CHAPTER – 2

A GENERAL CLASS OF POLYNOMIALS AND ITS PROPERTIES–I

2.1 INTRODUCTION

With a view to study certain properties of several polynomials, H.W. Gould [6], J.P. Singhal and Savita Kumari [1], Rekha Panda [1] and others proposed the unifications of polynomials under consideration and derived general generating functions, inverse series relations, differential recurrence relations etc. for the generalized polynomial constructed. Here, in this chapter, one of the extensions of the Jacobi polynomial, denoted by $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a),(b)\ x]$ (1.1.18) which was introduced by H.M. Srivastava and M.A. Pathan [1] is put into a general form using the notation $S_n(l,m,\alpha,\beta;x)$ The explicit representation of this is defined here by:

$$S_n(l,m,\alpha,\beta;x)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\sigma_k\, x^k}{\Gamma(1+\beta-n\alpha+lk)(n-mk)!}, \qquad \textbf{(2.1.1)}$$

where σ_k is a general sequence in k (not involving n); and for this polynomial the integral representations, θ-form differential equation (for particular σ_k) and inverse series relation will be derived.

The explicit representation (2.1.1) is considered for studying the properties of the polynomials belonging to it. It is interesting to see that the polynomial (2.1.1) gives rise to the extensions of the polynomials of Laguerre, Jacobi, and the biorthogonal polynomials $Z_n^{\alpha}(x;k)$ and $W_n^{(\alpha,\beta)}(x;k),\ k \in N$, which are as mentioned below.

$$L_{n,m}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}\, x^j}{(1+\alpha)_j\, j!} \qquad \textbf{(2.1.2)}$$

$$P_{n,m}^{(\beta-n,\alpha)}(x) = \frac{(1+\beta-n)_n}{n!} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}(1+\alpha+\beta)_j}{(1+\beta-n)_j\, j!} \left(\frac{1-x}{2}\right)^j \qquad \textbf{(2.1.3)}$$

$$Z_{n,m}^{\alpha}(dx;k) = \frac{(1+\alpha)_{nk}}{n!} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}\,(x/k)^{kj}\, d^j}{(1+\alpha)_{kj}\, j!}, \quad \text{where } d=k^k \qquad \textbf{(2.1.4)}$$

$$W_{n,m}^{(\beta-n,\alpha)}(cx,k) = \frac{(1+\beta-n)_n}{n!} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}\,(1+\alpha+\beta)_{kj}\, c^j \left(\frac{1-x}{2}\right)^{kj}}{(1+\beta-n)_{kj}\, j!}, \quad c=m^m.$$

(2.1.5)

In the first place, the integral form for (2.1.1) will be obtained in section 2.2 which will be subsequently used to further derive a transformation formula.

Section 2.3 deals with θ-form differential equation and the remaining five sections incorporate the inverse series relations. All these properties will be illustrated in section 2.9 for the polynomials belonging to the class $\{S_n(l,m,\alpha,\beta;x)\}$.

2.2 INTEGRAL FORMS AND TRANSFORMATION FORMULA

In this section two integral forms of the polynomial (2.1.1) are derived. Later a transformation formula will be obtained.

First the polynomial $S_n(l,m,\alpha,\beta;x)$ will be expressed in hypergeometric function form. For that replacing $S_n(l,m,\alpha,\beta,x)$ by $\frac{S_n(l,m,\alpha,\beta;x)}{\Gamma(1+\beta-n\alpha)n!}$ first and then choosing $\sigma_k = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k\, k!}$ one gets a particular case $\mathcal{H}^{(\alpha,\beta)}_{n,l,m}[(a),(b).x]$ of $S_n(l,m,\alpha,\beta,x)$.

Now

$$\mathcal{H}^{(\alpha,\beta)}_{n,l,m}[(a);(b)\ x] = \sum_{k=0}^{[n/m]} \frac{(-1)^{mk}(a_1)_k \cdots (a_p)_k\, \Gamma(1+\beta-n\alpha)\, n!\, x^k}{(b_1)_k \cdots (b_q)_k\, \Gamma(1+\beta-n\alpha+lk)(n-mk)!\, k!}$$

$$= \sum_{k=0}^{[n/m]} \frac{(a_1)_k \cdots (a_p)_k (-n)_{mk}\, x^k}{(b_1)_k \cdots (b_q)_k (1+\beta-n\alpha)_{lk}\, k!}$$

$$= \sum_{k=0}^{[n/m]} \frac{(a_1)_k \cdots (a_p)_k\, \Delta(m;-n)(cx^k)}{(b_1)_k \cdots (b_q)_k\, \Delta(l;1+\beta-n\alpha)k!}$$

where c is constant.

Thus,

$$\mathcal{H}^{(\alpha,\beta)}_{n,l,m}[(a);(b):x] = {}_{m+p}F_{l+q}\left[\begin{matrix} a_1,\dots,a_p,\Delta(m;-n); \\ b_1,\dots,b_q,\Delta(l;1+\beta-n\alpha); \end{matrix}\ cx\right]. \tag{2.2.1}$$

The hypergeometric function ${}_pF_q[z]$ has the following integral form (Rainville [1]):

$$
{}_pF_q\left[\begin{matrix} a_1,\dots,a_p, \\ b_1,\dots,b_q; \end{matrix} z\right] = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} {}_{p-1}F_{q-1}\left[\begin{matrix} a_2,\ \ ,a_p, \\ b_2,\ \ ,b_q, \end{matrix} zt\right] dt.
$$

(2.2.2)

Using the integral (2.2.2) one can write integral representation for $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a),(b).x]$.

In fact,

$$
\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b).x] = {}_{m+p}F_{l+q}\left[\begin{matrix} a_1,\dots,a_p, \Delta(m;-n), \\ b_1,\dots,b_q, \Delta(l;1+\beta-n\alpha); \end{matrix} cx\right]
$$

$$
= \frac{\Gamma\left(\frac{1+\beta-n\alpha}{l}\right)}{\Gamma(a_1)\Gamma\left(\frac{1+\beta-n\alpha}{l}-a_1\right)} \int_0^1 t^{a_1-1}(1-t)^{\frac{1+\beta-n\alpha}{l}-a_1-1}.
$$

$$
\cdot {}_{m+p-1}F_{l+q-1}\left[\begin{matrix} a_2,\dots,a_p, \Delta(m,-n), \\ b_1,\dots,b_q, \frac{2+\beta-n\alpha}{l},\dots,\frac{\beta-n\alpha+l}{l}; \end{matrix} cx\right],
$$

(2.2.3)

where m = 2,3,4,...

Another integral representation for $S_n(l,m,\alpha,\beta;x)$ is obtained using the well know relation

$$
\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad \mathrm{Re}(x)>0, \ \ \mathrm{Re}(y)>0. \qquad \textbf{(2.2.4)}
$$

First replacing $S_n(l,m,\alpha,\beta,x)$ by $\dfrac{S_n(l,m,\alpha,\beta,x)}{\Gamma(1+\beta-n\alpha)}$ and assuming

Re$(1+\beta-n\alpha)>0$, Re$(lk)>0$, we have

$$S_n(l,m,\alpha,\beta;x)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\,\Gamma(1+\beta-n\alpha)\,\sigma_k\,x^k}{\Gamma(1+\beta-n\alpha+lk)\,(n-mk)!}$$

$$=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\,\sigma_k\,x^k}{(n-mk)!\,\Gamma(lk)}\,\frac{\Gamma(1+\beta-n\alpha)\,\Gamma(lk)}{\Gamma(1+\beta-n\alpha+lk)}$$

$$=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\sigma_k\,x^k}{(n-mk)!\Gamma(lk)}\int_0^1 t^{\beta-n\alpha}(1-t)^{lk-1}\,dt,$$

$$=\int_0^1 S_n(l,m;x)\,t^{\beta-n\alpha}(1-t)^{lk-1}\,dt, \qquad \textbf{(2.2.5)}$$

wherein

$$S_n(l,m,x)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\sigma_k x^k}{\Gamma(lk)(n-mk)!}.$$

Consider,

$$\int_0^1 t^{(\lambda+n+1)-1}(1-t)^{\mu-1}\,S_n(l,m,\alpha,\beta,xt)\,dt$$

$$=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\sigma_k\,x^k}{\Gamma(1+\beta-n\alpha+lk)\,(n-mk)!}\int_0^1 t^{(\lambda+n+k+1)-1}(1-t)^{\mu-1}\,dt$$

$$=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\sigma_k\,x^k}{\Gamma(1+\beta-n\alpha+lk)(n-mk)!}\,\frac{\Gamma(\lambda+n+k+1)\Gamma(\mu)}{\Gamma(\mu+\lambda+n+k+1)} \qquad \textbf{(2.2.6)}$$

Since $e^t.e^{-t}=1$, one gets from (2.2.6),

$$\int_0^1 t^{(\lambda+n+1)-1}(1-t)^{\mu-1}\,S_n(l,m,\alpha,\beta;xt)\,dt$$

$$=\int_0^1 t^{(\lambda+n+1)-1}(1-t)^{\mu-1}\,e^{xt}\,e^{-xt}\,S_n(l,m,\alpha,\beta;xt)\,dt. \qquad \textbf{(2.2.7)}$$

Using (2.2.6) the left member which we denote by L, becomes

$$L = \Gamma(\mu) \sum_{k=0}^{[n/m]} \frac{(-1)^{mk} \sigma_k x^k \Gamma(\lambda+n+k+1)}{(n-mk)!\, \Gamma(1+\beta-n\alpha+lk)\Gamma(\mu+\lambda+n+k+1)} \tag{2.2.8}$$

and the right member R is

$$R = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^{i+j}}{i!\, j!} \int_0^1 t^{(\lambda+n+i+j+1)-1} (1-t)^{\mu-1} S_n(l,m,\alpha,\beta;xt)\, dt.$$

Now using (2.2.6), one obtains

$$R = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^{i+j}\, \Gamma(\mu)}{i!\, j!} \sum_{k=0}^{[n/m]} \frac{(-1)^{mk} \sigma_k x^k \Gamma(\lambda+n+k+i+j+1)}{(n-mk)!\Gamma(1+\beta-n\alpha+lk)\Gamma(\mu+\lambda+n+k+i+j+1)}. \tag{2.2.9}$$

In view of (2.2.8) and (2.2.9), (2.2.7) yields the transformation in the form:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-1)^{mk} \sigma_k \Gamma(\lambda+n+k+i+j+1)\, x^{i+j+k}}{i!\, j!(n-mk)!\, \Gamma(1+\beta-n\alpha+lk)\Gamma(\mu+\lambda+n+k+i+j+1)}$$

$$= \sum_{k=0}^{[n/m]} \frac{(-1)^{mk} \sigma_k x^k \Gamma(\lambda+n+k+1)}{(n-mk)!\Gamma(1+\beta-n\alpha+lk)\Gamma(\mu+\lambda+n+k+1)} \tag{2.2.10}$$

2.3 DIFFERENTIAL EQUATION (θ-FORM)

Here the particular case $\sigma_k = \frac{1}{k!}$ of $S_n(l,m,\alpha,\beta;x)$, denoted by $T_n^m(x)$ will be used to get the (θ-form) differential equation.

It is known that (E.D. Rainville [1]), the θ-form differential equation satisfied by

$$y = {}_pF_q\left[\begin{matrix} a_1, \ldots a_p; \\ \\ b_1, \ldots b_q \end{matrix} \; z\right]$$

is

$$[\theta \prod_{j=1}^{q} (\theta + b_j - 1) - z \prod_{i=1}^{p} (\theta + a_i)]y = 0, \text{ where } \theta = z\frac{d}{dz} \; . \qquad \textbf{(2.3.1)}$$

Therefore, the differential equation satisfied by $T_n^m(x)$ will be:

$$[\theta(\theta + \frac{1+\beta-n\alpha}{l} - 1)(\theta + \frac{2+\beta-n\alpha}{l} - 1) \ldots (\theta + \frac{\beta-n\alpha+l}{l} - 1) - cx(\theta - \frac{n}{m})(\theta + \frac{-n+1}{m}) \; .$$

$$(\theta + \frac{-n+m-1}{m})]\, w = 0 \; . \qquad \textbf{(2.3.2)}$$

where

$$W = T_n^m(x) = {}_mF_l\left[\begin{matrix} \frac{-n}{m}, \frac{-n+1}{m}, \ldots, \frac{-n+m-1}{m}, \\ \\ \frac{1+\beta-n\alpha}{l}, \frac{2+\beta-n\alpha}{l}, \ldots, \frac{\beta-n\alpha+l}{l}, \end{matrix} \; cx\right].$$

2.4 INVERSE SERIES RELATIONS

Amongst several general classes of polynomials (basic and ordinary) available in the literature such as $\{g_n^c(x,r,s)\}, \{g_n^c(x,r,s\,|\,q)\}$, $\{f_n^c(x,y,r,m)\}$, $\{f_n^c(x,y,r,m\,|\,q\}$, $\{F_{n,l,m}^{(\alpha,\beta)}(\alpha_1 \ldots\ldots \alpha_p; \beta_1 \ldots \beta_q; x)\}$ and $\{\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[\alpha_1, \ldots \alpha_p; \beta_1 \ldots \beta_q; x]\}$ studied by H.M.Srivastava [5], R.Panda [1], J.P.Singhal and Savita Kumari [1], B.I.Dave ([1]) etc., the extended Jacobi polynomial $\{\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[\alpha_1, \ \alpha_p, \beta_1 \ \ \beta_q, x]\}$ is considered here with a view to derive its inverse series relations. The explicit form of this polynomial is

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[\alpha_1, \ \alpha_p; \beta_1 \ .. \beta_q : x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} (\alpha_1)_k \cdot \ldots\ldots (\alpha_p)_k}{k! (1-n\alpha+\beta)_{lk} (\beta_1)_k \ldots\ldots (\beta_q)_k}. \tag{2.4.1}$$

Motivated by the study mentioned above, an effort is made here to find an inverse series of this polynomial. The investigation of inverse series relation of this polynomial resulted finally into the construction of a general inversion pair which is stated in the form of

Theorem-1.

If n is a non-negative integer l and m are positive integers (α and β are arbitrary parameters) then,

$$f(n) = \sum_{k=0}^{[n/m]} \frac{(-1)^{n-mk} \ g(k)}{(n-mk)! \, \Gamma(1+\beta+lk-n\alpha)} \tag{2.4.2}$$

implies

$$g(n) = \sum_{k=0}^{mn} \frac{(\beta+ln-\alpha mn)\Gamma(\beta+ln-\alpha k)}{(mn-k)!} f(k) \tag{2.4.3}$$

provided that $l \leq m$. The proof of this is given in section 2.5.

While working on the question "whether the converse of theorem-1 exists ?" – it was found that the above inverse relation does not hold conversely unless the second series is modified and then tested. An attempt made in this direction led to the following theorem in which the pair of inverse series relation is such that each of the series relation implies the other.

Theorem–2.

If

$$G(n) = \sum_{k=0}^{[n/m]} a(n,k,m)\, F(k) \qquad \textbf{(2.4.4)}$$

and

$$F(n) = \sum_{k=0}^{mn} b(n,k,m)\, G(k) \qquad \textbf{(2.4.5)}$$

then

$$a(n,k,m) = \frac{1}{\Gamma(\beta + mk\alpha - n\alpha + 1)\,(n-mk)!} \qquad \textbf{(2.4.6)}$$

implies and is implied by

$$b(n,k,m) = (-1)^{mn-k} \frac{\beta\, \Gamma(\beta + mn\alpha - k\alpha)}{(mn-k)!} \qquad \textbf{(2.4.7)}$$

and

$$\sum_{k=0}^{n} b(n,k,1)\, G(k) = 0, \qquad \text{when } n \neq ms,\ s \in N. \qquad \textbf{(2.4.8)}$$

Theorem–2 is proved in section-2.7. The following theorem implied by theorem-2 is also useful.

Theorem-3.

If

$$\mu(n) = \sum_{k=0}^{[n/m]} c(n,k,m)\, v(k) \tag{2.4.9}$$

and

$$v(n) = \sum_{k=0}^{mn} d(n,k,m)\, \mu(k) \tag{2.4.10}$$

then with $0 \le \alpha \le 1$, $m\alpha \in N$,

$$c(n,k,m) = \frac{(-1)^{n-mk}\,\beta\,\Gamma(\beta + n\alpha - mk\alpha)}{(n-mk)!} \tag{2.4.11}$$

$$\Leftrightarrow \qquad d(n,k,m) = \frac{1}{\Gamma(\beta + k\alpha - mn\alpha + 1)\,(mn-k)!} \tag{2.4.12}$$

and

$$\sum_{k=0}^{n} d(n,k,1)\, \mu(k) = 0 \qquad \text{when } n \ne ms. \tag{2.4.13}$$

From this, it is possible to invert the Brafman polynomial. A few combinatorial identities are also occurring as the special cases. Section-2.8 contains the proof of this theorem.

2.5 PROOF OF THEOREM-1

In order to prove the theorem, let the right hand member of (2.4.3) be denoted by, say ϕ, then

$$\phi=\sum_{k=0}^{mn}\frac{(\beta+ln-\alpha mn)\,\Gamma(\beta+ln-\alpha k)}{(mn-k)!}f(k).$$

Now on making use of (2.4.2) one gets,

$$\phi=\sum_{k=0}^{mn}\frac{(\beta+ln-\alpha mn)\,\Gamma(\beta+ln-\alpha k)}{(mn-k)!}\sum_{r=0}^{[k/m]}\frac{(-1)^{k-mr}\,g(r)}{\Gamma(1+\beta+lr-\alpha k)\,(k-mr)!}$$

$$=\sum_{k=0}^{mn}\sum_{r=0}^{[k/m]}\frac{(-1)^{k-mr}(\beta+ln-\alpha mn)\,\Gamma\big(\beta+ln-\alpha k\big)\,g(r)}{(mn-k)!\,\Gamma(1+\beta+lr-\alpha k)\,(k-mr)!}.$$

Using the double series relation:

$$\sum_{i=0}^{mn}\sum_{j=0}^{[i/m]}A(i,j)=\sum_{j=0}^{n}\sum_{i=0}^{mn-mj}A(i+mj,j) \qquad \textbf{(2.5.1)}$$

it becomes

$$\phi=\sum_{r=0}^{n}\sum_{k=0}^{mn-mr}\frac{(-1)^{k}(\beta+ln-\alpha mn)\,\Gamma(\beta+ln-\alpha k-\alpha mr)\,g(r)}{(mn-mr-k)!\;k!\;\Gamma(1+\beta+lr-\alpha k-\alpha mr)}$$

$$=g(n)+(\beta+ln-\alpha mn)\sum_{r=0}^{n-1}g(r)\sum_{k=0}^{mn-mr}(-1)^{k}\binom{mn-mr}{k}$$

$$\frac{\Gamma(\beta+ln-\alpha k-\alpha mr)}{(mn-mr)!\;\Gamma(1+\beta+lr-\alpha k-\alpha mr)}.$$

But

$$\frac{\Gamma(\beta+ln-\alpha k-\alpha mr)}{\Gamma(1+\beta+lr-\alpha k-\alpha mr)}=\prod_{i=1}^{ln-lr-1}(\beta-\alpha k-\alpha mr+ln-i)$$

$$=\prod_{i=1}^{ln-lr-1}a_i\,k^{i}\text{, say}$$

which is a polynomial in k of degree ($ln-lr-1$) and $l\leq m$, therefore the inner series in k above is the $(mn-mr)^{th}$ difference of a polynomial in k of degree less than mn-mr, and hence it is zero. Thus, $\phi = g(n)$ which completes the proof of theorem-1.

2.6 AN AUXILIARY INVERSE SERIES RELATION

For proving the theorems-2 and 3 the following inversion pair will be used.

$$U_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{\beta\,\Gamma(\beta + n\alpha - k\alpha)\, V_k}{(n-k)!} \tag{2.6.1}$$

$\Leftrightarrow$

$$V_n = \sum_{k=0}^{n} \frac{U_k}{\Gamma(\beta + k\alpha - n\alpha + 1)\,(n-k)!}. \tag{2.6.2}$$

Proof

First it will be proved that (2.6.1) $\Rightarrow$ (2.6.2). If the right hand side of (2.6.2) is denoted by T_n that is

$$T_n = \sum_{k=0}^{n} \frac{U_k}{\Gamma(\beta + k\alpha - n\alpha + 1)\,(n-k)!},$$

Then using (2.6.1) one gets

$$T_n = \sum_{k=0}^{n} \sum_{r=0}^{k} \frac{(-1)^{k-r}\,\beta\,\Gamma(\beta + k\alpha - r\alpha)\, V_r}{(k-r)!\,(n-k)!\,\Gamma(\beta + k\alpha - n\alpha + 1)}.$$

In view of the relation

$$\sum_{k=0}^{n} \sum_{j=0}^{k} A(k,j) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} A(k+j,j) \tag{2.6.3}$$

one further gets

$$T_n = \sum_{r=0}^{n} \sum_{k=0}^{n-r} \frac{(-1)^k \beta\, \Gamma(\beta + k\alpha)\, V_r}{\Gamma(\beta + k\alpha + r\alpha - n\alpha + 1)\,(n-r-k)!\,k!}$$

$$= V_n + \sum_{r=0}^{n-1} \sum_{k=0}^{n-r} \frac{(-1)^k \beta\, \Gamma(\beta + k\alpha)\, V_r}{\Gamma(\beta + k\alpha + r\alpha - n\alpha + 1)\,(n-r-k)!\,k!}.$$

Here

$\dfrac{\Gamma(\beta + k\alpha)}{\Gamma(\beta + k\alpha + r\alpha - n\alpha + 1)} = \sum_{j=0}^{n\alpha - r\alpha - 1} c_j\, k^j$ which is a polynomial in k of degree $n\alpha - r\alpha - 1$

Thus,

$$T_n = V_n + \sum_{r=0}^{n-1} \sum_{k=0}^{n-r} \frac{(-1)^k \beta\, V_r}{(n-r-k)!\,k!} \sum_{j=0}^{n\alpha - r\alpha - 1} c_j\, k^j$$

therefore, $T_n = V_n$ as the inner series in k on the right hand side above is the $(n\text{-}r)^{th}$ difference of a polynomial in k of degree less than (n-r) (precisely a polynomial of degree $n\alpha - r\alpha - 1$). Thus, (2.6.1) $\Rightarrow$ (2.6.2) which completes the first part of the proof. To prove: (2.6.2) implies (2.6.1), it is sufficient to show that the diagonal elements of the coefficient matrices in (2.6.1) and (2.6.2) are all non-zero. In fact if the diagonal elements are denoted by a_{nn} and b_{nn} for (2.6.1) and (2.6.2) respectively then

$$a_{nn} = \Gamma(\beta + 1) \neq 0 \quad \text{and} \quad b_{nn} = \frac{1}{\Gamma(\beta + 1)} \neq 0$$

whence (2.6.2) $\Rightarrow$ (2.6.1). Thus, (2.6.1) $\Leftrightarrow$ (2.6.2). This completes the proof of the auxiliary inversion pair.

2.7 PROOF OF THEOREM-2

In order to prove the theorem, it will be first proved that (2.4.6) $\Rightarrow$ (2.4.7) and (2.4.8).

Put

$$t(n)=\sum_{k=0}^{mn}(-1)^{mn-k}\frac{\beta\,\Gamma(\beta+mn\alpha-k\alpha)\,G(k)}{(mn-k)!}$$

Then in view of (2.4.4) and (2.4.6) t(n) becomes,

$$t(n)=\sum_{k=0}^{mn}(-1)^{mn-k}\frac{\beta\,\Gamma(\beta+mn\alpha-k\alpha)}{(mn-k)!}\sum_{j=0}^{[k/m]}\frac{F(j)}{\Gamma(\beta+mj\alpha-k\alpha+1)\,(k-mj)!}$$

$$=\sum_{k=0}^{mn}\sum_{j=0}^{[k/m]}\frac{(-1)^{mn-k}\beta\,\Gamma(\beta+mn\alpha-k\alpha)\,F(j)}{(mn-k)!\,(k-mj)!\,\Gamma(\beta+mj\alpha-k\alpha+1)}$$

The double series identity (2.5.1) now leads us to,

$$t(n)=\sum_{j=0}^{n}\sum_{k=0}^{mn-mj}\frac{(-1)^{mn-mj-k}\beta\,\Gamma(\beta+mn\alpha-mj\alpha-k\alpha)\,F(j)}{(mn-mj-k)!\,k!\,\Gamma(\beta-k\alpha+1)}$$

$$=F(n)+\sum_{j=0}^{n-1}\frac{(-1)^{mn-mj}\beta\,F(j)}{(mn-mj)!}\sum_{k=0}^{mn-mj}(-1)^{k}\binom{mn-mj}{k}\cdot\frac{\Gamma(\beta+mn\alpha-mj\alpha-k\alpha)}{\Gamma(\beta-k\alpha+1)}.$$

Here,

$$\frac{\Gamma(\beta+mn\alpha-mj\alpha-k\alpha)}{\Gamma(\beta-k\alpha+1)}=\sum_{r=0}^{mn\alpha-mj\alpha-1}c_r\,k^r=P(k),\ \text{say}$$

which is a polynomial in k of degree $mn\alpha-mj\alpha-1$, and so,

$$t(n)=F(n)+\sum_{j=0}^{n-1}\frac{(-1)^{mn-mj}\beta\,F(j)}{(mn-mj)!}\sum_{k=0}^{mn-mj}(-1)^{k}\binom{mn-mj}{k}P(k).$$

If $0 \leq \alpha \leq 1$ and $m\alpha$ is an integer then the inner series in k is the $(mn-mj)^{th}$ difference of a polynomial in k of degree less than mn-mj, and therefore t(n)=F(n)+0=F(n).

So the proof of (2.4.6) $\Rightarrow$ (2.4.7) is complete. To prove that (2.4.6) also implies (2.4.8) put

$$\phi(n) = \sum_{k=0}^{n} \frac{(-1)^{n-k}\, \beta\, \Gamma(\beta + n\alpha - k\alpha)}{(n-k)!} G(k) \qquad \textbf{(2.7.1)}$$

and substitute for G(k) using (2.4.4) and (2.4.6), then

$$\phi(n) = \sum_{k=0}^{n} \sum_{j=0}^{[k/m]} (-1)^{n-k} \frac{\beta\, \Gamma(\beta + n\alpha - k\alpha)\, F(j)}{\Gamma(\beta + mj\alpha - k\alpha + 1)\,(n-k)!\,(k-mj)!}\,.$$

Now using the double series relation,

$$\sum_{k=0}^{n} \sum_{j=0}^{[k/m]} A(k,j) = \sum_{j=0}^{[n/m]} \sum_{k=0}^{n-mj} A(k+mj, j)\,, \qquad \textbf{(2.7.2)}$$

$\phi(n)$ reduces to,

$$\phi(n) = \sum_{j=0}^{[n/m]} \sum_{k=0}^{n-mj} \frac{(-1)^{n-k-mj}\, \beta\, \Gamma(\beta + n\alpha - k\alpha - mj\alpha)\, F(j)}{\Gamma(\beta - k\alpha + 1)\,(n-k-mj)!\, k!}$$

$$= \sum_{j=0}^{[n/m]} \frac{(-1)^{n-mj}\, \beta\, F(j)}{(n-mj)!} \sum_{k=0}^{n-mj} (-1)^k \binom{n-mj}{k} \frac{\Gamma(\beta + n\alpha - k\alpha - mj\alpha)}{\Gamma(\beta - k\alpha + 1)}\,.$$

As shown above, here also the ratio of the gamma functions in the inner series is a polynomial in k of degree $(n\alpha - mj\alpha - 1) < (n - mj)$, that is

$$\frac{\Gamma(\beta + n\alpha - k\alpha - mj\alpha)}{\Gamma(\beta - k\alpha + 1)} = \sum_{r=0}^{n\alpha - mj\alpha - 1} c_r\, k^r = P(k);\ n \neq mj\,.$$

Therefore, the inner series in k above, being the (n-mj$)^{th}$ difference of a polynomial in k of degree $n\alpha - mj\alpha - 1$, is zero. Thus, $\phi(n) = 0$, if n≠mj, which completes the proof of the "only if" part.

To prove the converse part it is assumed that the relations (2.4.7) and (2.4.8) viz.

$b(n,k,m) = (-1)^{mn-k} \dfrac{\beta\,\Gamma(\beta + mn\alpha - k\alpha)}{(mn-k)!}$ and $\sum\limits_{k=0}^{n} b(n,k,1)\, G(k) = 0$, when n≠ms,

s = 1,2,3,...... hold true.

Now in view of (2.7.1) and (2.4.8) one readily gets

$$\phi(n) = 0, \; n \neq sm \; s = 1,2,3,....., \tag{2.7.3}$$

and also by comparing (2.7.1) and (2.4.5) with (2.4.7), one finds $\phi(nm) = \phi(mn) = F(n)$.

Thus, it follows that with (2.7.3),

$$\phi(n) = \sum_{k=0}^{mn} (-1)^{mn-k} \frac{\beta\,\Gamma(\beta + mn\alpha - k\alpha)\, \psi(k)}{(mn-k)!}$$

implies

$$\psi(k) = \sum_{k=0}^{[n/m]} \frac{\phi(k)}{\Gamma(\beta + mk\alpha - n\alpha + 1)\,(n-mk)!}$$

completing the proof of the 'if' part, and hence that of the theorem.

2.8 PROOF OF THEOREM-3

To prove the "only if" part, it will be first shown that (2.4.11) implies (2.4.12).

Beginning just as in section-2.7, with the notation μ^* (instead of t(n) there) one finds

$$\mu^* = \sum_{k=0}^{mn} \sum_{j=0}^{[k/m]} \frac{(-1)^{k-mj} \beta\, \Gamma(\beta+k\alpha-mj\alpha)\, v(j)}{\Gamma(\beta+k\alpha-mn\alpha+1)\,(k-mj)!\,(mn-k)!}$$

which by means of (2.5.1), reduces to

$$\mu^* = \sum_{j=0}^{n} \frac{\beta\, v(j)}{(mn-mj)!} \sum_{k=0}^{mn-mj} (-1)^k \binom{mn-mj}{k} \frac{\Gamma(\beta+k\alpha)}{\Gamma(\beta+k\alpha+mj\alpha-mn\alpha+1)}$$

$$= v(n) + \sum_{j=0}^{n-1} \frac{\beta\, v(j)}{(mn-mj)!} \sum_{k=0}^{mn-mj} (-1)^k \binom{mn-mj}{k} \frac{\Gamma(\beta+k\alpha)}{\Gamma(\beta+k\alpha+mj\alpha-mn\alpha+1)}.$$

But,

$$\frac{\Gamma(\beta+k\alpha)}{\Gamma(\beta+k\alpha+mj\alpha-mn\alpha+1)} = \sum_{r=0}^{mn\alpha-mj\alpha-1} c_r\, k^r = P(k), \text{ (say)}$$

is a polynomial in k of degree precisely $mn\alpha - mj\alpha - 1$. Replacing the ratio of the gamma functions above by this polynomial P(k), the inner series in k becomes the (mn-mj)th difference of a polynomial in k of degree $(mn\alpha - mj\alpha - 1) < (mn - mj)$ as mα is an integer, and therefore the inner series vanishes.

Thus, $\mu^* = v(n)$. This completes the proof of (2.4.11) $\Rightarrow$ (2.4.12).

To complete the proof of the first part it suffices to show that (2.4.11) implies (2.4.13), when n/m is not an integer.

For this set,

$$B_n = \sum_{k=0}^{n} \frac{\mu(k)}{\Gamma(\beta+k\alpha-n\alpha+1)\,(n-k)!}. \qquad \textbf{(2.8.1)}$$

On using (2.4.9) with (2.4.11) this becomes,

$$B_n = \sum_{k=0}^{n} \sum_{j=0}^{[k/m]} \frac{(-1)^{k-mj} \beta\, \Gamma(\beta + k\alpha - mj\alpha)\, v(j)}{\Gamma(\beta + k\alpha - n\alpha + 1)(n-k)!(k-mj)!}.$$

With an appeal to (2.7.2), this reduces to

$$B_n = \sum_{j=0}^{[n/m]} \frac{\beta\, v(j)}{(n-mj)!} \sum_{k=0}^{n-mj} (-1)^k \binom{n-mj}{k} \frac{\Gamma(\beta + k\alpha)}{\Gamma(\beta + k\alpha - n\alpha + mj\alpha + 1)}.$$

Once again,

$$\frac{\Gamma(\beta + k\alpha)}{\Gamma(\beta + k\alpha - n\alpha + mj\alpha + 1)} = \sum_{s=0}^{n\alpha - mj\alpha - 1} C_s\, k^s = P(k)$$

a polynomial in k of degree $n\alpha - mj\alpha - 1$ where $n\alpha$ and $m\alpha$ are integers, $0 \le \alpha \le 1$.

Therefore,

$$B_n = \sum_{j=0}^{[n/m]} \frac{\beta\, v(j)}{(n-mj)!} \sum_{k=0}^{n-mj} (-1)^k \binom{n-mj}{k} P(k) = 0$$

And thus it is proved that (2.4.11) implies (2.4.13) which completes the proof of the first part.

To prove the 'if' part, it is to be shown that (2.4.12) and (2.4.13) together imply (2.4.11), wherein (2.4.9) and (2.4.10) are used.

First it is to be noted that in view of the inversion pair of section-2.6, the inverse relation of (2.8.1) (taking $B_n = V(n)$ and $\mu(n) = U(n)$) is given by

$$U(n) = \sum_{k=0}^{n} \frac{(-1)^{n-k} \beta\, \Gamma(\beta + n\alpha - k\alpha)}{(n-k)!} V(k). \qquad \textbf{(2.8.2)}$$

Since it is assumed that (2.4.13) holds good, V(n)=0 if n≠ms. If n = ms then from (2.4.12) it follows that V(mn) = V(nm) = $v(n)$. Thus the inversion pair (2.8.1) and (2.8.2) assumes the form:

$$v(n)=\sum_{k=0}^{mn}\frac{U(k)}{\Gamma(\beta+k\alpha-mn\alpha+1)\,(mn-k)!}$$

implies

$$U(n)=\sum_{k=0}^{[n/m]}\frac{(-1)^{n-mk}\,\beta\,\Gamma(\beta+n\alpha-mk\alpha)\,v(k)}{(n-mk)!}$$

which with U(n)=μ(n) proves the if part. And this completes the proof of theorem-3.

2.9 PARTICULAR CASES

As mentioned in section-2.1, the general class $\{S_n(l,m,\alpha,\beta,x)$; n = 0,1,2,....} will be particularized to different polynomials such as the extended Jacobi polynomial $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a),(b)\cdot x]$, the Brafman polynomial $B_n^m[(a);(b):x]$,the well known Laguerre and Jacobi polynomials, the biorthogonal polynomials $Z_n^{\alpha}(x;k)$ and $W_n^{(\alpha,\beta)}(x,k)$ Then the above studied properties (in sections-2.2 to 2.8) will be illustrated in this section for these specialized polynomials.

In the first place, the polynomial $S_n(l,m,\alpha,\beta;x)$ will be particularized to the extended Jacobi polynomial $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b)\cdot x]$ (H.M.Srivastava & M.A.Pathan [1]) by choosing $\sigma_k=\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k\,k!}$, denoting $m\alpha$ by l and replacing $S_n(l,m,\alpha,\beta,x)$ by $\dfrac{S_n(l,m,\alpha,\beta,x)}{\Gamma(1+\beta-n\alpha)\,n!}$

Thus,

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a),(b):x]=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\Gamma(1+\beta-n\alpha)\,n!}{\Gamma(1+\beta-n\alpha+lk)\,(n-mk)!}\,\frac{(a_1)_k\;\cdots(a_p)_k\;x^k}{(b_1)_k\;\cdots(b_q)_k\;k!}. \quad \textbf{(2.9.1)}$$

To obtain the Brafman polynomial, take

$\sigma_k=\dfrac{(a_1)_k\cdots\cdots(a_p)_k}{(b_1)_k\cdots\cdots(b_q)_k\,k!}$, l = 0 and replace $S_n(l,m,\alpha,\beta,x)$ by $\dfrac{S_n(l,m,\alpha,\beta;x)}{\Gamma(1+\beta-n\alpha)\,n!}$.

Then

$$B_n^m[a_1,\ldots a_p;b_1,\ldots b_q:x]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}\,(a_1)_k\cdots(a_p)_k\;x^k}{(b_1)_k\cdots(b_q)_k\;k!}. \quad \textbf{(2.9.2)}$$

The Brafman polynomial with variable cx, where c is a constant, can also be obtained using another set of values for the parameters. This reducibility is of interest, for it will be useful in obtaining another integral form of the Brafman polynomial.

For that taking α = 0, replacing $S_n(l,m,0,\beta;x)$ by $\dfrac{S_n(l,m,0,\beta;x)}{\Gamma(1+\beta)\,n!}$, and choosing

$$\sigma_k=\frac{(a_1)_k\cdots\cdots(a_p)_k\left(\frac{1+\beta}{l}\right)_k\left(\frac{2+\beta}{l}\right)_k\cdots\cdots\left(\frac{l+\beta}{l}\right)_k}{(b_1)_k\cdots(b_q)_k\;k!},\qquad l\in N, \quad \textbf{(2.9.3)}$$

the polynomial $S_n(l,m,0,\beta;x)$ would reduce to

$$B_n^m[a_1,\ldots a_p,\Delta(l;1+\beta);b_1,\ldots\ldots b_q:cx],\qquad c=l^l. \quad \textbf{(2.9.4)}$$

Next, an extension of the biorthogonal polynomial $Z_n^{\alpha}(x,k)$, k∈N, denoted here by $Z_{n,m}^{\alpha}(x,k)$ is obtained from (2.1.1) by putting l=0,

$\sigma_j = \dfrac{1}{j!\left(\frac{\alpha+1}{k}\right)_j\left(\frac{\alpha+2}{k}\right)_j\cdots\cdots\left(\frac{\alpha+k}{k}\right)_j}$ and replacing $S_n(l,m,\alpha,\beta;x)$ by

$\dfrac{S_n(l,m,\alpha,\beta;x)}{\Gamma(1+\beta-n\alpha)\,(1+\alpha)_{kn}}$ and x by $\left(\dfrac{x}{k}\right)^k$. **(2.9.5)**

Thus,

$$Z_{n,m}^{\alpha}(dx,k) = \frac{(1+\alpha)_{kn}}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,(x/k)^{kj}\,d^j}{(1+\alpha)_{kj}\;j!}, \text{ where } d = k^k. \quad \textbf{(2.9.6)}$$

This polynomial can also be obtained from $S_n(l,m,\alpha,\beta,x)$ by taking

α=0, l=k, k∈N, $\sigma_j = \dfrac{1}{j!}$. **(2.9.7)**

And making the same replacements for $S_n(l,m,\alpha,\beta,x)$ and x, as above. When k=1, this polynomial gets reduced to an extended version of the Laguerre polynomial $L_n^{(\alpha)}(x)$ which is denoted here by $L_{n,m}^{(\alpha)}(x)$, given by

$$L_{n,m}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,x^j}{(1+\alpha)_j\;j!}. \quad \textbf{(2.9.8)}$$

The other biorthogonal polynomial $W_n^{(\alpha,\beta)}(x;k)$admits an exterision from (2.1.1), which is denoted here by $W_{n,m}^{(\alpha,\beta)}(x;k), k\in$N. In fact, a particular case $W_{n,m}^{(\beta-n,\alpha)}(cx,k)$is obtained as follows.

Letting l=k, k∈N and replacing x by $\left(\frac{1-x}{2}\right)^k$, and also replacing $S_n(l,m,\alpha,\beta;x)$ by $\frac{S_n(\ell,m,\alpha,\beta,x)}{(1+\beta-n)_n\,\Gamma(1+\beta-n\alpha)}$, one gets

$$S_n(\ell,m,\alpha,\beta,x)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,\Gamma(1+\beta-n\alpha)\,(1+\beta-n)_n\,\sigma_j}{\Gamma(1+\beta-n\alpha+kj)\,n!}\left(\frac{1-x}{2}\right)^{kj}$$

$$=\frac{(1+\beta-n)_n}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,\sigma_j}{(1+\beta-n\alpha)_{kj}}\left(\frac{1-x}{2}\right)^{kj}$$

Now taking α=1 and then choosing $\sigma_j=\frac{\Delta(k,1+\alpha+\beta)}{j!}$, one gets the extension in the form:

$$W_{n,m}^{(\beta-n,\alpha)}(cx;k)=\frac{(1+\beta-n)_n}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,(1+\alpha+\beta)_{kj}\,c^j}{(1+\beta-n)_{kj}\,j!}\left(\frac{1-x}{2}\right)^{kj}. \qquad \textbf{(2.9.9)}$$

A worth mentioning special case of this polynomial occurs if k=1. It is indeed an extended version of the Jacobi polynomial that bears here the symbol $P_{n,m}^{(\alpha,\beta)}(x)$. It is given by

$$P_{n,m}^{(\beta-n,\alpha)}(x)=\frac{(1+\beta-n)_n}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,(1+\alpha+\beta)_j}{(1+\beta-n)_j\,j!}\left(\frac{1-x}{2}\right)^j \qquad \textbf{(2.9.10)}$$

The other properties involving these polynomials will be illustrated as follows.

(A) Special cases of integrals.

(B) Differential equation (θ-form) for the particular polynomials.

(C) Polynomial special cases of the general inversion formulas.

(D) Combinatorial identities.

With this sequence, the first comes

(A) Special Cases of Integrals.

The integral representations (2.2.3) and (2.2.5), in the light of the above mentioned specialized polynomials (2.9.1), (2.9.2), (2.9.4), (2.9.6), (2.9.8), (2.9.9), and (2.9.10) get reduced to the forms which are listed below.

For m=2, 3, 4, ...,

$$\mathcal{H}_{n,\ell,m}^{(\alpha,\beta)}[(a);(b)\cdot cx]=\frac{\Gamma\left(\frac{1+\beta-n\alpha}{\ell}\right)}{\Gamma(a_1)\,\Gamma\left(\frac{1+\beta-n\alpha}{\ell}-a_1\right)}\cdot$$

$$\cdot\int_0^1 t^{a_1-1}\cdot(1-t)^{\frac{1+\beta-n\alpha}{\ell}-a_1-1}\;{}_{m+p-1}F_{\ell+q-1}\left[\begin{matrix}\Delta(m;-n),a_2,\dots,a_p;\\ \frac{2+\beta-n\alpha}{\ell},\dots,\frac{\beta-n\alpha+\ell}{\ell},b_1,\dots,b_q;\end{matrix}\;cx\right]dt,$$

(2.9.11)

which is essentially the integral given in (2.2.3),

$$B_n^m[a_1,..a_p;b_1,\dots,b_q:x]=\frac{\Gamma(b_1)}{\Gamma(a_1)\,\Gamma(b_1-a_1)}\cdot$$

$$\int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1}\;{}_{m+p-1}F_{q-1}\left[\begin{matrix}\Delta(m;-n),a_2,\dots,a_p;\\ b_2,\dots,b_q;\end{matrix}\;cx\right]dt,\quad \textbf{(2.9.12)}$$

$$W_{n,m}^{(\beta-n,\alpha)}(cx,k)=\frac{(1+\beta-n)_n\Gamma\left(\frac{1+\beta-n}{k}\right)}{n!\Gamma\left(\frac{1+\alpha+\beta}{k}\right)\Gamma\left(\frac{1+\beta-n}{k}-\frac{1+\alpha+\beta}{k}\right)}.$$

$$\cdot\int_0^1 t^{\frac{1+\alpha+\beta}{k}-1}(1-t)^{\frac{-(n+\alpha)}{k}-1}.$$

$$\cdot\,{}_{m+k-1}F_{k-1}\left[\begin{matrix}\Delta(m;-n),\frac{2+\alpha+\beta}{k},\ ,\frac{k+\alpha+\beta}{k};\\ \frac{2+\beta-n}{k},\dots,\frac{k+\beta-n}{k},\end{matrix}\ c\left(\frac{1-x}{2}\right)^k\right]dt \tag{2.9.13}$$

$$P_{n,m}^{(\beta-n,\alpha)}(x)=\frac{(1+\beta-n)_n\Gamma(1+\beta-n)}{n!\Gamma(\alpha+\beta)\Gamma(-n-\alpha)}.$$

$$\int_0^1 t^{\alpha+\beta-1}(1-t)^{\beta-n+\frac{n}{m}-1}\ {}_mF_0\left[\begin{matrix}\Delta(m,-n),\\ \underline{\qquad\qquad};\end{matrix}\ \frac{1-x}{2}\right]dt. \tag{2.9.14}$$

The following are the specializations of (2.2.5).

$$\mathcal{H}_{n,\ell,m}^{(\alpha,\beta)}[(a);(b):x]=\int_0^1\mu_n(\ell,m;x)t^{\beta-n\alpha-1}(1-t)^{\ell k-1}dt,$$

where

$$\mu_n(\ell,m;x)=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(a_1)_k\dots(a_p)_k x^k}{(b_1)_k\dots(b_q)_k\ \ k!\ \ \Gamma(\ell k+1)}. \tag{2.9.15}$$

In view of the substitutions mentioned for (2.9.2), one gets

$$B_n^m[a_1,\dots,a_p;b_1,\dots b_q\cdot x]=\int_0^1\lambda_n(\ell,m,x)\ t^{\beta-n}(1-t)^{\ell k}dt,$$

wherein

$$\lambda_n(\ell,m,x)=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(a_1)_k\ \ (a_p)_k(1+\beta)_{\ell k}(cx)^k}{\Gamma(\ell k+1)(b_1)_k\,..(b_q)_k\ \ k!},\quad c=\ell^{\ell} \tag{2.9.16}$$

as before; whereas the choices given in (2.9.3) leads us to the integral form:

$$B_n^m[a_1,\ \ .,a_p;b_1,..\ \ b_q\ \cdot x]=\int_0^1\Delta_n(m,x)\ \ t^{\beta-n\alpha-1}dt,$$

in which

$$\Delta_n(m;x)=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(a_1)_k...(a_p)_k\ \ x^k}{(b_1)_k\ \ (b_q)_k\ \ k!}. \tag{2.9.17}$$

For the extension of the biorthogonal polynomial $Z_n^\alpha(x,k)$, two sets of values of parameters are already discussed in (2.9.5) and (2.9.7). The cases given in (2.9.5) would reduce the formula (2.2.5) to

$$Z_{n,m}^\alpha(dx;k)=\int_0^1\upsilon_n(m;x)\ \ t^{\beta-n\alpha-1}dt, \tag{2.9.18}$$

where

$$\upsilon_n(m;x)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}d^j}{(\alpha+1)_{kj}j!}\left(\frac{x}{k}\right)^{kj};\quad d=k^k,$$

and (2.9.7) leads us to:

$$Z_{n,m}^\alpha(x;k)=\int_0^1\phi_n(k,m;x)\ \ t^{\beta-1}(1-t)^{kj}\,dt, \tag{2.9.19}$$

wherein

$$\phi_n(k,m;x)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{\Gamma(kj+1)j!}\left(\frac{x}{k}\right)^{kj}$$

Taking k=1 in (2.9.20) and (2.9.21), one immediately obtains the following two integrals

$$L_{n,m}^{(\alpha)}(x)=\int_0^1 \varepsilon_n(m,x)\ t^{\beta-n\alpha-1}\ dt, \qquad \textbf{(2.9.20)}$$

in which

$$\varepsilon_n(m,x)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,x^j}{(\alpha+1)_j\,j!}$$

and

$$L_{n,m}^{(\alpha)}(x)=\int_0^1 \gamma_n(m,x)\ t^{\beta-1}(1-t)^j\,dt, \qquad \textbf{(2.9.21)}$$

$$\gamma_n(m,x)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\,x^j}{\Gamma(j+1)j!}$$

respectively.

The integral representations for the biorthogonal polynomial $W_{n,m}^{(\beta-n,\alpha)}(cx;k)$ and the Jacobi polynomial $P_{n,m}^{(\beta-n,\alpha)}(x)$ are

$$W_{n,m}^{(\beta-n,\alpha)}(cx;k)=\int_0^1 \psi_n(k,m,x)\ t^{\beta-n-1}(1-t)^{kj}\,dt, \qquad \textbf{(2.9.22)}$$

with

$$\psi_n(k,m,x)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}(1+\alpha+\beta)_{kj}\,x^j d^j}{\Gamma(kj+1)j!},\ where\ d=k^k$$

and

$$P_{n,m}^{(\beta-n,\alpha)}(x)=\int_0^1 \delta_n(m;x)\ t^{\beta-n-1}(1-t)^j\,dt, \qquad \textbf{(2.9.23)}$$

where

$$\delta_n(m;x)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}(1+\alpha+\beta)_j x^j}{\Gamma(j+1)j!}.$$

(B) <u>Differential equation (θ-form) for the particular polynomials</u>

The differential equation (θ-form) in (2.3.2) which was derived for a particular case $T_n^m(x)$ *of* $S_n(l,m,\alpha,\beta;x)$ likewise for other specialized polynomials, the differential equations may be obtained easily. All such particular differential equations are as given below.

$$[\theta\prod_{j=1}^{l+q}(\theta+\mu_j-1)-x\prod_{i=1}^{m+p}(\theta+\lambda_i)]\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b)\ x]=0, \qquad \textbf{(2.9.24)}$$

where $\mu_j=\dfrac{\beta-n\alpha+j}{l}$ *for* $j=1,2,\ ,l$ *and* $\mu_j=b_{j-l}$, *for* $j=l+1,.\ ,l+q,$

$\lambda_i=\dfrac{-n+i}{m}$ *for* $i=1,2,...,m$ *and* $\lambda_i=a_{i-m}$, *for* $i=m+1,m+2,..,m+p$.

$$[\theta\prod_{j=1}^{q}(\theta+\mu_j-1)-x\prod_{i=1}^{m+p}(\theta+\lambda_i)]\ B_n^m[a_1,...a_p,b_1,..,b_q:x]=0, \qquad \textbf{(2.9.25)}$$

where $\mu_j=b_j, j=1,2,...,q, \lambda_i=\dfrac{-n+i}{m}$ *for* $i=1,2,...,m$ *and*

$\lambda_i=a_{i-m}$, *for* $i=m+1,m+2,..,m+p$.

$$[\theta\prod_{j=1}^{k}(\theta+\mu_j-1)-(x/k)^k\prod_{i=1}^{m}(\theta+\lambda_i)]\ Z_{n,m}^{\alpha}(dx,k)=0, \qquad \textbf{(2.9.26)}$$

where

$$\mu_j = \frac{1+\alpha+j}{k}, j=1,2,\ldots,k \ ; \ \lambda_i = \frac{-n+i}{m}, i=1,2, \ ,m.$$

$$[\theta(\theta+\alpha)-x\prod_{i=1}^{m}(\theta+\lambda_i)]\, L_{n,m}^{(\alpha)}(x)=0, \tag{2.9.27}$$

where $\lambda_i = \frac{-n+i}{m}$, $i=1,2,.\ ,m$.

$$[\theta(\theta+\alpha)-x(\theta-n)]\, L_{n,m}^{(\alpha)}(x)=0. \tag{2.9.28}$$

$$[\theta\prod_{j=1}^{k}(\theta+\mu_j-1)-b\left(\frac{1-x}{2}\right)^k \prod_{i=1}^{m+k}(\theta+\lambda_i)]\, W_{n,m}^{(\beta-n,\alpha)}(cx;k)=0, \tag{2.9.29}$$

where $\mu_j = \frac{1+\beta-n+j}{k}, j=1,2,\ldots,k, \quad \lambda_i = \frac{-n+i}{m}, i=1,2,..,m$ and

$$\lambda_i = \frac{\alpha+\beta+i-m}{k} \quad for \quad i=m+1,m+2, \ ,m+k. \quad ,$$

$$[\theta(\theta+\beta-n)-\left(\frac{1-x}{2}\right)\prod_{i=1}^{m+1}(\theta+\lambda_i)]\, P_{n,m}^{(\beta-n,\alpha)}(x)=0, \tag{2.9.30}$$

where $\lambda_i = \frac{-n+i}{m}$, $for \ i=1,2,..,m$ and $\lambda_{m+1}=1+\alpha+\beta$.

$$[\theta(\theta+\beta-n)-\left(\frac{1-x}{2}\right)(\theta+n)(\theta+1+\alpha+\beta)]\, P_n^{(\beta-n,\alpha)}(x)=0, \tag{2.9.31}$$

(C) Polynomial special cases of the general inversion formulas

The explicit representation (2.1.1) when compared with (2.4.6) of theorem-2, one finds immediately its inverse series in view of (2.4.7) in the form

$$\sigma_n x^n = \sum_{k=0}^{mn} \frac{(-1)^k \ \beta\,\Gamma(\beta+mn\alpha-k\alpha)}{(mn-k)!} S_k(\ell,m,\alpha,\beta;x), \tag{2.9.32}$$

subject to the condition

$$\sum_{k=0}^{n} \frac{(-1)^k\, \Gamma(\beta+n\alpha-k\alpha)}{(n-k)!} S_k(l,m,\alpha,\beta;x)=0,$$ n≠ms, s∈N.

From this or directly from theorem-2, the inverse series of the extended Jacobi polynomial, Brafman polynomial, extended biorthogonal polynomial $Z_{n,m}^{(\alpha)}(x;k)$ and $W_{n,m}^{(\beta-n,\alpha)}(x;k)$, extended Laguerre polynomial $L_{n,m}^{(\alpha)}(x)$, $P_{n,m}^{(\beta-n,\alpha)}(x)$ are easily obtainable in the forms

$$x^n = \frac{n!(b_1)_n \cdots (b_q)_n}{(a_1)_n \cdots (a_p)_n\, c^n} \sum_{k=0}^{mn} \frac{(-1)^k \beta\, \Gamma(\beta+ln-k\alpha)}{k!(mn-k)!\,\Gamma(\beta-k\alpha+1)} \mathcal{H}_{k,\ell,m}^{(\alpha,\beta)}[(a),(b),x] \quad \textbf{(2.9.33)}$$

subject to

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} \beta\, \Gamma(\beta+n\alpha-k\alpha)}{k!\,(n-k)!\,\Gamma(\beta-k\alpha+1)} \mathcal{H}_{k,\ell,m}^{(\alpha,\beta)}[(a);(b);x]=0,$$

when n≠ms, s∈N.

$$x^n = \frac{(b_1)_n \cdots (b_q)_n\, n!}{(a_1)_n \cdots (a_p)_n (mn)!} \sum_{k=0}^{mn} \frac{(-1)^{mn}(-mn)_k\, \beta}{k!(\beta-k\alpha)} B_k^m[a_1,\ldots,a_p;b_1,\ldots,b_q : x]$$

(2.9.34)

$$x^{nk} = \frac{n!(1+\alpha)_{nk}}{(mn)!} \sum_{j=0}^{mn} \frac{(-1)^{mn}(-mn)_j\, \beta}{k!\,(1+\alpha)_{kj}\,(\beta-k\alpha)} Z_{j,m}^{(\alpha)}(x;k) \quad \textbf{(2.9.35)}$$

$$\left(\frac{1-x}{2}\right)^{nk} = \frac{n!(-1)^{mn} k^{nk}}{(1+\alpha+\beta)_{nk}} \sum_{j=0}^{mn} \frac{(-mn)_j\, \beta\, \Gamma(\beta-j\alpha+nk)\, W_{j,m}^{(\beta-n,\alpha)}(x;k)}{(mn)!\,(1+\beta-j)_j\, \Gamma(1+\beta-j\alpha)} \quad \textbf{(2.9.36)}$$

$$x^n = \frac{n!(1+\alpha)_n}{(mn)!} \sum_{j=0}^{mn} \frac{(-1)^{mn}(-mn)_j \, \beta}{k!\,(1+\alpha)_j (\beta - j\alpha)} L_{j,m}^{(\alpha)}(x) \tag{2.9.37}$$

$$\left(\frac{1-x}{2}\right)^n = \frac{n!(-1)^{mn}}{(1+\alpha+\beta)_n} \sum_{j=0}^{mn} \frac{(-mn)_j \, \beta\Gamma(\beta+n-j\alpha)\, P_{j,m}^{(\beta-n,\alpha)}(x)}{(mn)!(1+\beta-j)_j \Gamma(1+\beta-j\alpha)} \tag{2.9.38}$$

(D) Combinatorial identities

Theorem-2 gives rise to a couple of inverse series relations studied by J. Riordan[2,p.44,46]:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \qquad \Leftrightarrow \qquad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k$$

and

$$a_n = \sum_{k=0}^{n} \binom{n+p}{k+p} b_k \qquad \Leftrightarrow \qquad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+p}{k+p} a_k,$$

when α=0, m=1, β=1.

The theorem also provides the inverse series of the compinatorial identities due to J. Riordan[2, p.71,72,74]

$$\frac{2^{-n}}{n!}\binom{2n}{n} = \sum_{k=0}^{[n/2]} \frac{2^{-2k}}{(n-2k)!\,k!\,k!},$$

$$a_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} b_{2k}$$

and

$$\beta_n = \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!\,k!\,k!}$$

in the forms:

$$\frac{2^{-2n}}{n!\,n!} = \sum_{k=0}^{2n} \frac{(-1)^k (2k)!\, 2^{-k}}{(2n-k)!\, k!\, k!\, k!},$$

$$b_{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} a_k ,$$

and

$\frac{1}{n!\,n!} = \sum_{k=0}^{2n} \frac{(-1)^k \beta_k}{(2n-k)!\,k!}$ respectively when $\alpha=0$, m=2 and $\beta=1$.