SUMMARY OF THE THESIS ENTITLED

GENERALIZATIONS OF CERTAIN ORDINARY AND BASIC POLYNOMIALS SYSTEMS AND THEIR PROPERTIES

submitted to

The Maharaja Sayajirao University of Baroda

for the Award of the Degree of

Doctor of Philosophy

in

Mathematics

By

MANISHA DALBHIDE

Department of Mathematics

Institute of Technology

Nirma University of Science and Technology

Ahmedabad - 382481

Under the Guidance of

DR. B. I. DAVE

Reader in Mathematics

Department of Mathematics

Faculty of Science

The M. S. University of Baroda

Baroda - 390002

July, 2004

SUMMARY OF THE THESIS ENTITLED

GENERALIZATIONS OF CERTAIN ORDINARY AND BASIC POLYNOMIALS SYSTEMS AND THEIR PROPERTIES

submitted to
The Maharaja Sayajirao University of Baroda
for the Award of the Degree of

Doctor of Philosophy

in

Mathematics

By
MANISHA DALBHIDE
Department of Mathematics
Institute of Technology
Nirma University of Science and Technology
Ahmedabad - 382481

Under the Guidance of
DR. B. I. DAVE
Reader in Mathematics
Department of Mathematics
Faculty of Science
The M. S. University of Baroda
Baroda - 390002

July, 2004

The field of special functions is rich in polynomials, all of which are hypergeometric in character. The polynomials of Laguerre, Hermite, Legendre, Jacobi, Gegenbauer etc. were studied from the point of view of examining various properties such as generating functions, orthogonality, Rodrigue's formula, recurrence relations, zeros, inverse series relations, integral representations, summation formulas, differential equations etc. by number of researchers (Olver [6], Rainville [7], Wang and Guo [9], Chihara [2]).

The aforementioned classical polynomials and many others were later on generalized by the researchers; of these generalizations, the extended Jacobi polynomial (H.M. Srivastava [8])

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[a_1,\ldots,a_p;b_1,\ldots,b_q \;\; x]=m+{}_pF_{q+l}\left[\begin{matrix}\Delta(m;n),a_1,\ldots,a_p; \\ \Delta(l;1-n\alpha+\beta),b_1,\ldots,b_q;\end{matrix}\;\; x\right]$$

is taken up for a further study; particularly for obtaining inverse series, integral forms, differential equation etc. and then deriving such properties for a q-analogue of this polynomial. Also, this polynomial is put into a more general form:

$$S_n(l,m,\alpha,\beta;x)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\,\sigma_k\,x^k}{\Gamma(1+\beta-n\alpha+lk)\,(n-mk)!}, \qquad \textbf{(1)}$$

and its properties as mentioned above are studied.

Having given a brief introduction on the subject matter in Chapter-1, Chapter-2 incorporates the polynomial (1) and its properties some of which are given below.

Integral Representation:

$$S_n(l,m,\alpha,\beta,x)=\int_0^1 t^{\beta-n\alpha}\,(1-t)^{lk-1}\,\rho_n(l,m,x)\,dt,$$

where $\rho_n(l,m;x)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\,\sigma_k\,x^k}{\Gamma(lk)\,(n-mk)!}$, and Re(β-nα) > 0, Re($lk$) > 0.

Differential Equation (θ-form):

$$\left[\theta\prod_{j=1}^{l}(\theta+b_j-1)-cx\prod_{i=1}^{m}(\theta+a_i)\right]w=0,$$

where $a_i=\frac{-n+i-1}{m}, b_j=\frac{\beta-n\alpha+j}{l}, \theta\equiv x\frac{d}{dx},$ and, $w=T_n^m(x)$ is a special case $\sigma_k=\frac{1}{k!}$ of $S_n(l,m,\alpha,\beta,x)$

Inverse Series Relation:

Theorem-1. When n is a non-negative integer, l and m are positive integers, α and β are arbitrary parameters, l=mα, and if

$$f(n)=\sum_{k=0}^{[n/m]}a(n,k,m)\,g(k) \qquad \text{and} \qquad g(n)=\sum_{k=0}^{mn}b(n,k,m)\,f(k)$$

then

$$a(n,k,m)=\frac{1}{\Gamma(1+\beta-n\alpha+lk)\,(n-mk)!}$$

if and only if

$$b(n,k,m)=(-1)^{mn-k}\,\frac{\beta\,\Gamma(\beta+ln-k\alpha)}{(mn-k)!} \text{ and } \sum_{k=0}^{n}b(n,k,1)\,f(k)=0, \text{ if } n\neq ms,\ s\in N.$$

The q-analogues of these properties are obtained in Chapter-3 for the polynomial $S_n(l,m,\alpha,\beta,x\,|\,q)$, which is defined in the form:

$$S_n(l,m,\alpha,\beta,x|p)=\sum_{k=0}^{[n/m]}(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\,\frac{(\beta q^{lk-n\alpha+1},p)_\infty\,\sigma_k\,x^k}{(p,p)_{n-mk}},\textbf{(2)}$$

where $p=q^{\alpha}$, $\alpha\neq0$.

Some of the properties of (2) studied in this chapter, are as listed below.

q-Integral Representations:

For Re $(\beta\text{-}n\alpha) > 0$, $1+lk\neq0,-1,-2,\ldots$, (Gasper and Rahman [3])

$$S_n(l,m,\alpha,\beta;x|p)=\int_0^1 t^{\beta-n\alpha}\,(tp;p)_\infty\,\Omega(n,l,m,t,x)\,d_p t,$$

where

$$\Omega(n,l,m,t,x)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\,\sigma_k\,x^k}{(1-p)^{lk}(p;p)_{n-mk}\,\Gamma_p(lk)(tp^{lk};p)_\infty}.$$

q-Difference Equation:

$$\left[\theta\left\{\prod_{j=0}^{l-1}\prod_{k=0}^{l-1}(\theta+w^k\,p^{1-\frac{1+\beta-n\alpha+j\alpha}{l}}-1)\right\}+xp^{m(n\alpha-n-\beta)+m(m+3)/2}\right.$$

$$\left.\cdot\left\{\prod_{s=0}^{m-1}\prod_{r=0}^{m-1}(\theta+w^r q^{(n-s)/m}-1)\right\}\right]R_n^m(x|p)=0,$$

where w is the m[th] roots of unity and $R_n^m(x|p)$ is a particular case

$\sigma^k=\frac{1}{(p,p)_k}$ of $S_n(l,m,\alpha,\beta,x|p)$.

Inverse Series Relation:

Theorem-2. For n=0,1,2,..., and m=1,2,3..., if

$$F(n)=\sum_{k=0}^{[n/m]} a(n,k,m,q)\,G(k) \quad \text{and} \quad G(n)=\sum_{k=0}^{mn} b(n,k,m,q)\,F(k)$$

then

$$a(n,k,m,q)=(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\,\frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(p;p)_{n-mk}}$$

implies and is implied by

$$b(n,k,m,q)=\frac{(-1)^k\,p^{k(k-1)/2}\,(1-\beta q^{1-\alpha})}{(\beta q^{nl-k\alpha+1-\alpha};p)_\infty\,(p;p)_{mn-k}}$$

and $\sum_{k=0}^{n} b(n,k,1,q)\,F(k)=0$, if $n\neq mj$, $j\in N$.

Chapter-4 begins with the introduction of the polynomial

$$M_n(s,A,\beta;x)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\,\Gamma(A+sk+n\beta)}{(n-sk)!}\,\Psi_k\,x^k, \tag{3}$$

which is suggested by the inverse series relation due to Gessel and Stanton [4]. For this polynomial various properties are derived. The following are few of them.

Integral Form:

$$M_n(s,A,\beta;x)=\int_0^1 t^{A+n\beta-1}\,(1-t)^{A-1}\,\xi_n(s,A,x,t)\,dt,$$

where

$$\xi_n(s,A,x,t)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\,\Psi_k\,x^k}{(n-sk)!\,\Gamma(A-sk)}\left(\frac{t}{1-t}\right)^{sk}$$,and $Re(A-sk)>0$, $Re(A+n\beta)>0$.

Inverse Series Relation:

Theorem-3. For n=0,1,2,..., and s=1,2,3,..., if

$$A(n)=\sum_{k=0}^{[n/s]} a(n,k,s)\,B(k) \quad \text{and} \quad B(n)=\sum_{k=0}^{sn} b(n,k,s)\,A(k)$$

then

$$a(n,k,s)=\frac{(-1)^{sk}\,\Gamma(A+sk+n\beta)}{(n-sk)!} \quad \Leftrightarrow \quad b(n,k,s)=\frac{(-1)^{k}\,(A+k+k\beta)}{\Gamma(A+sn+k\beta+1)\,(sn-k)!}$$

and $\sum_{k=0}^{n} b(n,k,1)\,A(k)=0$, if $n \neq sj$, $j \in N$.

The q-analogues of these results are derived in Chapter-5 with the help of a q-analogue of (3), given by

$$M_n(s,A,\beta,x\,|\,q)=\sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\,q^{-snk}\,(Aq^{sk+sk\beta};q^{\beta})_{n-sk}}{[q]_{n-sk}}\,\xi_k\,x^k. \qquad \textbf{(4)}$$

Some of the main properties that are obtained for this basic polynomial, are listed below.

q-Integral Representations:

(I) $M_n(s,A,\beta;x\,|\,q)=\int_{-1}^{1} (-tq,q_2)_\infty\,(tq,q_2)_\infty\,\delta_n(s,A,\beta,x,t\,|\,q)\,d_{q_2}t,$

where

$$\delta_n(s,A,\beta;x,t\,|\,q)=\sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\,q^{-snk}\,(1-q_2)^{A+n\beta-1}\,\xi_k\,x^k}{[q]_{n-sk}\,\Gamma_{q_2}(A+sk+sk\beta)\,(-1;q_2)_\infty}\cdot$$

$$\cdot\frac{\Gamma_{q_2}(2A+sk+sk\beta+n\beta)\,(q^{2A+2sk\beta};q_2^2)_\infty\,(-q^{A+sk+n\beta};q_2)_\infty}{(q_2^2,q_2^2)_\infty\,(-tq^{A+sk+n\beta},q_2)_\infty\,(-tq^{A+sk\beta}q_2)_\infty}, \text{ and } q_2=q^{\beta}.$$

(II) $M_n(s,A,\beta;xt^{-s}|q)=\int_0^1 t^{A+n\beta-1}\, E_{q_2}(tq_2)\,\Delta_n(s,A,\beta,x|q)\, d(q_2,t)$,

where

$$\Delta_n(s,A,\beta;x|q)=\frac{1}{(q_2;q_2)}\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\, q^{-snk}\,(Aq^{sk+sk\beta},q_2)_\infty\, \xi_k\, x^k}{[q]_{n-sk}\,(1-q_2)}.$$

q-Difference Equation:

$$\left\{\theta\prod_{j=0}^{2s-1}\prod_{m=0}^{2s-1}(\theta+\sigma^m q^{1-(A+j)/2s}-1)+x\,q^{2s^2+3s-As}\,.\right.$$

$$\left.\cdot\left[\prod_{p=0}^{s-1}\prod_{v=0}^{s-1}(\theta+w^v q^{(n-p)/s}-1)(\theta+w^v q^{-(A+n+p)/s}-1)\right]\right\}G_n^s(x|q)=0,$$

where σ is $(2s)^{th}$ and w is s^{th} roots of unity.

Inverse Series Relation:

Theorem-4. For n=0,1,2,..., and s=1,2,3,..., if

$$U(n)=\sum_{k=0}^{[n/s]} a(n,k,s)\,V(k) \qquad \text{and} \qquad V(n)=\sum_{k=0}^{sn} b(n,k,s)\,U(k)$$

then

$$a(n,k,s)=\frac{(-1)^{sk}\, q^{-snk}\,(Aq^{sk+sk\beta};q^{\beta})_{n-sk}}{[q]_{n-sk}}$$

if and only if

$$b(n,k,s)=\frac{(-1)^k\, q^{(sn-k)(sn-k+1)/2+snk}\,(1-Aq^{k+k\beta})}{[q]_{sn-k}}\,(Aq^{sn+(sn-1)\beta};q^{-\beta})_{sn-k-1}$$

and $\sum_{k=0}^{n} b(n,k,1)\,U(k)=0$, if $n\neq sj$, $j\in N$.

Some of the q-integrals used in Chapter-3 and 5 are further exploited to derive basic transformation formulas in Chapter-6. One such transformation formula occurs in the form:

$$\sum_{i,j=0}^{\infty} \frac{(-1)^j q^{i+j+j(j-1)/2} x^{i+j} (p^u;p)_{i+j}}{(p;p)_i (p;p)_j} \sum_{k=0}^{[n/m]} p^{mk(mk+1)/2-mnk} \cdot \frac{(\beta q^{lk-n\alpha+1},p)_\infty (p^{u+i+j};p)_k \sigma_k x^k}{(p;p)_{n-mk}}$$

$$= \sum_{k=0}^{[n/m]} p^{mk(mk+1)/2-mnk} \frac{(\beta q^{lk-n\alpha+1};p)_\infty (p^u;p)_k \sigma_k x^k}{(p;p)_{n-mk}}$$

In Chapter-7 some miscellaneous results are incorporated which include summation formulas, and n^{th} order q-derivatives of the polynomials cited in (2) and (4).

REFERENCES

1. Andrews, G.E. and Askey, R., Another q-extension of Beta function, Proc. Amer. Math. Soc. 81(1981), 97-100.
2. Chihara, T.S., *An Introduction to Orthogonal Polynomials*, Gordon and Breach Science Publishers, New York, London, Paris, 1978.
3. Gasper, G. and Rehman, M., *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.
4. Gessel, I. and Stanton, D., Applications of q-Lagrange inversion to basic hypergeometric series, Trans. of Amer. Math. Soc., 277, No. 1(1983), 173-201.
5. Hahn, W., Uber Orthogonal polynome, die q-Differenzengleichungen genugen, Math. Nachr., 2(1949), 4-34.
6. Olver, F.W.J., *Introduction to asymptotics and Special Functions*, Academic Press, New York and London, 1974.
7. Rainville, E.D., *Special Functions*, Macmillan Co., New York, 1960.
8. Srivastava, H.M., The Weyl fractional integral of a general class of polynomials, Boll. Un. Mat. Ital., (6)2B (1983), 219-228.
9. Wang, Z.X. and Guo, D.R., *Special Functions*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1989.