CHAPTER V

CURTAILED THREE CLASS ATTRIBUTES SAMPLING PLAN - SINGLE SAMPLING PLAN

5.1 In this chapter we consider single three class attributes sampling plan in which the unit of a lot submitted for the inspection is classified as either bad, marginal or good. Semi-curtailed and fully-curtailed forms of single three class attributes sampling plan are introduced. The maximum likelihood estimators of the proportion of bad units and that of marginal units under these forms are obtained. The efficiency of these estimators in terms of the saving in inspection is discussed.

5.2 Three Class Attributes Sampling Plan and Curtailment in the Inspection :

Recently Bray, Lyon, and Burr [4] introduced a new type of acceptance sampling plan called a three class attributes sampling plan where the unit of a lot submitted for inspection is classified into one of the three classes: bad, marginal, and good. The sampling plan is defined by four numbers : the lot size N, the sample size n, the acceptance numbers a_1 and

a_2. The decision rule is to accept the lot if the number of bad units in the sample does not exceed a_2 and the sum of the numbers of bad units and marginal units in the sample does not exceed a_1. The lot is rejected otherwise.

It is possible to have curtailment in the inspection in case of a three class attributes sampling plan. When the inspection is in progress one can know before hand whether a lot is going to be rejected or accepted having observed a sufficient number of bad units, or nongood units, or good units and nonbad units etc. Here, a nongood unit is defined as a unit which is classified as either bad or marginal and a nonbad unit is defined as a unit which is classified as either good or marginal. It is desirable, as explained in Chapter I, to use curtailed sampling when it is necessary to know only whether a lot is to be rejected or accepted or when the inspection is destructive or expensive.

In the following sections of this chapter we introduce semi-curtailed and fully-curtailed single three class attributes sampling plans. The expressions for the average sample number (ASN) for these curtailed sampling plans are obtained. Furthermore, we give the maximum likelihood estimators of the proportion of bad units and the proportion of marginal units. The asymptotic variances of these estimators are also obtained. The relation between the percent saving in inspection and the efficiency of the estimators is established. A numerical example to illustrate the percent saving in inspection is provided.

5.3 Notations and Assumptions :

For convenience, we give in this section the important symbols, notations etc. used through out in this chapter and state the assumptions made.

N = Number of units in a lot.

n = Maximum number of units to be inspected.

Y = Number of units actually inspected.

D_2 = Number of bad units found in Y.

D_1 = Number of marginal units found in Y.

$D_0 = Y-D_1-D_2$ = Number of good units found in Y.

p_2 = Proportion of bad units in the production run.

p_1 = Proportion of marginal units in the production run.

$p_0 = 1-p_1-p_2$ = Proportion of good units in the production run.

a_2 = Maximum allowable number of bad units in the sample.

a_1 = Maximum allowable number of nongood units in the sample.

$k_2 = a_2+1$ = Rejection number for bad units.

$k_1 = a_1+1$ = Rejection number for nongood units.

I = Indicator Variable.

$p'_1 = p_1/(p_1+p_o)$.

$p''_1 = p_1/(p_1+p_2)$

$p'_o = p_o/(p_1+p_o)$

$p''_2 = p_2/(p_1+p_2)$.

m = Number of lots submitted for inspection.

(TB) = Total number of bad units observed in m lots.

(TM) = Total number of marginal units found in m lots.

(TG) = Total number of good units found in m lots.

(TU) = (TB) + (TM) + (TG)

= Total number of units inspected in m lots.

$$B(r;n,p) = \sum_{x=o}^{r} \binom{n}{x} p^x q^{n-x} \quad .$$

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x} = B(x;n,p)-B(x-1;n,p).$$

We have assumed that p_1 and p_2 remain constant over the entire production run, the lot size is large, and the probability of any inspected unit to be marginal is p_1 and to be bad is p_2. This applies to the Type B situation of Dodge and Romig [10]; hence, the lot size N does not subsequently appear in this chapter.

5.4 Semi-Curtailed Single Three Class Attributes Sampling Plan :

5.4.1 Statement of the Plan and the Probability Function :

The statement of the semi-curtailed single three class attributes sampling plan is as given below :

Inspect randomly selected units of a lot one at a time till one of the following four mutually exclusive and exhaustive events $E_i (i=1,2,3,4)$ occurs :

(E_1) n units are inspected and $D_1+D_2 \leq k_1-1$ and $D_2 \leq k_2-1$,

(E_2) k_2 bad units are observed and $D_1 \leq k_1-k_2-1$,

(E_3) k_1 nongood units are observed and $D_2 \leq k_2-1$,

(E_4) k_2 bad units and k_1-k_2 marginal units are observed.

The decision rule is to accept the lot if E_1 occurs and to reject the lot if any one of the events E_i (i=2,3,4) occurs. Here k_1, k_2 and n are predetermined integers such that $1 \leq k_2 \leq k_1 << n$. The curtailment in the inspection is caused by finding either enough number of bad units or enough number of nongood units.

The probability function associated with the random phenomenon prevailing in the above curtailed sampling plan is as given below :

$$P(Y=y,\ D_1=d_1,\ D_2=d_2,\ I=i) = \begin{cases} f_1(y,d_1,d_2;p_1,p_2) & y=n \\ & d_1 = 0,1,\ldots,k_1-1-d_2 \\ & d_2 = 0,1,\ldots,k_2-1 \\ & i = 1 \\ f_2(y,d_1,d_2;p_1,p_2) & y = k_2,k_2+1,\ldots,n \\ & d_1 = 0,1,\ldots,k_1-k_2-1 \\ & d_2 = k_2 \\ & i = 2 \\ f_3(y,d_1,d_2;p_1,p_2) & y = k_1,k_1+1,\ldots,n \\ & d_1 = k_1-d_2 \\ & d_2 = 0,1,\ldots,k_2-1 \\ & i = 3 \\ f_4(y,d_1,d_2;p_1,p_2) & y = k_1,k_1+1,\ldots,n \\ & d_1 = k_1-k_2 \\ & d_2 = k_2 \\ & i = 4 \end{cases} \qquad \ldots(5.4.1)$$

where

$$f_1(y,d_1,d_2;p_1,p_2) = \frac{n!\ p_o^{n-d_1-d_2}\ p_1^{d_1}\ p_2^{d_2}}{d_1!\ d_2!(n-d_1-d_2)!}$$

$$f_2(y,d_1,d_2;p_1,p_2) = \frac{(y-1)!\,p_o^{y-k_2-d_1}\ p_1^{d_1}\ p_2^{k_2}}{d_1!(k_2-1)!(y-k_2-d_1)!}$$

$$f_3(y,d_1,d_2;p_1,p_2) = \frac{k_1(y-1)!\ p_o^{y-k_1}\ p_1^{k_1-d_2}\ p_2^{d_2}}{(k_1-d_2)!\ d_2!\ (y-k_1)!}$$

and

$$f_4(y,d_1,d_2;p_1,p_2) = \frac{(y-1)!\,p_o^{y-k_1}\,p_1^{k_1-k_2}\ p_2^{k_2}}{(k_1-k_2)!(k_2-1)!(y-k_1)!}.$$

While calculating the probabilities from the above probability function one assigns probability zero whenever factorial of a negative quantity, say $r!$ with $r < 0$ appears. In fact, this remark holds for any term that may appear in further development.

Furthermore, it may be noted that the function f_1 is obtained considering the trinomial distribution and the function f_j $(j=2,3,4)$ are obtained considering the negative trinomial distributions with appropriate parameters. For instance, f_3 is obtained by addition of the following probabilities: (A) Probability that one has to inspect y units to have (i) d_2 $(1 \leq d_2 \leq k_2-1)$ bad units, (ii) k_1-d_2 marginal units, and (iii) $y-k_1$ good units such that the yth unit inspected happens to be a bad unit and (B) probability that one has to inspect y units to have (i) $d_2 (0 \leq d_2 \leq k_2 - 1)$ bad units, (ii) $k_1 - d_2$ marginal units, and (iii) $y - k_1$ good units such that the yth unit inspected happens to be a marginal unit.

5.4.2 Operating Characteristic Function :

The probability of acceptance of a lot gives the operating characteristic function of a given plan. In case of a semi-curtailed single three class attributes sampling plan event E_1 leads to an acceptance of a lot. Denoting the probability of acceptance of a lot for the plan under consideration by $P_a(\text{semi})$, the operating characteristic function is given as

$$P_a(\text{semi}) = P(E_1)$$

$$= \sum_{d_2=o}^{k_{2-1}} \sum_{d_1=o}^{k_1-1-d_2} f_1(y, d_1, d_2; p_1, p_2)$$

$$= \sum_{d_2=o}^{k_2-1} b(d_2; n, p_2)\, B(k_1-d_2-1; n-d_2, p_1') \qquad \ldots(5.4.2)$$

The expression (5.4.2) can be calculated with the use of the usual binomial tables such as [34], [43]. It may be noted that the expression (5.4.2) involves an individual term of the binomial distribution which can be evaluated as a difference between the two cumulated binomial probabilities as introduced in the Section 5.3 on notations and assumptions. Furthermore, it may be noted that the expression for P_a(semi) given by (5.4.2) is the same as (3) of [4] which ought to be the case since the events which lead to the acceptance of a lot under the uncurtailed sampling plan and the semi-curtailed sampling plan are identical. Hence the probability of acceptance under uncurtailed single three class attributes sampling plan is equal to that under semi--curtailed single three class attributes sampling plan. This fact leads to the following identity which gives a relation between a trinomial and a negative trinomial distributions :

$$\sum_{y=k_2}^{n} \sum_{d_1=o}^{k_1-k_2} \frac{(y-1)!\, p_o^{y-k_2-d_1}\, p_1^{d_1}\, p_2^{k_2}}{d_1!\,(k_2-1)!\,(y-k_2-d_1)!}$$

$$+\sum_{y=k_1}^{n} \sum_{d_2=o}^{k_2-1} \frac{k_1(y-1)!\, p_o^{y-k_1}\, p_1^{k_1-d_2}\, p_2^{d_2}}{(k_1-d_2)!\, d_2!\,(y-k_1)!}$$

$$= \sum_{d_2=o}^{k_2-1} \sum_{d_1=k_1-d_2}^{n-d_2} \frac{n!\, p_o^{n-d_1-d_2}\, p_1^{d_1}\, p_2^{d_2}}{d_1!\, d_2!\,(n-d_1-d_2)!}$$

$$+\sum_{d_2=k_2}^{n} \sum_{d_1=o}^{n-d_2} \frac{n!\, p_o^{n-d_1-d_2}\, p_1^{d_1}\, p_2^{d_2}}{d_1!\, d_2!\,(n-d_1-d_2)!} \qquad \ldots(5.4.3)$$

5.4.3 The Average Sample Number :

The average sample number is defined as ASN=E(Y). It follows from (5.4.1) that

$$E(Y) = \sum_{y} y \sum_{i} \sum_{d_1} \sum_{d_2} f_i(y,d_1,d_2;p_1,p_2)$$

$$= n \sum_{d_1} \sum_{d_2} f_1(n,d_1,d_2;p_1,p_2)$$

$$+ \sum_{y} y \sum_{d_1} \sum_{d_2} f_2(y,d_1,d_2;p_1,p_2)$$

$$+ \sum_{y} y \sum_{d_1} \sum_{d_2} f_3(y,d_1,d_2;p_1,p_2)$$

$$+ \sum_{y} y \sum_{d_1} \sum_{d_2} f_4(y,d_1,d_2\cdot;p_1,p_2)$$

On further evaluation we have the following expression for the ASN :

$$\text{ASN(semi)} = n \sum_{d_2=o}^{k_2-1} b(d_2;n,p_2)\ B(k_1-d_2-1;n-d_2,p_1')$$

$$+\frac{k_2}{p_2''} \sum_{t=k_2+1}^{n+1} \{B(k_2;t-1,p_2)-B(k_2;t,p_2)\}$$

$$\cdot B(k_1-k_2;t-k_2-1,p_1')$$

$$+\frac{k_1}{1-p_o}\ B(k_2-1;k_1,p_2'')\{1-B(k_1;n+1,1-p_o)\}$$

...(5.4.4)

where ASN(semi) denotes the ASN under semi-curtailed single three class attributes sampling plan. The expression (5.4.4) can be evaluated with the help of usual binomial tables such as [34], [43] etc.

5.4.4 The Maximum Likelihood Estimators and the Asymptotic Variance :

Let m lots have undergone the inspection in accordance with the semi-curtailed three class attributes sampling plan. Let the event E_i (i=1,2,3,4) has occured m_i (i=1,2,3,4) times. Thus, m_1 lots are accepted and $m-m_1$ lots are rejected. Among $m-m_1$ lots m_i (i=2,3,4) lots are rejected due to the occurrence of the event E_i (i=2,3,4) m_i times. The m 4-tuple observations associated with the random variables Y, D_1, D_2, I be displayed in four groups for convenience as given below :

$$\begin{array}{ll} (Y_{1j}, D_{11j}, D_{21j}, 1) & j = 1,2,\ldots,m_1 \\ (Y_{2j}, D_{12j}, D_{22j}, 2) & j = 1,2,\ldots,m_2 \\ (Y_{3j}, D_{13j}, D_{23j}, 3) & j = 1,2,\ldots,m_3 \\ (Y_{4j}, D_{14j}, D_{24j}, 4) & j = 1,2,\ldots,m_4 \end{array} \qquad \ldots(5.4.5)$$

Then the likelihood function, L, for the above sample of m observations is as follows :

$$L = \prod_{i=1}^{4} \prod_{j=1}^{m_i} f_i(y_{ij}, d_{1ij}, d_{2ij}; p_1, p_2) \qquad \ldots(5.4.6)$$

Taking logarithm of the likelihood, differentiating the logarithm partially with respect to p_1 and p_2 and equating the partial derivatives to zero, we get the following maximum likelihood equations in p_1 and p_2 with little algebra :

$$[(TG) + (TM)]\, p_1 + (TM)\, p_2 = (TM) \qquad \ldots(5.4.7)$$

$$(TB)\, p_1 + [(TG) + (TB)]\, p_2 = (TB) \qquad \ldots(5.4.8)$$

where the physical meaning of (TG), (TB), (TM) is as given in the Section 5.3. However, the expressions of these quantities in terms of the sample display (5.4.5) are given below :

$$(TG) = (nm_1 - \sum_{j=1}^{m_1} d_{11j} - \sum_{j=1}^{m_1} d_{21j}) + (\sum_{j=1}^{m_2} y_{2j} - k_2 m_2 - \sum_{j=1}^{m_2} d_{12j}) + (\sum_{j=1}^{m_3} y_{3j} - k_1 m_3) + (\sum_{j=1}^{m_4} y_{4j} - k_1 m_4)$$

$$(TM) = \sum_{j=1}^{m_1} d_{11j} + \sum_{j=1}^{m_2} d_{12j} + (k_1 m_3 - \sum_{j=1}^{m_3} d_{23j}) + (k_1 - k_2) m_4$$

$$(TB) = \sum_{j=1}^{m_1} d_{21j} + k_2 m_2 + \sum_{j=1}^{m_3} d_{23j} + k_2 m_4 \; .$$

Solving the equations (5.4.7) and (5.4.8) for p_1 and p_2, the maximum likelihood estimators for p_1 and p_2 are

$$\hat{p}_1 = (TM)/(TU) \qquad \ldots(5.4.9)$$

$$\hat{p}_2 = (TB)/(TU) \qquad \ldots(5.4.10)$$

We note the following results about the expectations of (TB), (TM), (TG) which are useful in finding the asymptotic variances and covariance of the $\hat{p}_1$ and $\hat{p}_2$.

$$E(TB) = p_2 \; E(TU) = p_2 \; m(ASN)$$

$$E(TM) = p_1 \; E(TU) = p_1 \; m(ASN)$$

$$E(TG) = p_o \; E(TU) = p_o \; m(ASN)$$

where ASN = ASN (semi) given by the expression (5.4.4) when semi-curtailed three class attributes sampling plan is under consideration.

Then we find that

$$-E\left(\frac{\partial^2 \log L}{\partial p_1^2}\right) = \frac{p_o^2 \; E(TM) + p_1^2 \; E(TG)}{p_o^2 \; p_1^2}$$

$$= \frac{m(ASN)(1-p_2)}{p_o p_1}$$

$$= \phi_{11}$$

$$-E\left(\frac{\partial^2 \log L}{\partial p_2^2}\right) = \frac{p_o^2 \; E(TB) + p_2^2 \; E(TG)}{p_o^2 \; p_2^2}$$

$$= \frac{m(ASN) \; (1-p_1)}{p_o \; p_2}$$

$$= \phi_{22}$$

$$- E \left(\frac{\partial^2 \log L}{\partial p_1 \partial p_2} \right) = \frac{E(TG)}{p_o^2}$$

$$= \frac{m(ASN)}{p_o}$$

$$= \phi_{12} = \phi_{21}$$

$$\therefore \Delta = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix}$$

$$= \frac{m^2(ASN)^2}{p_o p_1 p_2} \qquad \ldots(5.4.11)$$

The asymptotic variance and covariance matrix of $\hat{p}_1$ and $\hat{p}_2$, $[\phi_{ij}]^{-1}$, leads to the following asymptotic variances and covariance

$$V(\hat{p}_1) = \frac{\phi_{22}}{\Delta} = \frac{p_1(1-p_1)}{m(ASN)} \qquad \ldots(5.4.12)$$

$$V(\hat{p}_2) = \frac{\phi_{11}}{\Delta} = \frac{p_2(1-p_2)}{m(ASN)} \qquad \ldots(5.4.13)$$

and $Cov(\hat{p}_1, \hat{p}_2) = - \phi_{12}/\Delta$

$$= - \frac{p_1 p_2}{m(ASN)} \qquad \ldots(5.4.14)$$

5.5 Fully-curtailed Single Three Class Attributes Sampling Plan :

5.5.1 Statement of the Plan and the Probability Function :

The statement of the fully-curtailed single three class attributes sampling plan is as follows :

Inspect randomly selected units of a lot one at a time till one of the following six mutually exclusive and exhaustive events $e_i (i=1,2,3,\ldots,6)$ occurs :

(e_1) $n-k_2+1$ nonbad units are observed and $D_0 > n-k_1+1$,

(e_2) $n-k_1+1$ good units are observed and $D_1 > k_1-k_2$,

(e_3) $n-k_1+1$ good units and $n-k_2+1$ nonbad units are observed,

(e_4) k_2 bad units are observed and $D_1 \leq k_1-k_2-1$,

(e_5) k_1 nongood units are observed and $D_2 \leq k_2-1$,

(e_6) k_2 bad units and k_1-k_2 marginal units are observed.

Events e_4, e_5, and e_6 are respectively the events, E_2, E_3, and E_4 of the semi-curtailed single three class attributes sampling plan described in ~~in~~ Section 5.4.1. The decision rule is to accept the lot if any one of the events e_i $(i=1,2,3)$ occurs and to reject the lot if any one of the events e_i $(i=4,5,6)$ occurs.

Though not apparent, it is evident that not more than n units of a lot are required to be inspected to accept or reject the lot.

The probability function associated with the random phenomenon prevailing in the above curtailed sampling plan is as given below :

$$P(Y=y,\ D_o=d_o,\ D_1=d_1,\ D_2=d_2,\ I=i) =$$

$$\begin{cases}
h_1(y,d_o,d_1;p_1,p_2) & y=n-k_2+1,\ldots,n \\
 & d_o=n-k_1+2,\ldots,n-k_2+1 \\
 & d_1=n-k_2+1-d_o \\
 & d_2=y-(n-k_2+1) \\
 & i=1 \\
h_2(y,d_o,d_1;p_1,p_2) & y=n-k_2+1,\ldots,n \\
 & d_o=n-k_1+1 \\
 & d_1=k_1-k_2+1,\ldots,n \\
 & d_2=y-(n-k_1+1)-d_1 \\
 & i=2 \\
h_3(y,d_o,d_1;p_1,p_2) & y=n-k_2+1,\ldots,n \\
 & d_o=n-k_1+1 \\
 & d_1=k_1-k_2 \\
 & d_2=y-(n-k_2+1) \\
 & i=3 \\
h_4(y,d_1,d_2;p_1,p_2) & y=k_2,\ldots,n \\
 & d_o=y-d_1-k_2 \\
 & d_1=0,1,\ldots,k_1-k_2-1 \\
 & d_2=k_2 \\
 & i=4
\end{cases}$$

cont...

$$\begin{cases} h_5(y,d_1,d_2;p_1,p_2) & y=k_1,\ldots,n \\ & d_o=y-k_1 \\ & d_1=k_1-d_2 \\ & d_2=0,1,\ldots,k_2-1 \\ & i=5 \\ h_6(y,d_1,d_2;p_1,p_2) & y=k_1,\ldots,n \\ & d_o=y-k_1 \\ & d_1=k_1-k_2 \\ & d_2=k_2 \\ & i=6 \end{cases} \qquad \ldots(5.5.1)$$

where

$$h_1(y,d_o,d_1;p_1,p_2) = \frac{(n-k_2+1)(y-1)!\; p_o^{d_o}\; p_1^{n-k_2+1-d_o}\; p_2^{y-(n-k_2+1)}}{(n-k_2+1-d_o)!\,[y-(n-k_2+1)]!\; d_o!}$$

$$h_2(y,d_o,d_1;p_1,p_2) = \frac{(y-1)!\; p_o^{n-k_1+1}\; p_1^{d_1}\; p_2^{y-(n-k_1+1)-d_1}}{d_1!\,[y-(n-k_1+1)-d_1]!\,(n-k_1)!}$$

$$h_3(y,d_o,d_1;p_1,p_2) = \frac{(n-k_2+1)(y-1)!\; p_o^{n-k_1+1}\; p_1^{k_1-k_2}\; p_2^{y-(n-k_2+1)}}{(k_1-k_2)!\,[y-(n-k_2+1)]!\;(n-k_1+1)!}$$

and

$$h_j(y,d_1,d_2;p_1,p_2) = f_{j-2}(y,d_1,d_2;p_1,p_2) \text{ for } j=4,5,6$$

where $f_{j-2}(\cdot)$ $(j=4,5,6)$ are defined in the Section 5.4.1. It may be noted that the function $h_j(\cdot)$ $(j=1,2,3)$ are obtained considering the negative trinomial distributions with appropriate parameters as is done in obtaining $f_i(.)(i=2,3,4)$.

5.5.2 Operating Characteristic Function :

Let P_a(fully) be the probability of acceptance for a fully-curtailed single three class attributes sampling plan. Thus,

$$P_a(\text{fully}) = P(e_1) + P(e_2) + P(e_3)$$

$$= \sum_{y=n-k_2+1}^{n} \sum_{d_o=n-k_1+1}^{n-k_2+1} \frac{(n-k_2+1)(y-1)!\; p_o^{d_o}\; p_1^{n-k_2+1-d_o}\; p_2^{y-(n-k_2+1)}}{(n-k_2+1-d_o)!\,[y-(n-k_2+1)]!\; d_o!}$$

$$+ \sum_{y=n-k_2+1}^{n} \sum_{d_1=k_1-k_2+1}^{n} \frac{(y-1)!\; p_o^{n-k_1+1}\; p_1^{d_1}\; p_2^{y-(n-k_1+1)-d_1}}{d_1!\,[y-(n-k_1+1)-d_1]!\;(n-k_1)!}$$

...(5.5.2)

We have noted that the events e_4, e_5, and e_6 of a fully--curtailed single three class attributes sampling plan are, respectively, identical with the events E_2, E_3, and E_4 of a semi-curtailed sampling plan. This will result into the fact that the probability of acceptance in case of a fully--curtailed sampling plan is equal to the probability of acceptance in case of a semi-curtailed sampling plan. But we have observed that the probability of acceptance for a semi-curtailed and an uncurtailed sampling plan is same, Section 5.4.2. This implies that the probability of acceptance for a fully-curtailed and an uncurtailed sampling plan is also same. Thus in case of a three class attributes sampling plan the probability of acceptance remains same for all the three forms of the inspection. This fact leads to the following identity, similar to (5.4.3):

$$\sum_{y=n-k_2+1}^{n} \sum_{d_0=n-k_1+1}^{n-k_2+1} \frac{(n-k_2+1)(y-1)!\, p_0^{d_0}\, p_1^{n-k_2+1-d_0}\, p_2^{y-(n-k_2+1)}}{(n-k_2+1-d_0)!\,[y-(n-k_2+1)]!\, d_0!}$$

$$+ \sum_{y=n-k_2+1}^{n} \sum_{d_1=k_1-k_2+1}^{n} \frac{(y-1)!\, p_0^{n-k_1+1}\, p_1^{d_1}\, p_2^{y-(n-k_1+1)-d_1}}{d_1!\,[y-(n-k_1+1)-d_1]!\,(n-k_1)!}$$

$$= \sum_{d_2=0}^{k_2-1} \sum_{d_1=0}^{k_1-1-d_2} \frac{n!\, p_0^{n-d_1-d_2}\, p_1^{d_1}\, p_2^{d_2}}{d_1!\, d_2!\,(n-d_1-d_2)!} \qquad \ldots(5.5.3)$$

It may be noted that in addition to the following obvious relation

$$\sum_{d_2=0}^{k_2-1} \sum_{d_1=0}^{n-d_2} \frac{n!\, p_0^{n-d_1-d_2}\, p_1^{d_1}\, p_2^{d_2}}{d_1!\, d_2!\,(n-d_1-d_2)!}$$

$$= \sum_{y=n+1}^{\infty} \sum_{d_1=0}^{y-k_2} \frac{(y-1)!\, p_0^{y-k_2-d_1}\, p_1^{d_1}\, p_2^{k_2}}{d_1!\,(k_2-1)!\,(y-k_2-d_1)!} \qquad \ldots(5.5.4)$$

between trinomial and negative trinomial distributions, the relations given by (5.4.3) and (5.5.3) may be regarded as additional relations.

5.5.3 The Average Sample Number :

Following the procedure of the Section 5.4.3, the ASN of a fully-curtailed single three class attributes sampling plan can be derived. Denote the ASN of a fully-curtailed single three class attributes sampling plan by ASN(fully). Then, the expression for the ASN is as given below :

$$\begin{aligned}\text{ASN(fully)} = {} & \frac{n-k_2+1}{1-p_2}\{1-B(n-k_1;n-k_2+1,p_o')\} \\ & .\{1-B(n-k_2+1;n+1,1-p_2)\} \\ & + \frac{n-k_1+1}{p_o}\sum_{t=n-k_2+2}^{n+1}\{B(n-k_1+1;t-1,p_o) \\ & -B(n-k_1+1;t,p_o)\} \\ & .\{B(k_1-1;t-(n-k_1+2),p_1'')-B(k_1-k_2;t-(n-k_1+2),p_1'')\} \\ & + \frac{k_2}{p_2}\sum_{t=k_2+1}^{n+1}\{B(k_2;t-1,p_2)-B(k_2;t,p_2)\} \\ & B(k_1-k_2;t-k_2-1,p_1') \\ & + \frac{k_1}{1-p_o}B(k_2-1;k_1,p_2'')\{1-B(k_1;n+1,1-p_o)\} \quad \ldots(5.5.5)\end{aligned}$$

Here the relation $B(r;n,p) = 1-B(n-r-1;n,1-p)$ may be found useful while using the binomial table [43] to calculate the ASN.

5.5.4 The Maximum Likelihood Estimators and Asymptotic Variance :

As usual, consider the m lots for inspection according to the fully-curtailed single three class attributes sampling plan. The m 4-tuple observations associated with the random variables Y, D_0, D_1, D_2, I be displayed in 6 groups for convenience as given below :

$(Y_{1j}, D_{o1j}, D_{11j}, 1)$ $j=1,2,\ldots,m_1$

$(Y_{2j}, D_{o2j}, D_{12j}, 2)$ $j=1,2,\ldots,m_2$

$(Y_{3j}, D_{o3j}, D_{13j}, 3)$ $j=1,2,\ldots,m_3$

$(Y_{4j}, D_{14j}, D_{24j}, 4)$ $j=1,2,\ldots,m_4$

$(Y_{5j}, D_{15j}, D_{25j}, 5)$ $j=1,2,\ldots,m_5$

$(Y_{6j}, D_{16j}, D_{26j}, 6)$ $j=1,2,\ldots,m_6$ ...(5.5.6)

where m_i (i=1,2,3) lots are accepted due to the occurrence of the event e_i (i=1,2,3) m_i times and m_i (i=4,5,6) lots are rejected due to the occurrence of event e_i (i=4,5,6) m_i times such that $\sum_{i=1}^{6} m_i = m$. Then the likelihood function, L, for the sample of m observations is as follows :

$$L = \prod_{i=1}^{3} \prod_{j=1}^{m_i} h_i(y_{ij}, d_{oij}, d_{1ij}; p_1, p_2)$$

$$\prod_{i=4}^{6} \prod_{j=1}^{m_i} h_i(y_{ij}, d_{1ij}, d_{2ij}; p_1, p_2) \qquad \ldots(5.5.7)$$

Then, following the procedure of the Section 5.4.4, the expressions for the maximum likelihood estimators of p_1 and p_2 and for the elements of the asymptotic variance-covariance matrix are exactly same as those given by (5.4.9),(5.4.10), (5.4.12), (5.4.13), and (5.4.14). Of course, the quantities (TB), (TM) and (TU), though they carry the same physical meaning as is prevailed there, are to be expressed in terms of the sample display (5.5.6) and they are as follow :

$$(TG) = \sum_{j=1}^{m_1} d_{o1j} + (n-k_1+1)m_2 + (n-k_1+1)\, m_3$$
$$+ \left[\sum_{j=1}^{m_4} y_{4j} - k_2 m_4 - \sum_{j=1}^{m_4} d_{14j}\right]$$
$$+ \left[\sum_{j=1}^{m_5} y_{5j} - k_1 m_5\right] + \left[\sum_{j=1}^{m_6} y_{6j} - k_1 m_6\right]$$

$$(TB) = \left[\sum_{j=1}^{m_1} y_{1j} - (n-k_2+1)m_1\right] + \left[\sum_{j=1}^{m_2} y_{2j} - (n-k_1+1)m_2 - \sum_{j=1}^{m_2} d_{12j}\right]$$
$$+ \left[\sum_{j=1}^{m_3} y_{3j} - (n-k_2+1)m_3\right] + k_2 m_4 + \sum_{j=1}^{m_5} d_{25j} + k_2 m_6$$

$$(TM) = \left[(n-k_2+1)m_1 - \sum_{j=1}^{m_1} d_{01j}\right] + \sum_{j=1}^{m_2} d_{12j} + (k_1-k_2)m_3 + \sum_{j=1}^{m_4} d_{14j}$$
$$+ (k_1 m_5 - \sum_{j=1}^{m_5} d_{25j}) + (k_1-k_2)\, m_6 \ .$$

5.6 Saving in Inspection and Loss in Efficiency :

It is obvious that the ASN for the single three class attributes sampling plan is n. Furthermore, it can be easily varified that the ASN's of the curtailed sampling plans obtained in sections 5.4.3 and 5.5.3 satisfy the following relation

$$ASN(fully) \leq ASN(semi) \leq ASN(uncu) \qquad ...(5.6.1)$$

Then the percent saving in inspection as one passes from an uncurtailed sampling plan to a semi-curtailed or to a fully-curtailed sampling plan can be defined as

$$S(uncu,semi) = 100(n-ASN(semi))/n \qquad ...(5.6.2)$$

$$S(uncu,fully) = 100(n-ASN(fully))/n \qquad ...(5.6.3)$$

Furthermore, it may be noted that the variances of the MLE $\hat{p}_i$ (i=1,2) based on the inspection of m lots in accordance with the uncurtailed sampling plan are given as

$$V(\hat{p}_i\ (uncu)) = \frac{p_i(1-p_i)}{mn} \quad i=1,2 \qquad ...(5.6.4)$$

Then the efficiency of the MLE $\hat{p}_i$ (semi)(i=1,2) with respect to the MLE $\hat{p}_i$(uncu) (i=1,2) is obtained as

$$\frac{V(\hat{p}_i(\text{uncu}))}{V(\hat{p}_i(\text{semi}))} = \frac{ASN(\text{semi})}{n} \quad i=1,2 \qquad \ldots(5.6.5)$$

and the efficiency of the MLE $\hat{p}_i$ (fully) (i=1,2) with respect to the MLE $\hat{p}_i$(uncu) (i=1,2) is obtained as

$$\frac{V(\hat{p}_i(\text{uncu}))}{V(\hat{p}_i(\text{fully}))} = \frac{ASN(\text{fully})}{n} \quad i=1,2 \qquad \ldots(5.6.6)$$

Hence the expressions for the percent loss in efficiency in estimation of p_i(i=1,2) as one passes from the uncurtailed sampling plan to the semi-curtailed sampling plan and to the fully-curtailed sampling plan are given by the right hand sides of (5.6.2) and (5.6.3) respectively. Thus the price in reduction in inspection is paid by proportional increase in the variance of the estimator.

5.7 Numerical Example :

It is needless to state that the trends in the characteristics such as the probability of acceptance, saving in inspection, loss in efficiency, etc., depend upon the actual values taken by the parameters p_1 and p_2. They are, therefore, illustrated with the help of the following numerical example.

Consider the sampling plan with $n=40$, $k_1=8$, $k_2=3$. In standard notation the sampling plan ensures the producer's risk, consumer's risk as follows :

Good lot = (p_{10} = 5 %, p_{20}= 2%),

Producer's risk = α = 5%.

Bad lot = (p_{11} = 20%, p_{21} = 8%),

Consumer's risk = β = 7%

where $1-\alpha = P_a(p_{10}, p_{20})$, $\beta = P_a(p_{11}, p_{21})$ and $P_a(p_1, p_2)$ is given by (5.4.2)

In Table 5.1 one finds that column (3) gives $P_a(p_1, p_2)$ for all sampling plans: semi-curtailed, fully-curtailed as well as uncurtailed. Columns (4) and (6) give ASN's which are obtained using (5.4.4) and (5.5.5). Columns (5) and (7) give percent saving in inspection which also reflects the loss in efficiency in estimation. These columns are obtained using (5.6.2) and (5.6.3). It may be revealed that there are instances where one finds a mentionable saving in inspection (loss in efficiency in estimation) as one passes from an uncurtailed sampling plan to a semi-curtailed sampling plan but not an appreciable saving in inspection as one passes from a semi-curtailed sampling plan to a fully-curtailed sampling plan.

Table 5.1

$n=40,\ k_1=8,\ k_2=3$

p_2	p_1	$P_a(p_1,p_2)$	ASN(semi)	S(uncu, semi)	ASN(fully)	S(uncu, fully)
1	2	3	4	5	6	7
0.02	0.05	0.951052	39.5177	1.21	38.2642	4.34
	0.10	0.871304	39.2739	1.82	38.2971	4.26
	0.15	0.618814	37.6799	5.80	36.9757	7.56
	0.20	0.315923	34.0645	14.84	33.5508	16.12
0.04	0.05	0.780695	37.4736	6.32	36.7784	8.05
	0.10	0.688429	37.3397	6.65	36.8598	7.85
	0.15	0.449754	33.6748	15.81	33.3002	16.75
	0.20	0.207331	30.3297	24.18	30.1966	24.51
0.06	0.05	0.561672	34.1948	14.51	34.1148	14.71
	0.10	0.480158	32.8636	17.84	32.3867	19.03
	0.15	0.293239	30.8641	22.84	30.6188	23.45
	0.20	0.124068	28.4911	28.77	28.3709	29.07
0.08	0.05	0.365494	30.4780	23.81	30.2981	24.25
	0.10	0.304034	29.8074	25.48	29.6544	25.86
	0.15	0.175054	28.4047	28.99	28.2880	29.28
	0.20	0.068581	24.7648	38.09	24.7395	38.15