CHAPTER - 7

BASIC DIFFERENTIATION FORMULAS AND SUMMATION FORMULAS

7.1 INTRODUCTION

In this chapter, the q-differentiation operator defined by (Gasper and Rahman [1])

$$D_q f(x) = \frac{f(x) - f(xq)}{x(1-q)} \tag{7.1.1}$$

will be applied to the polynomials $S_n(l,m,\alpha,\beta.x\,|\,q)$ and $M_n(s,A,\beta:x\,|\,q)$ defined in Chapters 2 and 5 respectively. By applying the operator D_q successively, the higher order derivatives are obtained. Next, using a summation formula due to D.S. Moak [1], the summation formulas for these two polynomials will also be obtained. The derivations are given in section-7.2.

In the subsequent section, another simple summation formulas are derived for these two polynomials.

7.2 HIGHER ORDER q-DERIVATIVE FORMULA AND ASSOCIATED SUMMATION FORMULAS

For the purpose of finding the basic differentiation, a particular case of the polynomial

$$S_n(l,m,\alpha,\beta;x\,|\,q) = \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-mnk} \frac{(\beta q^{\ell k-n\alpha+1};p)_\infty}{(p;p)_{n-mk}} \sigma_k x^k$$

is considered here by taking $\sigma_k = \frac{1}{(p;p)_k}$, where $p \equiv q^{\alpha}$.

This particular polynomial will be denoted by $V_n(x|p)$. Thus,

$$V_n(x\,|\,p) = \sum_{k=0}^{[n/m]} (-1)^{mk}\, p^{mk(mk+1)/2-mnk} \frac{(\beta q^{\ell k-n\alpha+1};p)_\infty x^k}{(p,p)_{n-mk}(p;p)_k}. \qquad \textbf{(7.2.1)}$$

Now, applying the operator D_p given in (7.1.1) on $V_n(x|q)$, one gets

$$D_p\, V_n(x\,|\,p) = \sum_{k=0}^{[n/m]} (-1)^{mk}\, p^{mk(mk+1)/2-mnk} \frac{(\beta q^{\ell k-n\alpha+1};p)_\infty\, x^k(1-p^k)}{(p;p)_{n-mk}(p,p)_k\, x(1-p)}$$

$$= \frac{1}{1-p} \sum_{k=1}^{[n/m]} (-1)^{mk}\, p^{mk(mk+1)/2-mnk} \frac{(\beta q^{\ell k-n\alpha+1};p)_\infty x^{k-1}}{(p;p)_{n-mk}(p,p)_{k-1}}$$

$$= \frac{1}{1-p} \sum_{k=0}^{[(n-m)/m]} (-1)^{m+km}\, p^{m(k+1)[m(k+1)+1]/2-mn(k+1)} \cdot \frac{(\beta q^{\ell k+\ell-n\alpha+1};p)_\infty x^k}{(p;p)_{n-m-mk}(p;p)_k}$$

$$= \frac{(-1)^m\, p^{m(m+1)/2-mn}}{1-p} \sum_{k=0}^{[n/m]} (-1)^{mk}\, p^{mk(mk+1)/2-mk(n-m)} \cdot \frac{(\beta q^{\ell k-(n-m)\alpha+1};p)_\infty x^k}{(p;p)_{n-m-mk}(p,p)_k}.$$

Thus, the first order basic derivative occurs in the form:

$$D_p\, V_n(x\,|\,p) = \frac{(-1)^{mk}\, p^{m(m+1)/2-mn}}{1-p} V_{n-m}(x\,|\,p). \qquad \textbf{(7.2.2)}$$

Again operating by D_p on (7.2.2), one obtains

$$D_p^2\, V_n(x \mid p) = \frac{(-1)^m p^{m(m+1)/2-mn}}{1-p} .$$

$$\cdot \sum_{k=0}^{[(n-m)/m]} \frac{(-1)^{mk} p^{mk(mk+1)/2-mk(n-m)} (\beta q^{\ell k-(n-m)\alpha+1}, p)_\infty x^k (1-p^k)}{(p;p)_{n-m-mk}(p;p)_k (1-p)x}$$

$$= \frac{(-1)^m p^{m(m+1)/2-mn}}{1-p} \sum_{k=1}^{[(n-m)/m]} \frac{(-1)^{mk} p^{mk(mk+1)/2-mk(n-m)}}{(p;p)_{n-m-mk}(p;p)_{k-1}} . \cdot (\beta q^{\ell k-(n-m)\alpha+1}; p)_\infty x^{k-1}$$

$$= \frac{(-1)^m p^{m(m+1)/2-mn}}{1-p} \sum_{k=0}^{[(n-2m)/m]} (-1)^{mk+m} p^{m(k+1)[m(k+1)+1]/2-m(n-m)(k+1)} \frac{(\beta q^{\ell k+\ell-(n-m)\alpha+1}; p)_\infty x^k}{(p;p)_{n-2m-mk}(p;p)_k}$$

$$= \frac{(-1)^{2m} p^{2m(m+1)/2-2mn+m^2}}{(1-p)^2} \sum_{k=0}^{[(n-2m)/m]} (-1)^{mk} p^{mk(mk+1)/2-mk(n-2m)} . \frac{(\beta q^{\ell k-(n-2m)\alpha+1}, p)_\infty x^k}{(p;p)_{n-2m-mk}(p;p)_k}$$

Thus,

$$D_p^2\, V_n(x \mid p) = \frac{(-1)^{2m} p^{2m(m+1)/2-2mn+m^2}}{(1-p)^2} V_{n-2m}(x \mid p) . \quad \textbf{(7.2.3)}$$

Similarly it can be shown that

$$D_p^3\, V_n(x \mid p) = \frac{(-1)^{3m} p^{3m(m+1)/2-3mn+3m^2}}{(1-p)^2} V_{n-3m}(x \mid p) .$$

In general, for a positive integer r,

$$D_p^r\, V_n(x \mid p) = \frac{(-1)^{rm} p^{rm(m+1)/2-rmn+\mu_r m^2}}{(1-p)^r} V_{n-rm}(x \mid p) , \quad \textbf{(7.2.4)}$$

where μ_r is the r[th] term of sequence $\{\mu_n\}$ with $\mu_n=n-1+\mu_{n-1}$, $\mu_1=0$, $n>1$.

In view of the definition (7.1.1) of the q-derivative operator we also have, with $p=q^{\alpha}$ as usual,

$$D_p V_n(x\mid p)=\frac{V_n(x\mid p)-V_n(xp\mid p)}{x(1-p)}.$$

Therefore, from (7.2.2) above,

$$\frac{(-1)^m p^{m(m+1)/2-mn}}{1-p} V_{n-m}(x\mid p)=\frac{V_n(x\mid p)-V_n(xp\mid p)}{x(1-p)}$$

that is

$$(-1)^m xp^{m(m+1)/2-mn} V_{n-m}(x\mid p)=V_n(x\mid p)-V_n(xp\mid p). \qquad \textbf{(7.2.5)}$$

The formula (7.2.5) will now be generalized with the help of a summation formula (D.S. Moak [1])

$$x^m(1-q)^m D_q^m f(x)=(-1)^m q^{-m(m-1)/2}\sum_{k=0}^{m}(-1)^k q^{k(k-1)/2}\begin{bmatrix} m \\ k \end{bmatrix} f(xq^{m-k})$$

(7.2.6)

involving m[th] order basic derivative.

In deed, if $f(x)=V_n(x|p)$ and m is replaced by r, then the substitution of the formula (7.2.4) on the left hand side of (7.2.6) yields

$$(-1)^{rm} p^{rm(m+1)/2-rmn+\mu_r m^2} x^r V_{n-rm}(x\mid p)$$

$$=(-1)^r p^{-r(r-1)/2}\sum_{k=0}^{r}(-1)^k p^{k(k-1)/2}\begin{bmatrix} r \\ k \end{bmatrix} V_k(xp^{r-k}\mid p) \qquad \textbf{(7.2.7)}$$

which provides a generalization of the relation in (7.2.5).

In order to obtain a summation formula similar to (7.2.7) for the polynomial introduced in Chapter-5 viz.

$$M_n(s,A,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}\,\xi_k\,x^k}{[q]_{n-sk}}$$

where $q_2 \equiv q^\beta$, we take $\xi_k=\frac{1}{(q;q)_k}$. This particular case of $M_n(s,A,\beta:x\,|\,q)$ will be denoted here by $F_n(s,A,\beta;x\,|\,q)$. Hence

$$F_n(s,A,\beta:x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}\,x^k}{[q]_{n-sk}[q]_k}. \qquad \textbf{(7.2.8)}$$

Upon operating the polynomial $F_n(s,A,\beta\cdot x\,|\,q)$ by the operator D_q as defined in (7.1.1), we get,

$$D_q\,F_n(s,A,\beta\cdot x\,|\,q)=\sum_{k=1}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}}{[q]_{n-sk}[q]_k}\frac{x^k(1-q^k)}{x(1-q)}$$

$$=\frac{1}{1-q}\sum_{k=1}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}\,x^{k-1}}{[q]_{n-sk}[q]_{k-1}}$$

$$=\frac{(-1)^s}{1-q}\sum_{k=0}^{[(n-s)/s]}\frac{(-1)^{sk}q^{-snk-sn}x^k}{[q]_{n-s-sk}[q]_k}.\,(Aq^{sk+sk\beta+s+s\beta};q_2)_{n-s-sk}$$

$$=\frac{(-1)^s q^{-sn}}{1-q}\sum_{k=0}^{[(n-s)/s]}\frac{(-1)^{sk}q^{-sk(n-s)}x^k q^{-s^2k}}{[q]_{n-s-sk}[q]_k}.(Aq^{sk+sk\beta+s+s\beta};q_2)_{n-s-sk}$$

Thus, it is shown that

$$D_qF_n(s,A,\beta;x\,|\,q)=\frac{(-1)^s q^{-sn}}{1-q}F_{n-s}(s,A+s+s\beta;\beta,xq^{-s^2}\,|\,q)\,. \qquad \textbf{(7.2.9)}$$

The second order basic derivative may be obtained as follows.

$D_q^2\, F_n(s,A,\beta:x|q)$

$$=\frac{(-1)^s q^{-sn}}{1-q}\sum_{k=0}^{[(n-s)/s]}\frac{(-1)^{sk}q^{-sk(n-s)}(Aq^{sk+sk\beta+s+s\beta};q_2)_{n-s-sk}}{[q]_{n-s-sk}[q]_k}\cdot\frac{q^{-s^2k}x^k(1-q^k)}{x(1-q)}$$

$$=\frac{(-1)^s q^{-sn}}{(1-q)^2}\sum_{k=0}^{[(n-s)/s]}\frac{(-1)^{sk}q^{-sk(n-s)}(Aq^{sk+sk\beta+s+s\beta},q_2)_{n-s-sk}}{[q]_{n-s-sk}[q]_{k-1}}.x^{k-1}q^{-s^2k}$$

$$=\frac{(-1)^{2s} q^{-2sn}}{(1-q)^2}\sum_{k=0}^{[(n-2s)/s]}\frac{(-1)^{sk}q^{-sk(n-s)-s^2k}}{[q]_{n-2s-sk}[q]_k}(Aq^{sk+sk\beta+2s+2s\beta};q_2)_{n-2s-sk}\,q^{-2s^2k}x^k$$

Thus,

$$D_q^2\,F_n(s,A,\beta;x\,|\,q)=\frac{(-1)^{2s}q^{-2sn}}{(1-q)^2}F_{n-2s}(s,A+2s+2s\beta;\beta;xq^{-2s^2}\,|\,q)\,.$$

Similarly, it can be shown that

$$D_q^3\,F_n(s,A,\beta,x\,|\,q)=\frac{(-1)^{3s}q^{-3sn}}{(1-q)^3}F_{n-3s}(s,A+3s+3s\beta;\beta;xq^{-3s^2}\,|\,q)\,.$$

Repeating the above process m times, one gets

$$D_q^m\,F_n(s,A,\beta;x\,|\,q)=\frac{(-1)^{ms}q^{-msn}}{(1-q)^m}F_{n-ms}(s,A+ms+ms\beta;\beta;xq^{-ms^2}\,|\,q)\,.$$

(7.2.10)

Now by definition (7.1.1),

$$D_q F_n(s,A,\beta,x\,|\,q)=\frac{F_n(s;A,\beta,x\,|\,q)-F_n(s,A,\beta;xq\,|\,q)}{x(1-q)}$$

which in view of the q-derivative (7.2.9) yields

$$\frac{(-1)^s q^{-sn}}{1-q}F_{n-s}(s,A+s+s\beta;\beta;xq^{-s^2}\,|\,q)=\frac{F_n(s,A,\beta;x\,|\,q)-F_n(s,A,\beta;xq\,|\,q)}{x(1-q)}$$

therefore,

$$(-1)^s xq^{-sn}F_{n-s}(s,A+s+s\beta,\beta;xq^{-s^2}\,|\,q)=F_n(s,A,\beta;x\,|\,q)-F_n(s,A,\beta,xq\,|\,q). \tag{7.2.11}$$

This can be further generalized by using the Moak's formula (7.2.6) above, as follows.

Using the result of (7.2.10) in (7.2.6), we get the summation formula

$$(-1)^{ms}x^m F_{n-ms}(s,A+ms+ms\beta,\beta;xq^{-ms^2}\,|\,q)$$

$$=(-1)^m q^{msn-m(m-1)/2}\sum_{k=0}^{m}(-1)^k q^{k(k-1)/2}\begin{bmatrix} m \\ k \end{bmatrix}F_n(s,A,\beta;xq^{m-k}\,|\,q).$$

This is a generalization of the formula (7.2.11) because setting m=1 in this, it reduces to the one given by (7.2.11).

7.3 q-SUMMATION FORMULAS

Consider first

$$S_n(l,m,\alpha,\beta;x\,|\,p)=\sum_{k=0}^{[n/m]}(-1)^{mk}p^{mk(mk+1)/2-mnk}\frac{(\beta q^{\ell k-n\alpha+1};p)_\infty\,\sigma_k x^k}{(p,p)_{n-mk}}.$$

Here $\ell=m\alpha$ and $p=q^{\alpha}$ gives,

$$(\beta q^{\ell k-n\alpha+1};p)_\infty \frac{(\beta q,p)_\infty}{(\beta q;p)_\infty} = \frac{(\beta q;p)_\infty}{(\beta q,p)_{-n+mk}}$$

which in view of the formula

$$(a;q)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n (q/a;q)_n}$$

becomes

$$\frac{(\beta q^{\ell k-n\alpha+1};p)_\infty (\beta q;p)_\infty}{(\beta q,p)_\infty} = \frac{(\beta q;p)_\infty (-1)^{n-mk} q^{(\beta+1)(n-mk)}}{p^{(n-mk)(n-mk+1)/2}} \cdot$$

$$(q^{\alpha-\beta-1};p)_{n-mk}$$

therefore

$$S_n(\ell,m,\alpha,\beta;x\,|\,p) = \frac{(-1)^n (\beta q;p)_\infty}{p^{n(n+1)/2}} \sum_{k=0}^{[n/m]} \frac{p^{mk} q^{(\beta+1)(n-mk)} \sigma_k x^k}{(p;p)_{n-mk}}$$

$$\cdot (q^{\alpha-\beta-1},p)_{n-mk}$$

$$= \frac{(-1)^n q^{n(\beta+1)} (\beta q;p)_\infty}{p^{n(n+1)/2}} \sum_{k=0}^{[n/m]} \frac{p^{mk} q^{-mk(\beta+1)} \sigma_k x^k}{(p;p)_{n-mk}} \cdot$$

$$\cdot (q^{\alpha-\beta-1};p)_{n-mk}$$

Let

$$\frac{(-1)^n q^{n(\beta+1)} (\beta q;p)_\infty}{p^{n(n+1)/2}} S_n(\ell,m,\alpha,\beta;x\,|\,p) = T_n(\ell,m,\alpha,\beta;x\,|\,p)$$

then

$$\sum_{j=0}^{n} T_j(\ell, m, \alpha, \beta; q^{m(\beta+1)}x \mid p)p^j$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{[n/m]} \frac{(q^{\alpha-\beta-1}, p)_{j-mk}\, \sigma_k\, x^k\, p^{j+mk}}{(p;p)_{j-mk}}$$

which on using the relation

$$\sum_{j=0}^{n} \sum_{k=0}^{[n/m]} A(j,k) = \sum_{k=0}^{[n/m]} \sum_{j=0}^{n-mk} A(j+mk,k), \tag{7.3.1}$$

takes the form

$$\sum_{j=0}^{n} T_j(\ell, m, \alpha, \beta; q^{m(\beta+1)}x \mid p)p^j = \sum_{k=0}^{[n/m]} p^{2mk}\, \sigma_k\, x^k \sum_{j=0}^{n-mk} \frac{(q^{\alpha-\beta-1}, p)_j\, p^j}{(p,p)_j}$$

$$= \sum_{k=0}^{[n/m]} p^{2mk} \sigma_k x^k \frac{(q^{2\alpha-\beta-1}; p)_{n-mk}}{(p,p)_{n-mk}}$$

$$= T_n(\ell, m, \alpha, \beta-\alpha; q^{m(\alpha+\beta+1)}x \mid p),$$

which follows from a simple summation formula

$$1 + \sum_{n=1}^{j} \frac{[a]_n}{[q]_n} q^n = \frac{[aq]_j}{[q]_j}. \tag{7.3.2}$$

Thus,

$$T_n(\ell, m, \alpha, \beta-\alpha; q^{m(\alpha+\beta+1)}x \mid p) = \sum_{j=0}^{n} T_j(\ell, m, \alpha, \beta; q^{m(\beta+1)}x \mid p)p^j,$$

which is the relation wherein the polynomial T_n is expressed in a series of itself.

A similar result for the polynomial

$$M_n(s,A,\beta,x|q) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk} q^{-snk} (Aq^{sk+sk\beta}, q^{\beta})_{n-sk}}{[q]_{n-sk}} \xi_k x^k$$

may be obtained by taking β=1, that is, for the explicit form

$$M_n(s,A,1;x|q) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk} q^{-snk} [Aq^{2sk}]_{n-sk}}{[q]_{n-sk}} \xi_k x^k .$$

For that consider

$$\sum_{n=0}^{j} M_n(s,A,1;xq^{sn}|q) q^n = \sum_{n=0}^{j} \sum_{k=0}^{[n/s]} \frac{[Aq^{2sk}]_{n-sk}}{[q]_{n-sk}} \xi_k x^k q^n ,$$

this, in view of (7.3.1) becomes

$$\sum_{n=0}^{j} M_n(s,A,1;xq^{sn}|q) q^n = \sum_{k=0}^{[j/s]} \sum_{n=0}^{j-sk} \frac{[Aq^{2sk}]_n}{[q]_n} \xi_k x^k q^{n+sk}$$

$$= \sum_{k=0}^{[j/s]} \xi_k x^k q^{sk} \sum_{n=0}^{j-sk} \frac{[Aq^{2sk}]_n}{[q]_n} q^n .$$

Now applying the summation formula (7.3.2) the inner series above simplifies to the sum $\frac{[Aq^{2sk+1}]_{j-sk}}{[q]_{j-sk}}$ which leads us to

$$\sum_{n=0}^{j} M_n(s,A,1;xq^{sn}|q) \; q^n = \sum_{k=0}^{[j/s]} \frac{[Aq^{2s+1}]_{j-sk}}{[q]_{j-sk}} \xi_k (xq^s)^k$$

$$= M_j(s,A+1,1;xq^s|q) .$$

Thus the result may be stated as

$$M_j(s,A+1,1;xq^s|q) = \sum_{n=0}^{j} M_n(s,A,1;xq^{sn}|q) \; q^n .$$