p}$ is the new approximation of the MLE. Using this new value of t, find $(\partial \log L/\partial p), V(\hat{p})$, and hence obtain $\hat{p}$ from (2.5.19). Compare $\hat{p}$ obtained at the second iteration with that of first iteration. If the difference is negligible we stop iteration and take this last $\hat{p}$ as the MLE. If the difference is large repeat the iterative procedure till the difference between the two consecutive approximations of $\hat{p}$ is negligible. Once this is achieved, the MLE is the value of $\hat{p}$ obtained at the final iteration. Using this value of $\hat{p}$ one gets the estimate of the asymptotic variance. The method converges rapidly for large m [29]."

In the Appendix of this chapter we have given a SUBROUTINE, written in programming language FORTRAN IV, to evaluate the MLE and the asymptotic variance using the method given above for both the cases discussed in Sections 2.5.1 through 2.5.6. Some important points about the SUBROUTINE are given below :

(1) SUBROUTINE is called by the main program through the name AMLE.

(2) The input parameters of the SUBROUTINE AMLE are explained below :

i) L1 = size of the first sample.

ii) L2 = size of the second sample.

iii) R1 = number of defectives required for the rejection of a lot on the basis of first sample.

iv) R2 = accumulated total of defectives required for the rejection of a lot on the basis of second sample.

v) G1 = number of nondefectives required for the acceptance of a lot on the basis of first sample.

vi) G2 = number of nondefectives required for the acceptance of a lot on the basis of second sample.

vii) TD = Total number of defectives observed when m lots have undergone the inspection under Case-I. It is regarded as zero under Case-II.

viii) TND= Total number of defectives observed when m lots have undergone the inspection under Case-II. It is regarded as zero under Case-I.

ix) J1 = Number of rejected lots when a lot is rejected on the basis of first sample.

x) J2 = number of rejected lots when a lot is rejected on the basis of second sample.

xi) J3 = number of accepted lots when a lot is accepted on the basis of first sample.

xii) J4 = number of accepted lots when a lot is accepted on the basis of second sample.

xiii) AD = number of defectives observed in the accepted lots under Case-I. (Assign a zero value to this parameter under Case-II) .

xiv) RND= number of nondefectives observed in the rejected lots under Case-II. (Assign a zero value to this parameter in Case-I)

(3) The output parameters of the SUBROUTINE AMLE are :

i) P = the MLE of p.

ii) AV = the asymptotic variance of the MLE.

iii) I = the total number of iterations.

(4) We have considered both the cases in the SUBROUTINE. The value of AD will distinguish these two cases. When AD≠0 evaluation will be according to Case-I. When AD = 0, evaluation will be according to Case-II.

The main program is also given along with the SUBROUTINE.

2.5.8 Numerical Example :

Two examples are worked out, one for each case, using the SUBROUTINE given in the appendix. For this purpose we have used EC 1030 computer available at Operations Research Group, Baroda. We have considered the following fully curtailed DSP:

$$n_1 = 5,\ n_2=10,\ r_1=3,\ r_2=5,\ g_1=4,\ g_2=11.$$

Using the model sampling method the above plan was administered on 25 lots each with fraction defective equal to 0.2. The results of the sampling inspection under Case-I and Case-II are respectively given in Table 2.1 and Table 2.2.

Table 2.1 : Results of the Sampling Inspection under Case-I.

Lot No.	Number of defectives a.1	r.1	a.2	r.2	Lot No.	Number of Defectives a.1	r.1	a.2	r.2
1	-	3	-	-	14	1	-	-	-
2	1	-	-	-	15	1	-	-	-
3	1	-	-	-	16	-	-	4	-
4	0	-	-	-	17	0	-	-	-
5	1	-	-	-	18	-	3	-	-
6	-	3	-	-	19	1	-	-	-
7	0	-	-	-	20	1	-	-	-
8	-	-	-	5	21	1	-	-	-
9	0	-	-	-	22	1	-	-	-
10	1	-	-	-	23	0	-	-	-
11	0	-	-	-	24	-	-	2	-
12	0	-	-	-	25	-	-	2	-
13	0	-	-	-					

Table 2.2 : Results of the Sampling Inspection Under Case-II.

Lot No.	Number of Nondefectives a.1	r.1	a.2	r.2	Lot No.	Number of Nondefectives a.1	r.1	a.2	r.2
1	-	1	-	-	14	4	-	-	-
2	4	-	-	-	15	4	-	-	-
3	4	-	-	-	16	-	-	11	-
4	4	-	-	-	17	4	-	-	-
5	4	-	-	-	18	-	0	-	-
6	-	1	-	-	19	4	-	-	-
7	4	-	-	-	20	4	-	-	-
8	-	-	-	5	21	4	-	-	-
9	4	-	-	-	22	4	-	-	-
10	4	-	-	-	23	4	-	-	-
11	4	-	-	-	24	-	-	11	-
12	4	-	-	-	25	-	-	11	-
13	4	-	-	-					

Note: a.1 = lot accepted on the basis of first sample.
a.2 = lot accepted on the basis of second sample.
r.1 = lot rejected on the basis of first sample.
r.2 = lot rejected on the basis of second sample.

It is observed that $m_1=3$, $m_2=1$, $m_3=18$, and $m_4=3$. The numeric values of the input parameters of the SUBROUTINE obtained from the results of the sampling inspection and given plan are as given below :

L1 = 5 R1 = 3 G1 = 4 J1 = 3 J3 = 18

L2 = 10 R2 = 5 G2 = 11 J2 = 1 J4 = 3

Values of TD, AD, TND and RND under different cases are given in Table 2.3. In the same table values of the output parameters are also given.

Table 2.3: Results.

	TD	AD	TND	RND	$\hat{p}$	$V(\hat{p})$	Iteration
Case-I	32	18	-	-	0.21538216	0.00074314	8
Case-II	-	-	112	7	0.22211182	0.00199202	12

It may be noted that the initial value $\hat{p}_o$ (or t) for the first iteration can be given by the following expressions :

Case-I $\quad \hat{p}_o = TD/TU$

Case-II $\quad \hat{p}_o = 1 - \frac{TND}{TU}$

where TU = Total number of units inspected. But information on TU is not available under the Situation-B. Hence TU can be approximated by averaging the possible number of units inspected under events E_i (i=1,2,3,4) of the Situation-A. Let l_i be the average number of units inspected when E_i(i=1,2,3,4) occurs, then the total number of units inspected when m lots have undergone the inspection can be approximated as $\sum_{i=1}^{4} l_i m_i$.

2.5.9 Curtailed MSP under Situation-B :

We have noted in Section 2.4.4 (Situation-A) that the results of fully-curtailed DSP can be generalized to fully--curtailed MSP. Unlike this, it may be pointed out that the results of fully curtailed DSP cannot be generalized to fully-curtailed MSP in this situation, namely, Situation-B.

APPENDIX

```
C PROGRAM FOR COMPUTATION OF MLE OF FRACTION
C DEF P BY ITERATIVE PROCEDURE WHEN CENSORED
C SAMPLING OF TYPE I IS CONSIDERED UNDER FULLY
C CURTAILED DOUBLE SAMPLING PLAN=ASYMPTOTIC
C VAR OF THE MLE IS ALSO COMPUTED-FOR COMPUTA
C TION OF MLE AND ITS ASY VAR SUBROUTINE IS GI
C VEN-PROGRAM BY DKS-DPT OF STATISTICS
      REALN1,N2,NR1,NR2,NG1,NG2,NTD,NTND
      REALM1,M2,M3,M4,NAD,NRND
      READ(5,105)L
  105 FORMAT(1X,I2)
      DO 110 J =1,L
      READ(5,5)N1,N2,NR1,NR2,NG1,NG2,NTD
      READ(5,5)NTND,M1,M2,M3,M4,NAD,NRND
    5 FORMAT(1X,7F5.1)
      CALL AMLE(N1,N2,NR1,NR2,NG1,NG2,NTD,NTND,M1,M2,M3,M4,NAD,NRND,
     1P,V,I)
      WRITE(6,6) P,V,I
    6 FORMAT(1X,9HMLE OF P=,F12.8,2X,6HASYVP=,F14.8,2X,5HITER=,I4)
  110 CONTINUE
C MAIN PROGRAM IS OVER
      STOP
      END
      SUBROUTINE AMLE(L1,L2,R1,R2,G1,G2,TD,TND,J1,J2,J3,J4,AD,RND,P,AV,
     1 I)
C L1 IS THE SIZE OF FIRST SAMPLE, L2 IS THE SIZE OF SECOND SAMPLE.R1,R2,
C G1,G2 ARE THE PREDETERMINED NUMBERS OF A FULLY-CURTAILED DSP, TD IS
C TOTAL NUMBER OF DEFECTIVES AND TND IS TOTAL NUMBER OF NONDEFECTIVES
C OBSERVED IN M LOTS. OUT OF M LOTS J1 LOTS ARE REJECTED ON THE BASIS OF
C FIRST SAMPLE, J2 LOTS ARE REJECTED ON THE BASIS OF 2ND SAMPLES, J3 LOT
C S ARE ACCEPTED ON THE BASIS OF FIRST SAMPLE, AND J4 LOTS ARE ACCEPTED
C ON THE BASIS OF 2ND SAMPLES. J=J1+J2+J3+J4   . AD IS NUMBER OF DEFECTI-
C VES OBSERVED IN ACCEPTED LOTS UNDER CASE-I. RND IS THE NUMBER OF NOND-
C EFECTIVES OBSERVED IN REJECTED LOTS UNDER CASE II. P IS THE MLE AND AV
C IS THE ASYMPTOTIC VARIANCE OF THE MLE. I IS THE ITERATIONS REQUIRED.
      REAL L1,L2,J1,J2,J3,J4
      DIMENSION PR1(10),PL1(10),PI1(10)
      DIMENSION PR2(10),PL2(10),PI2(10)
      DIMENSION PR3(10),PL3(10),PI3(10)
      DIMENSION PR4(10),PL4(10),PI4(10)
      DIMENSION PR5(10),PL5(10),PI5(10)
      F1=(L1+R1)/2.
      F2=(R2-R1+L1+1.+L1+L2)/2.
      F3=(G1+L1)/2.
      F4=((G2-G1+L1+1.)+(L1+L2))/2.
      TTR=F1*J1+F2*J2+F3*J3+F4*J4
      IF(AD.EQ.0.) GOTO 50
      P=TD/TTR
      I=1
      X = 1.
   20 Q=1.-P
      NN=L1
      NN1=L1-1.
      NM=L2
      M11=R1
      M21=NN-M11+1
      M15=L1-G1+1.
      M25=NN-M15+1
      CALL BIN(P,M11,M21,QR1,QL1,QI1)
      CALL BIN(P,M15,M25,QR2,QL2,QI2)
      DELT1=QR1
      PHI1=QI1/DELT1
      T1=Q*G1*QL2*P
      T4=(P**2)*G1*QL2
```

```
      T61=((R1-1.)*Q+(1.-(L1-R1+1.)*(1.+PHI1))*P)
      T6=(L1-R1+1.)*P*PHI1*T61*DELT1
      S1=Ø.
      S2=Ø.
      DELT2=Ø.
      T21=Ø.
      T31=Ø.
      T51=Ø.
      T71=Ø.
      T81=Ø.
      T91=Ø.
      DO 11 J=1,K
      AJ=J
      M12=L1-G1+AJ+1.
      M22=NN-M11+1
      M13=L1-G1+AJ
      M23=NN1-M13+1
      M14=R2-L1+G1-AJ
      M24=NM-M14+1
      M16=L2-G2+G1-AJ+1.
      M26=NM-M16+1
      M17=L2-G2+G1-AJ
      M27=NM-M17+1
      CALL BIN(P,M12,M22,PR1(J),PL1(J),PI1(J))
      CALL BIN(P,M13,M23,PR2(J),PL2(J),PI2(J))
      CALL BIN(P,M14,M24,PR3(J),PL3(J),PI3(J))
      CALL BIN(P,M16,M26,PR4(J),PL4(J),PI4(J))
      CALL BIN(P,M17,M27,PR5(J),PL5(J),PI5(J))
      S=G2-G1+AJ
      S1=S1+S*PI1(J)*PI3(J)
      S2=S2+PI2(J)*PR3(J)
      DELT2=DELT2+PI1(J)*PR3(J)
      T21=T21+PI1(J)*(G2-G1+AJ)*PL5(J)
      T31=T31+PI1(J)*(L1-G1+AJ)*PL4(J)
      T51=T51+PI1(J)*PL4(J)
      T71=T71+S*PI1(J)*PI3(J)
      T81=T81+S*PI2(J)*PI3(J)
      T92=(L1-G1+AJ-1.)*Q-(G1-AJ)*P
      T91=T91+T92*PI2(J)*PR3(J)
   11 CONTINUE
      PHI2=(S1+L1*S2-L1*DELT2)/DELT2
      T2=Q*P*T21
      T3=(Q**2)*T31
      T5=(P**2)*G2*T51
      T7=P*((R1-1.)*Q-G2*P)*T71
      T8=(P**2)*L1*T81
      T9=L1*P*T91
      T1Ø=(P**2)*PHI2*DELT2*(PHI2+L1-1.)
      H1=T1+T2+T3+T4+T5-T6-T7-T8-T9+T1Ø
      AV=((P**2)*(Q**2))/((J1+J2+J3+J4)*H1)
      IF(X.EQ.Ø.) GO TO 21Ø
      DN1=G1*J3+G2*J4
      DN2=L1-R1+1
      BRK=J1*DN2*P*PHI1+J2*P*PHI2+AD*Q-DN1*P
      DERI=BRK/(P*Q)
      PE=P+DERI*AV
      DIF=ABS(P-PE)
      IF(DIF.LT.Ø.ØØØØØ5) GOTO 15
      P=PE
      I=I+1
      GOTO 2Ø
   15 P=PE
      X = Ø.
      I=I+1
```

```
      GO TO 20
  210 GOTO 100
   50 Q=TND/TTR
      P=1.-Q
      I=1
      X = 1.
   70 Q=1.-P
      NN=L1
      NN1=L1-1.
      NM=L2
      M11=L1-G1+1.
      M21=NN-M11+1
      M15=R1
      M25=NN-M15+1
      M16=R1+1.
      M26=NN-M16+1
      CALL BIN(P,M11,M21,QR1,QL1,QI1)
      CALL BIN(P,M15,M25,QR2,QL2,QI2)
      CALL BIN(P,M16,M26,QR3,QL3,QI3)
      DLT1=QL1
      T1=(Q**2)*R1*QR2
      SI1=QI1/DLT1
      T3=P*Q*R1*QR3
      T61=(L1-G1)*Q+(1.-G1+SI1)*P
      T6=DLT1*P*G1*(SI1*T41)
      S1=0.
      S2=0.
      DLT2=0.
      T21=0.
      T41=0.
      T51=0.
      T71=0.
      T81=0.
      T91=0.
      DO 13 J=1,K
      AJ=J
      M12=L1-G1+AJ+1.
      M22=NN-M12+1
      M13=L1-G1+AJ
      M23=NN1-M13+1
      M14=L2-G2+G1-AJ+1.
      M24=NM-M14+1
      M17=R2-L1+G1-AJ
      M27=NM-M12+1
      M18=R2-L1+G1-AJ+1.
      M28=NM-M18+1
      CALL BIN(P,M12,M22,PR1(J),PL1(J),PI1(J))
      CALL BIN(P,M13,M23,PR2(J),PL2(J),PI2(J))
      CALL BIN(P,M14,M24,PR3(J),PL3(J),PI3(J))
      CALL BIN(P,M17,M27,PR4(J),PL4(J),PI4(J))
      CALL BIN(P,M18,M28,PR5(J),PL5(J),PI5(J))
      S=(G2-G1+AJ)
      S1=S1+S*PI1(J)*PI3(J)
      S2=S2+PI2(J)*PL3(J)
      DLT2=DLT2+PI1(J)*PL3(J)
      T21=T21+PI1(J)*PR4(J)
      T41=T41+PI1(J)*(R2-L1+G1-AJ)*PR5(J)
      T51=T51+PI1(J)*(G1-AJ)*PR4(J)
      T72=((R1-1.)*Q-G2*P)*S
      T71=T71+T72*PI1(J)*PI3(J)
      T81=T81+S*PI2(J)*PI3(J)
      T92=((L1-G1+AJ-1.)*Q-(G1-AJ)*P)
      T91=T91+T92*PI2(J)*PL3(J)
   13 CONTINUE
```

```
      SI2=(S1-L1*S2+L1*DLT2)/DLT2
      T2=(Q**2)*R2*T21
      T4=P*Q*T41
      T5=(P**2)*T51
      T7=P*T71
      T8=L1*(P**2)*T81
      T9=L1*P*T91
      T10=(P**2)*DLT2*SI2*(SI2-L1+1.)
      H2=T1+T2+T3+T4+T5+T6+T7+T8-T9+T10
      AV=((P**2)*(Q**2))/((J1+J2+J3+J4)*H2)
      IF(X.EQ.0.) GO TO 100
      DN4=J1*R1+J2*R2
      DN5=J3*P*G1*SI1
      DN6=J4*P*SI2
      BRK=DN4*Q-P*RND-DN5-DN6
      DERI=BRK/(P*Q)
      PE=P+DERI*AV
      DIF=ABS(P-PE)
      IF(DIF.LT.0.000005) GOTO 60
      P=PE
      I=I+1
      GOTO 70
   60 P=PE
      X = 0.
      I=I+1
      GO TO 70
  100 RETURN
      END
      SUBROUTINE BIN(X,MM,NT,P,PP,PIND)
C     PROGRAM FOR CALCU INDIVIDUAL AND CUMULATIVE
C     PROBABILITY  OF BINOMIAL DISTRIBUTION
34    DIMENSION AA(301)
      DOUBLE PRECISION AA,RN,AANOT,RK
      NN=NT+MM-1
      RN=NN
      AANOT=(1.-X)**RN
      AA(1)=(RN*X*AANOT)/(1.-X)
      DO 25 K=2,NN
      RK=K
25    AA(K)=(X*(RN-RK+1.)*AA(K-1))/(RK*(1.-X))
      P=0
      DO 4 I=MM,NN
4     P=P+AA(I)
      PP=1.-P
      M=MM-1
      IF(M.EQ.0) GO TO 6
      PIND=AA(M)
      GO TO 7
6     PIND=AANOT
7     CONTINUE
C     P GIVES PROB FROM M+1 TO N
C     PP GIVES PROB  FROM 0 TO M
C     PIND GIVES PROB AT M
C     MM=M+1
C     NT=NUMBER OF TERMS IN SUM FROM M+1 TO N
C     NT=N-MM+1=N-M
C     MM+NT = BINOMIAL INDEX N +1
      RETURN
      END
```