# CHAPTER – 3

## A GENERAL CLASS OF POLYNOMIALS AND ITS PROPERTIES–II

### 3.1 INTRODUCTION

After having studied the properties of certain family of (ordinary) polynomials, it is natural to ask that – what are their possible q–analogues? In that context this chapter incorporates the q–analogues of these properties which are studied in chapter–2. In fact, some more q–integral formulas are also studied here.

. Just as the polynomials set $\{S_n(l,m,\alpha,\beta;x)$, n = 0,1,2...} was considered in chapter-2, here a q–analogue of this set of polynomials will be defined and the following properties will be studied.

(i) q–integral representations.

(ii) θ-form q–difference equations.

(iii) q–inverse series relations.

We define a q – analogue of $S_n(l,m,\alpha,\beta,x)$ in explicit form as:

$$S_n(l,m,\alpha,\beta;x\,|\,p)=\sum_{k=0}^{[n/m]}(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\,\frac{(\beta q^{lk-n\alpha+1};q)_\infty}{(p;p)_{n-mk}}\,\sigma_k\,x^k\,,$$

**(3.1.1)**

where $p\equiv q^{\alpha},\alpha\neq 0$.

This polynomial enables us to define the following q-polynomials.

(i) q-analogue of the extended Jacobi polynomial:

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[\alpha_1 \ldots, \alpha_s, \beta_1, \ldots \beta_r . x \mid q] = \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-mnk} .$$

$$\frac{(\beta q^{lk-n\alpha+1}, p)_\infty [\alpha_1]_k \cdot . [\alpha_s]_k \quad x^k}{(\beta q^{1-n\alpha}; q)_\infty [\beta_1]_k \cdots [\beta_r]_k (p,p)_{n-mk} (p,p)_k} \tag{3.1.2}$$

(ii) q-Brafman polynomial:

$$B_n^m[\alpha_1 \quad, \alpha_s, \beta_1, \quad \beta_r \cdot x \mid q] = \sum_{k=0}^{[n/m]} (-1)^{mk} q^{mk(mk+1)/2-mnk} .$$

$$\frac{[\alpha_1]_k \ldots \ldots [\alpha_s]_k \quad x^k}{[\beta_1]_k \cdots \ldots [\beta_r]_k (q,q)_{n-k} (q;q)_k} . \tag{3.1.3}$$

(iii) q- $Z_{n,m}^{(\alpha)}(x,k)$ :

$$Z_{n,m}^{(\alpha)}(x;k \mid q) = \frac{[\alpha q]_n}{(q^k, q^k)_n} \sum_{j=0}^{[n/m]} \frac{q^{kj(n+\alpha+1)+kj(kj-1)/2} (q^{-kn}; q^k)_{mj} \, x^{kj}}{(q^k;q^k)_{mj} \, [\alpha q]_{kj}} .$$

**(3.1.4)**

(iv) Basic analogue of $L_{n,m}^{(\alpha)}(x)$ :

$$L_{n,m}^{(\alpha)}(x \mid q) = \frac{[\alpha q]_n}{[q]_n} \sum_{j=0}^{[n/m]} \frac{q^{j(n+\alpha+1)+j(j-1)/2} (q^{-n})_{mj} \, x^j}{[q]_{mj} \, [\alpha q]_j} . \tag{3.1.5}$$

(v) q-analogue of $W_{n,m}^{(\alpha,\beta)}(x;k)$ :

$$W_{n,m}^{(\alpha,\beta-n)}(x;k \mid q) = \frac{[\alpha q]_n}{(q^k;q^k)_n} \sum_{j=0}^{[n/m]} \frac{q^{mkj} (q^{-nk}; q^k)_{mj} \, (\alpha\beta q; q)_{kj}}{(q^k;q^k)_j \, (\alpha q, q)_{kj}} \left(\frac{1-x}{2}\right)^{kj} .$$

**(3.1.6)**

(vi) q-analogue of the Jacobi polynomial:

$$p_{n,m}(x;\beta-n,\alpha;q)=\sum_{j=0}^{[n/m]}\frac{[q^{-n}]_{mj}\,[\alpha\beta q]_{mj}}{[q]_j\,[\beta q^{1-n}]_{lj}}(xq^m)^j\ . \qquad \textbf{(3.1.7)}$$

Amongst the properties which will be taken up here, the basic integral formulas will be derived in section-3.2. The q-difference equations in θ-form will be obtained in section-3.3. In sections-3.4 to 3.8, the basic inverse series relations will be proved. All these properties will be particularized for the above cited q-polynomials (3.1.2) to (3.1.7) in section-3.9.

## 3.2 q-INTEGRAL REPRESENTATIONS

The beta and gamma integrals are extended in the literature in terms of basic analogue in more than one form. In fact, Gasper and Rehman [1], Andrew and Askey [1] and W.Hahn [1] gave different q-analogues of these well known integrals. In this section these q-integrals are used as tools to obtain the q-integral forms of the generalized polynomial $S_n(l,m,\alpha,\beta;x\,|\,q)$.

The q-integrals that will be used here to derive the q-integrals are as follows:

$$\beta_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\,d_qt, \qquad \mathrm{Re}(x)>0,\ y\neq 0,-1,-2,\ldots$$

**(3.2.1)**

(Gasper and Rahman [1]),

$$\int_{-c}^{d} \frac{(-qx/c,q)_\infty (qx/d,q)_\infty}{(-xq^{\alpha}/c,q)_\infty (xq^{\beta}/d;q)_\infty} d_q x = \frac{\Gamma_q(x)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \frac{(-c/d,q)_\infty (-d/c,q)_\infty}{(-q^{\beta}c/d,q)_\infty (-q^{\alpha}d/c,q)_\infty} \quad \textbf{(3.2.2)}$$

(Andrews and Askey [1]),

In the above integrals $\Gamma_q(x)$ is the well known q-gamma function defined as:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \quad \textbf{(3.2.3)}$$

The q-polynomial $S_n(l,m,\alpha,\beta,x\,|\,q)$ defined by (3.1.1) when replaced by $S_n(l,m,\alpha,\beta;x\,|\,q)\,(\beta q^{1-n\alpha};p)_\infty$, then it becomes

$$S_n(l,m,\alpha,\beta;x\,|\,q) = \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-mnk} \frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(\beta q^{1-n\alpha},p)_\infty} \frac{\sigma_k\, x^k}{(p;p)_{n-mk}} \quad \textbf{(3.2.4)}$$

in which the ratio of the infinite products in view of (3.2.1), provides a q-integral as shown below.

$$\frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(\beta q^{1-n\alpha};p)_\infty} \cdot \frac{(p,p)_\infty}{(p;p)_\infty} = \frac{\Gamma_p(1+\beta-n\alpha)(1-p)^{\beta-n\alpha}}{\Gamma_p(1+\beta-n\alpha+lk)(1-p)^{\beta-n\alpha+lk}}$$

$$= \frac{\Gamma_p(1+\beta-n\alpha)}{\Gamma_p(1+\beta-n\alpha+lk)(1-p)^{lk}} \frac{\Gamma_p(lk)}{\Gamma_p(lk)} \quad \textbf{(3.2.5)}$$

$$= \frac{1}{(1-p)^{lk}\Gamma_p(lk)} \int_0^1 t^{\beta-n\alpha} \frac{(tp;p)_\infty}{(tp^{lk};p)_\infty} d_p t, \quad \textbf{(3.2.6)}$$

where Re$(1+\beta-n\alpha)>0,\ lk \neq 0,-1,-2,\ldots\ \ldots$.

On making use of (3.2.6) in the q-polynomial (3.2.4), one finds

$$S_n(l,m,\alpha,\beta,x\,|\,q) = \sum_{k=0}^{[n/m]} \frac{(-1)^{mk}\, p^{mk(mk+1)/2-mnk}\, \sigma_k\, x^k}{(p;p)_{n-mk}\, \Gamma_p(lk)\,(1-p)^{lk}} \,.$$

$$.\int_0^1 t^{\beta-n\alpha} \frac{(tp;p)_\infty}{(tp^{lk},p)_\infty}\, d_p t \,.$$

If

$$\sum_{k=0}^{[n/m]} \frac{(-1)^{mk}\, p^{mk(mk+1)/2-mnk}\, \sigma_k\, x^k}{(p,p)_{n-mk}\, \Gamma_p(lk)\,(1-p)^{lk}\,(tp^{lk},p)_\infty} = \Lambda(n,l,m,t;x)$$

then

$$S_n(l,m,\alpha,\beta;x\,|\,q) = \int_0^1 t^{\beta-n\alpha}\,(tp;p)_\infty\;\Lambda(n,l,m,t,x)\, d_p t \,, \qquad \textbf{(3.2.7)}$$

where $\mathrm{Re}(1+\beta-n\alpha)>0$ and $lk\neq 0,-1,-2,\ldots..$

For deriving another q-integral form of $S_n(l,m,\alpha,\beta;x\,|\,q)$ a particular case c=d=1 of (3.2.2) will be used which is given by the formula

$$\int_{-1}^{1} \frac{(-qx,q)_\infty (qx,q)_\infty}{(-xq^\alpha;q)_\infty (xq^\beta,q)_\infty}\, d_q x = \frac{\Gamma_q(\alpha)\,\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}\, \frac{1}{2}\, \frac{(-1,q)_\infty (-1;q)_\infty}{(-q^\beta;q)_\infty (-q^\alpha;q)_\infty} \,. \qquad \textbf{(3.2.8)}$$

The substitutions B for $\beta/\alpha$, v for $l/\alpha$ and *a* for $1/\alpha$ will be used, in the derivation of the formula.

Now

$$\frac{(\beta q^{1-n\alpha+lk},p)_\infty}{(\beta q^{1-n\alpha};p)_\infty}=\frac{(q^{\alpha(-n+(\beta/\alpha)+(lk/\alpha)+(1/\alpha))},p)_\infty}{(q^{\alpha((1/\alpha)+(\beta/\alpha)-n)};p)_\infty}$$

$$=\frac{(p^{a+B-n+vk};p)_\infty}{(p^{a+B-n},p)_\infty}$$

$$=\frac{(p;p)_\infty(1-p)^{1-a-B+n}}{(p^{a+B-n};p)_\infty(1-p)^{vk}}\frac{(p^{a+B-n+vk};p)_\infty}{(p;p)_\infty(1-p)^{1-a-B+n-vk}}$$

$$=\frac{\Gamma_p(a+B-n)\Gamma_p(vk)}{(1-p)^{vk}\,\Gamma_p(a+B-n+vk)\Gamma_p(vk)}$$

$$=\frac{2(-p^{a+B-n};p)_\infty\,(-p^{vk},p)_\infty}{(1-p)^{vk}\,\Gamma_p(vk)(-1;p)_\infty\,(-1;p)_\infty}$$

$$\cdot\frac{(-1,p)_\infty\,(-1;p)_\infty\,\Gamma_p(a+B-n)\Gamma_p(vk)}{2(-p^{a+B-n},p)_\infty\,(-p^{vk},p)_\infty\,\Gamma_p(a+B-n+vk)}$$

On making use of the formula (3.2.8) with α is replaced by $a$+B-n, β is replaced by vk, and $(1-p)^{vk}\,\Gamma_p(vk)$ is replaced by $(1-p)(p;p)_\infty/(p^{vk};p)_\infty$, this further simplifies to

$$\frac{(\beta q^{1-n\alpha+lk};p)_\infty}{(\beta q^{1-n\alpha};p)_\infty}=\frac{(p^{vk};p)_\infty(-p^{vk};p)_\infty(-p^{a+B-n};p)_\infty}{(1-p)(p;p)_\infty(-p;p)_\infty(-1;p)_\infty}\cdot$$

$$\cdot\int_{-1}^{1}\frac{(-py;p)_\infty(py;p)_\infty}{(-yp^{a+B-n};p)_\infty(yp^{vk};p)_\infty}\,d_py. \qquad \textbf{(3.2.9)}$$

By making an appeal to the formula

$$(x;p)_\infty(-x;p)_\infty=(x^2,p^2)_\infty,$$

one finds

$$\frac{(\beta q^{1-n\alpha+lk},p)_\infty}{(\beta q^{1-n\alpha},p)_\infty}=\frac{(p^{2vk};p^2)_\infty(-p^{a+B-n};p)_\infty}{(1-p)(p^2;p^2)_\infty(-1;p)_\infty}.$$

$$\cdot\int_{-1}^{1}\frac{(-py,p)_\infty(py;p)_\infty}{(-yp^{a+B-n};p)_\infty(yp^{vk};p)_\infty}d_py.$$

Hence from (3.2.9),

$$S_n(l,m,\alpha,\beta;x\,|\,q)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\;p^{mk(mk-2n+1)/2}}{(p;p)_{n-mk}}\sigma_k\,x^k\,\frac{(p^{2vk};p^2)_\infty(-p^{a+B-n};p)_\infty}{(1-p)(p^2,p^2)_\infty(-1,p)_\infty}$$

$$\cdot\int_{-1}^{1}\frac{(-py;p)_\infty(py;p)_\infty}{(-yp^{a+B-n};p)_\infty(yp^{vk};p)_\infty}d_py.$$

$$=\int_{-1}^{1}\frac{(py;p)_\infty(-py,p)_\infty}{(-yp^{a+B-n};p)_\infty}\delta_n(l,m,\alpha,\beta,x,y;p)\,d_py, \qquad \textbf{(3.2.10)}$$

where

$$\delta_n(l,m,\alpha,\beta,x,y;p)=\sum_{k=0}^{[n/m]}\frac{(-1)^{mk}\;p^{mk(mk-2n+1)/2}}{(yp^{vk};p)_\infty(p;p)_{n-mk}}.$$

$$\cdot\frac{(-p^{a+B-n};p)_\infty(p^{2vk};p^2)_\infty}{(1-p)(-1;p)_\infty(p^2;p^2)_\infty}\sigma_k\,x^k$$

which is an integral representation of $S_n(l,m,\alpha,\beta;x\,|\,q)$.

3.3 q-DIFFERENCE EQUATION

This section aims at the derivation of the q-difference equation for the polynomial $S_n(m,\alpha,\beta;x\,|\,p)$ which is the special instance $\ell$=mα of the polynomial $S_n(l,m,\alpha,\beta;x\,|\,p)$.

The useful tool for obtain the equation is that the function

$$y = {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r; q, z \\ b_1, b_2, \ldots, b_s; \end{matrix}\right] \qquad \textbf{(3.3.1)}$$

satisfies the equation

$$\{\theta(\theta+q^{1-b_1}-1)(\theta+q^{1-b_2}-1)\ldots(\theta+q^{1-b_s}-1)+zq^{a_1+\ldots+a_r-b_1\ldots-b_s+s}\,.$$

$$.(\theta+q^{-a_1}-1)(\theta+q^{a_2}-1)\ldots(\theta+q^{-a_r}-1)\}y=0, \qquad \textbf{(3.3.2)}$$

wherein θ is the q-difference operator defined by $\theta f(x)=f(x)-f(xq)$ and for the solution to exist one of the following conditions must hold.

i) r≤s in which case $|z|<\infty$, $|q|<1$

ii) r=s+1, $|z|<1$, $|q|<1$

iii) $|q|>1$, $|z|<|b_1\,b_2\ldots b_s|/|a_1\,a_2\ldots a_r|$.

The verification of the equation (3.3.2) is a routine calculations which is sketched below.

First consider

$$(\theta+q^{1-b_s}-1)y=(\theta+q^{1-b_s}-1)\sum_{n=0}^{\infty}\frac{[a_1]_n\ldots[a_r]_n z^n}{[b_1]_n\ldots[b_s]_n[q]_n}$$

$$=\sum_{n=0}^{\infty}\frac{[a_1]_n\ldots[a_r]_n z^n(1-q^n+q^{1-b_s}-1)}{[b_1]_n\ldots[b_s]_n[q]_n}$$

$$= \sum_{n=0}^{\infty} \frac{[a_1]_n \cdot [a_r]_n z^n (q^{1-b_s} - q^n)}{[b_1]_n \; [b_s]_n [q]_n}$$

$$= q^{1-b_s} \sum_{n=0}^{\infty} \frac{[a_1]_n \cdot [a_r]_n \; z^n (1-q^{n-1+b_s})}{[b_1]_n \ldots [b_{s-1}]_n \; [b_s]_n \; [q]_n}$$

Next,

$$(\theta + q^{1-b_s-1} - 1)(\theta + q^{1-b_s} - 1) y$$

$$= q^{1-b_s} (\theta + q^{1-b_{s-1}} - 1) \sum_{n=1}^{\infty} \frac{[a_1]_n \cdot \cdot [a_r]_n z^n}{[b_1]_n \; [b_{s-1}]_n [b_s]_n [q]_n}$$

$$= q^{2-b_s-b_{s-1}} \sum_{n=0}^{\infty} \frac{[a_1]_n \cdot [a_r]_n z^n (1-q^{n-1+b_{s-1}})(1-q^{n-1+b_s})}{[b_1]_n \cdot \cdot [b_{s-1}]_n \; [b_s]_n \; [q]_n}$$

Proceeding in this way, one gets

$$(\theta + q^{1-b_1} - 1)(\theta + q^{1-b_2} - 1) \ldots (\theta + q^{1-b_s} - 1) y$$

$$= q^{s-b_1-b_2-\ldots-b_s} \sum_{n=0}^{\infty} \frac{[a_1]_n \cdot [a_r]_n (1-q^{n-1+b_s}) \ldots (1-q^{n-1+b_1}) z^n}{[b_1]_n \ldots [b_s]_n \; [q]_n}$$

$$= q^{s-b_1-b_2-\ldots-b_s} \sum_{n=1}^{\infty} \frac{[a_1]_n \ldots [a_r]_n z^n (1-q^n)}{[b_1]_{n-1} \ldots [b_s]_{n-1} [q]_n}$$

$$= zq^{s-b_1-b_2-\ldots-b_s} \sum_{n=0}^{\infty} \frac{[a_1]_{n+1} \cdot \cdot [a_r]_{n+1} z^n}{[b_1]_n \ldots [b_s]_n [q]_n}. \qquad \textbf{(3.3.3)}$$

Now

$$(\theta+q^{-a_r}-1)y=(\theta+q^{-a_r}-1)\sum_{n=0}^{\infty}\frac{[a_1]_n\,.[a_r]_n z^n}{[b_1]_n\,..[b_s]_n[q]_n}$$

$$=\sum_{n=0}^{\infty}\frac{[a_1]_n..\,.[a_r]_n z^n(q^{-a_r}-q^n)}{[b_1]_n....[b_s]_n[q]_n}$$

$$=q^{-a_r}\sum_{n=0}^{\infty}\frac{[a_1]_n....[a_r]_n(1-q^{n+a_r})z^n}{[b_1]_n....[b_s]_n[q]_n}$$

$$=q^{-a_r}\sum_{n=0}^{\infty}\frac{[a_1]_n\,.[a_{r-1}]_n[a_r]_{n+1}z^n}{[b_1]_n..\,[b_s]_n\,[q]_n}\,.$$

Likewise

$$(\theta+q^{-a_{r-1}}-1)(\theta+q^{-a_r}-1)y$$

$$=q^{-a_r}(\theta+q^{-a_{r-1}}-1)\sum_{n=1}^{\infty}\frac{[a_1]_n..\,[a_{r-1}]_n[a_r]_n z^n}{[b_1]_n..[b_s]_n[q]_n}$$

$$=q^{-a_r-a_{r-1}}\sum_{n=0}^{\infty}\frac{[a_1]_n....[a_{r-1}]_n[a_r]_{n+1}(1-q^{n+a_{r-1}})z^n}{[b_1]_n...[b_s]_n[q]_n}$$

$$=q^{-a_r-a_{r-1}}\sum_{n=0}^{\infty}\frac{[a_1]_n....[a_{r-2}]_n[a_{r-1}]_{n+1}[a_r]_{n+1}z^n}{[b_1]_n....[b_s]_n[q]_n},$$

and in general,

$$(\theta+q^{-a_r}-1)(\theta+q^{-a_2}-1)....(\theta+q^{-a_r}-1)y$$

$$=q^{-a_1-a_2-\;\;-a_r}\sum_{n=0}^{\infty}\frac{[a_1]_{n+1}\;\;[a_r]_{n+1}z^n}{[b_1]_n...[b_s]_n[q]_n}$$

or

$$q^{a_1+a_2+..+a_r}(\theta+q^{-a_r}-1)(\theta+q^{-a_2}-1)\,..(\theta+q^{-a_r}-1)y$$

$$=\sum_{n=0}^{\infty}\frac{[a_1]_{n+1}\,..\,[a_r]_{n+1}\,z^n}{[b_1]_n\,...[b_s]_n\,[q]_n}. \qquad \textbf{(3.3.4)}$$

From (3.3.3) and (3.3.4) the q-difference equation (3.3.2) follows readily.

In order to find the q-difference equation for the polynomial $S_n(l,m,\alpha,\beta,x\mid p)$, first it is expressed in ${}_r\phi_s$-series form by replacing

$S_n(l,m,\alpha,\beta;x\mid p)$ by $\frac{S_n(m,\alpha,\beta;x\mid p)(\beta q^{1-n\alpha};p)_\infty}{(p;p)_n}$, where $\ell$=mα p=$q^\alpha$, α≠0,

and using the formula

$$(q,q)_{n-mk}=\frac{(-1)^{mk}\,q^{mk(mk-1)/2-mnk}\,[q]_n}{[q^{-n}]_{mk}}.$$

With all this, one finds

$$S_n(m,\alpha,\beta,x\mid p)=\sum_{k=0}^{[n/m]}\frac{p^{mk}\,(p^{-n},p)_{mk}\,(\beta q^{1-n\alpha+m\alpha k};p)_\infty\,\sigma_k\,x^k}{(\beta q^{1-n\alpha};p)_\infty}$$

$$=\sum_{k=0}^{[n/m]}\frac{p^{mk}\,(p^{-n};p)_{mk}\,\sigma_k\,x^k}{(\beta q^{1-n\alpha};p)_{mk}}. \qquad \textbf{(3.3.5)}$$

Now consider a particular case $\sigma_k=\frac{1}{(p;p)_k}$ of $S_n(m,\alpha,\beta,x\mid p)$, which is denoted by $R_n^m(x\mid p)$, that is,

$$R_n^m(x\mid p)=\sum_{k=0}^{[n/m]}\frac{(p^{-n};p)_{mk}\,(xp^m)^k}{(\beta q^{1-n\alpha},p)_{mk}(p,p)_k}. \quad \textbf{(3.3.6)}$$

It is to be noted here that $\xi_k$ which was chosen as $1/(p;p)_k$, can also be taken as $\frac{\prod_i (a_i,p)_k}{\prod_j (b_{j;p})_k (p;p)_k}$, that by introducing finite number of numerator and denominator parameters.

Here the q-factorial functions $(p^{-n};p)_{mk}$ and $(\beta q^{1-n\alpha};p)_{mk}$ occurring in the series of $R_n^m(x\mid p)$ are expressed in the form of the factorial functions in which mk is replaced by k.

In fact, in view of the formula:

$$(\alpha,q)_{mk}=(\alpha;q^m)_k(\alpha q,q^m)_k\cdots(\alpha q^{m-1};q^m)_k \quad \textbf{(3.3.7)}$$

where,

$(\alpha,q^m)_k=(\alpha^{1/m};q)_k(\alpha^{1/m}w;q)_k.\ .(\alpha^{1/m}w^2;q)_k\quad(\alpha^{1/m}w^{m-1},q)_k,$

$(\alpha q;q^m)_k=(\alpha^{1/m}q^{1/m},q)_k(\alpha^{1/m}q^{1/m}w,q)_k....(\alpha^{1/m}q^{1/m}w^{m-1};q)_k,$

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$(\alpha q^{m-1};q^m)_k=(\alpha^{1/m}q^{(m-1)/m};q)_k(\alpha^{1/m}q^{(m-1)/m}w,q)_k....(\alpha^{1/m}q^{(m-1)/m}w^{m-1};q)_k,$

**(3.3.8)**

w is $m^{th}$ roots of unity, the q-factorial functions $(p^{-n},p)_{mk}$ and $(\beta q^{1-n\alpha};p)_{mk}$ give rise to $r=m^m$ numerator parameters and same number of denominator parameters.

Thus,

$$(p^{-n};p)_{mk} = (p^{-n},p^m)_k(p^{-n+1};p^m)_k \;..(p^{-n+m-1},p^m)_k, \quad \textbf{(3.3.9)}$$

and further each of the products on the right side is expressed in k number of factors as follows. They are:

$$(p^{-n};p^m)_k = (p^{-n/m},p)_k(p^{-n/m}\,w,p)_k..\,(p^{-n/m}\,w^2;p)_k\,...\,(p^{-n/m}\,w^{m-1},p)_k,$$

$$(p^{-n+1},p^m)_k = (p^{(-n+1)/m};p)_k(p^{(-n+1)/m}\,w,p)_k\,(p^{(-n+1)/m}\,w^2;p)_k\,....$$

$$.....(p^{(-n+1)/m}\,w^{m-1};p)_k,$$

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$$(p^{-n+m-1};p^m)_k = (p^{(-n+m-1)/m};p)_k(p^{(-n+m-1)/m}\,w,p)_k.$$

$$.(p^{(-n+m-1)/m}\,w^2;p)_k...\,(p^{(-n+m-1)/m}\,w^{m-1};p)_k \quad \textbf{(3.3.10)}$$

Similarly,

$$(\beta q^{1-n\alpha};p)_{mk} = (\beta q^{1-n\alpha};p^m)_k(\beta q^{1-n\alpha}\,p;p^m)_k(\beta q^{1-n\alpha}\,p^2;p^m)_k..\,.$$

$$(\beta q^{1-n\alpha}\,p^{m-1};p^m)_k, \quad \textbf{(3.3.11)}$$

where

$$(\beta q^{1-n\alpha};p^m)_k=(q^{\frac{1+\beta-n\alpha}{m}};p)_k(q^{\frac{1+\beta-n\alpha}{m}}w,p)_k(q^{\frac{1+\beta-n\alpha}{m}}w^2,p)_k$$

$$\cdot\cdot\ (q^{\frac{1+\beta-n\alpha}{m}}w^{m-1};p)_k$$

$$(\beta q^{1-n\alpha}p,p^m)_k=(q^{\frac{1+\beta-n\alpha}{m}}p^{1/m};p)_k(q^{\frac{1+\beta-n\alpha}{m}}p^{1/m}w;p)_k(q^{\frac{1+\beta-n\alpha}{m}}p^{1/m}w^2;p)_k\cdots$$

$$\cdot\ (q^{\frac{1+\beta-n\alpha}{m}}p^{1/m}w^{m-1};p)_k,$$

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$$(\beta q^{1-n\alpha}p^{m-1};p^m)_k=(q^{\frac{1+\beta-n\alpha}{m}}p^{\frac{m-1}{m}},p)_k(q^{\frac{1+\beta-n\alpha}{m}}p^{\frac{m-1}{m}}w,p)_k$$

$$(q^{\frac{1+\beta-n\alpha}{m}}p^{\frac{m-1}{m}}w^2,p)_k\cdots(q^{\frac{1+\beta-n\alpha}{m}}p^{\frac{m-1}{m}}w^{m-1},p)_k \quad \textbf{(3.3.12)}$$

Here w is as before, the $m^{th}$ roots of unity.

Now, in the light of the equation (3.3.2), the q-difference equation satisfied by $R_n^m(x\,|\,p)$ (eq. (3.3.6)), is obtained with the aid of the formulas (3.3.8), (3.3.10) & (3.3.12), which is given by

$$\{\theta\prod_{j=0}^{m-1}\prod_{k=0}^{m-1}(\theta+p^{1-\frac{(1+\beta-n\alpha+j\alpha)}{m}}w^k-1)+x\,p^{m(n\alpha-n-\beta)+m(m+3)/2}\,.$$

$$.(\prod_{s=0}^{m-1}\prod_{r=0}^{m-1}[\theta+q^{\frac{n}{m}-\frac{s}{m}}w^r-1])\}\,R_n^m(x\,|\,p)=0\,. \qquad \textbf{(3.3.13)}$$

3.4 q-INVERSE SERIES RELATIONS

It will be interesting to examine the q-extensions of the inversion pairs proved in Chapter-2; for the special cases of such q-versions may provide q-analogues of the special instances of the theorems 1, 2 and 3. In fact, the q-analogues of the coefficients a(n,k,m) and c(n,k,m) will be first constructed; the coefficient b(n,k,m;q) and d(n,k,m;q) will be so designed that the inverse series relations would be "two sided". If a polynomial in $x$ is known in the explicit form then there is a natural question "what is its inverse series ?". The following results are proved with a view to obtain the inverse series of (3.1.1).

As before p stands for $q^{\alpha}$. To begin with, a q-analogue of theorem-1 which is the q-inverse series relation may be stated as:

**Theorem-4.**

For m = 1,2,... and a positive integer $l \le$ m, if

$$F(n) = \sum_{k=0}^{[n/m]} a(n,k,m,q) f(k) \tag{3.4.1}$$

and

$$f(n) = \sum_{k=0}^{mn} b(n,k,m,q) F(k) \tag{3.4.2}$$

then

$$a(n,k,m;q) = (-1)^{mk} p^{mk(mk-2n+1)/2} \frac{(\beta q^{lk-\alpha n};q)_\infty}{(p;p)_{n-mk}} \tag{3.4.3}$$

implies

$$b(n,k,m;q) = p^{k(k-1)/2} \frac{(1-\beta q^{ln-\alpha mn-1})}{(\beta q^{ln-\alpha k-1};q)_\infty (p;p)_{mn-k}}. \tag{3.4.4}$$

The proof of this theorem is given in section-3.5. The following inversion pair stated as theorem-5 provides "two sided" inverse series relation.

**Theorem-5.**

For n = 0,1,2,... if

$$G(n)=\sum_{k=0}^{[n/m]} a(n,k,m;q)g(k) \quad \textbf{(3.4.5)}$$

and

$$g(n)=\sum_{k=0}^{mn} b(n,k,m;q)\,G(k) \quad \textbf{(3.4.6)}$$

then

$$a(n,k,m;q)=(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\,\frac{(\beta q^{mk\alpha-n\alpha+1};p)_\infty}{(p,p)_{n-mk}} \quad \textbf{(3.4.7)}$$

implies and is implied by

$$b(n,k,m;q)=\frac{(-1)^k\,p^{k(k-1)/2}(1-\beta q^{1-\alpha})}{(\beta q^{mn\alpha-k\alpha+1-\alpha};p)_\infty(p;p)_{mn-k}} \quad \textbf{(3.4.8)}$$

and

$$\sum_{k=0}^{n} b(n,k,1;q)G(k)=0 \text{ if } n\neq mj,\ j\in N. \quad \textbf{(3.4.9)}$$

Section-3.6 contains the proof of this theorem. The inversion pair proved as theorem-3 in chapter-2 is extended in the form of

**Theorem-6.**

If

$$A(n) = \sum_{k=0}^{[n/m]} c(n,k,m,q)B(k) \tag{3.4.10}$$

and

$$B(n) = \sum_{k=0}^{mn} d(n,k,m;q)A(k) \tag{3.4.11}$$

then,

$$c(n,k,m,q) = \frac{(-1)^{n-mk} p^{mk(mk-1)/2}(1-\beta q^{1-\alpha})}{(\beta q^{n\alpha-mk\alpha+1-\alpha};p)_{\infty}(p,p)_{n-mk}} \tag{3.4.12}$$

implies and is implied by

$$d(n,k,m,q) = p^{k(k+1)/2-mnk} \frac{(\beta q^{k\alpha-mn\alpha+1},p)_{\infty}}{(p,p)_{mn-k}} \tag{3.4.13}$$

and

$$\sum_{k=0}^{n} d(n,k,1;q)A(k) = 0 \text{ when } n \neq ms, \; s \in N. \tag{3.4.14}$$

The proof is given in section-3.8.

## 3.5 PROOF OF THEOREM-4

In order to prove the theorem, let us denote the right hand member of (3.4.2) by $\Omega$, then substituting for F(k) from (3.4.1) where the coefficient a(n,k,m;q) and b(n,k,m;q) are as defined in (3.1.3) and (3.1.4), we get

$$\Omega = \sum_{k=0}^{mn} \sum_{j=0}^{[k/m]} \frac{(-1)^{k-mj} p^{k(k-1)/2} p^{mj(mj+1)/2-kmj} (1-\beta q^{ln-\alpha mn-1}) f(j)}{(p,p)_{mn-k} (p;p)_{k-mj} [\beta q^{lj-k\alpha}]_{\infty}^{-1} [\beta q^{ln-k\alpha-1}]_{\infty}}.$$

This in view of the relation

$$\sum_{k=0}^{mn} \sum_{j=0}^{[k/m]} A(k,j) = \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} A(k+mj,j) \qquad \textbf{(3.5.1)}$$

may further be written as

$$\Omega = \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} (-1)^k p^{(k+mj)(k+mj-1)/2} p^{mj(mj+1)/2-mj(k+mj)} .$$

$$\cdot \frac{(1-\beta q^{ln-\alpha mn-1}) [\beta q^{lj-k\alpha-mj\alpha}]_{\infty} f(j)}{(p;p)_k (p;p)_{mn-k-mj} [\beta q^{ln-k\alpha-mj\alpha-1}]_{\infty}}$$

$$= f(n) + \sum_{j=0}^{n-1} \frac{(1-\beta q^{ln-\alpha mn-1}) f(j)}{(p,p)_{mn-mj}} .$$

$$\cdot \sum_{k=0}^{mn-mj} (-1)^k p^{k(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{\alpha} \frac{[\beta q^{lj-k\alpha-mj\alpha}]_{\infty}}{[\beta q^{ln-k\alpha-mj\alpha-1}]_{\infty}}$$

Noticing that

$$\frac{[\beta q^{lj-k\alpha-mj\alpha}]_{\infty}}{[\beta q^{ln-k\alpha-mj\alpha-1}]_{\infty}} = (\beta q^{lj-k\alpha-mj\alpha};q)_{ln-lj-1} = \sum_{r=0}^{ln-lj-1} \mu_r q^{-\alpha kr},$$

we further have

$$\Omega = f(n) + \sum_{j=0}^{n-1} \frac{(1-\beta q^{ln-\alpha mn-1}) f(j)}{(p;p)_{mn-mj}} \sum_{k=0}^{mn-mj} (-1)^k p^{k(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{\alpha}$$

$$\sum_{r=0}^{ln-lj-1} \mu_r q^{-\alpha kr}$$

$$= f(n) + \sum_{j=0}^{n-1} \frac{(1-\beta q^{ln-\alpha mn-1}) f(j)}{(p,p)_{mn-mj}}.$$

$$\sum_{r=0}^{ln-lj-1} \mu_r \sum_{k=0}^{mn-mj} (-1)^k p^{k(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_\alpha p^{-kr}$$

But

$$\sum_{k=0}^{mn-mj} (-1)^k p^{k(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_\alpha p^{-kr} = \prod_{i=1}^{mn-mj} (1-q^{-r+i-1})$$

which vanishes for r = 0,1,2,....., $ln-lj-1$ provided that $l \le m$, is a positive integer.

Thus, $\Omega$ = f(n) when $l \le m$. This completes the proof of the theorem.

## 3.6 PROOF OF THEOREM-5

In order to prove the 'only if' part, consider the series

$$\sum_{n=0}^{\infty} (-1)^n p^{n(n-1)/2} G(n) t^n = S, \text{ say.}$$

Substituting for G(n) from (3.4.5) and using (3.4.7) for the coefficient a(n,k,m;q), one gets

$$S = \sum_{n=0}^{\infty} (-1)^n p^{n(n-1)/2} G(n)\, t^n$$
$$= \sum_{n=0}^{\infty} (-1)^n p^{n(n-1)/2}\, t^n \sum_{k=0}^{[n/m]} \frac{(-1)^{mk} p^{mk(mk+1)/2-mnk} (\beta q^{mk\alpha-n\alpha+1}; p)_\infty\, g(k)}{(p,p)_{n-mk}}.$$

Using the double series identity,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} A(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n+mk,k),$$

one arrives at, after some simplification

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n p^{n(n-1)/2}(\beta q^{1-n\alpha},p)_\infty g(k)\,t^{n+mk}}{(p;p)_n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n p^{n(n-1)/2}(\beta q^{1-n\alpha};p)_\infty t^n}{(p;p)_n}\sum_{k=0}^{\infty} g(k)\,t^{mk},$$

but since

$$(\beta q^{1-n\alpha};p)_\infty=(\beta q,p)_\infty\cdot\frac{(\beta q^{1-n\alpha},p)_\infty}{(\beta q,p)_\infty}$$

$$=(\beta q;p)_\infty(\beta q p^{-n},p)_n,$$

and

$$(aq^{-n};q)_n=(-1)^n a^n q^{-n(n+1)/2}(q/a;q)_n,$$

one gets

$$(\beta q^{1-n\alpha};p)_\infty=(\beta q;p)_\infty(-1)^n(\beta q)^n p^{-n(n+1)/2}(p/\beta q;p)_n.$$

Therefore,

$$S=(\beta q;p)_\infty\sum_{n=0}^{\infty}\frac{(p/\beta q,p)_n}{(p;p)_n}(t\beta q/p)^n\sum_{k=0}^{\infty}g(k)t^{mk}\quad.$$

Now, using the basic binomial theorem [Gasper and Rehman]

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q,q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty},|z|<1 \qquad \textbf{(3.6.1)}$$

with a and z replaced by $p/\beta q$ and $t\beta q/p$ respectively, one finally obtains,

$$\sum_{n=0}^{\infty}(-1)^n p^{n(n-1)/2}G(n)\,t^n=\frac{(\beta q;p)_\infty(t;p)_\infty}{(t\beta q/p;p)_\infty}\sum_{k=0}^{\infty}g(k)\,t^{mk},$$

which can also be written as

$$\sum_{n=0}^{\infty} g(n)\, t^{mn} = \frac{(t\beta q/p,p)_\infty}{(\beta q,p)_\infty (t;p)_\infty} \sum_{k=0}^{\infty} (-1)^k\, p^{k(k-1)/2}\, G(k)\, t^k \,.$$

Again, making use of (3.6.1) in

$$\frac{(t\beta q/p;p)_\infty}{(t;p)_\infty}$$

one gets

$$\frac{(t\beta q/p,p)_\infty}{(t,p)_\infty} = \sum_{n=0}^{\infty} \frac{(\beta q/p,p)_n}{(p,p)_n}\, t^n \,.$$

Hence,

$$\sum_{n=0}^{\infty} g(n)\, t^{mn} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\, p^{k(k-1)/2}\, (\beta q/p,p)_n G(k)\, t^{n+k}}{(\beta q,p)_\infty (p,p)_n}$$

which on making use of the double series relation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \qquad \textbf{(3.6.2)}$$

reduces to

$$\sum_{n=0}^{\infty} g(n)\, t^{mn} = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \frac{(-1)^k\, p^{k(k-1)/2}\, (\beta q/p,p)_{n-k} G(k)}{(\beta q,p)_\infty (p;p)_{n-k}} \right\} t^n \,.$$

If $t^m = u$ then $t^n = u^{n/m}$, and thus from above,

$$\sum_{n=0}^{\infty} g(n)\, u^n = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \frac{(-1)^k\, p^{k(k-1)/2} (\beta q/p;p)_{n-k} G(k)}{(\beta q;p)_\infty (p;p)_{n-k}} \right\} u^{n/m} \,. \qquad \textbf{(3.6.3)}$$

If n/m is a positive integer i.e. n=mr, then the comparison of the coefficients of integral powers of u on both the sides of (3.6.3) gives,

$$g(n)=\sum_{k=0}^{mr}\frac{(-1)^k p^{k(k-1)/2}(\beta q/p,p)_{mr-k}}{(\beta q,p)_\infty(p;p)_{mr-k}}G(k).$$

As since

$$\frac{(\beta q/p;p)_{mr-k}}{(\beta q;p)_\infty}=\frac{(1-\beta q^{1-\alpha})}{(\beta q^{1+mr\alpha-k\alpha-\alpha};p)_\infty},$$

one arrives at

$$g(n)=\sum_{k=0}^{mn}\frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})G(k)}{(p,p)_{mn-k}(\beta q^{1+mn\alpha-k\alpha-\alpha},p)_\infty}$$

which is (3.4.6).

Thus, it is proved that (3.4.7)⇒(3.4.8).

If n/m is not an integer then right hand member of (3.6.3) does not contain positive integral powers of u, so the coefficient of $u^{n/m}$ on the right hand side vanishes, therefore

$$\sum_{k=0}^{n}\frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})G(k)}{(p;p)_{n-k}(\beta q^{n\alpha-k\alpha+1-\alpha};p)_\infty}=0 \text{ where } n\neq ms,\ s=1,2,3,.....$$

which is the condition (3.4.9).

Thus, it is proved that (3.4.7)⇒(3.4.9), which completes the proof of the first part.

For proving the converse, the following notations are used

$$g(n,1)=\sum_{k=0}^{n}b(n,k,1;q)\,G(k,1), \qquad \textbf{(3.6.4)}$$

where

$$G(n,1)=\sum_{k=0}^{n} a(n,k,1,q)\; g(k,1). \qquad \textbf{(3.6.5)}$$

Now,

$$\sum_{n=0}^{\infty} g(n,1)\,t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})\,G(k,1)\,t^n}{(\beta q^{n\alpha-k\alpha+1-\alpha};p)_\infty (p;p)_{n-k}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})\,G(k,1)\,t^{n+k}}{(\beta q^{n\alpha+1-\alpha},p)_\infty (p;p)_n}$$

$$=\frac{(1-\beta q^{1-\alpha})}{(\beta q^{1-\alpha},p)_\infty}\sum_{k=0}^{\infty}(-1)^k\;p^{k(k-1)/2}\;G(k,1)\,t^k\sum_{n=0}^{\infty}\frac{(\beta q^{1-\alpha};p)_\infty\;t^n}{(\beta q^{n\alpha+1-\alpha},p)_\infty\,(p,p)_n}$$

$$=\frac{1}{(\beta q,p)_\infty}\sum_{k=0}^{\infty}(-1)^k\;p^{k(k-1)/2}\;G(k,1)\,t^k\sum_{n=0}^{\infty}\frac{(\beta q^{1-\alpha},p)_n\;t^n}{(p,p)_n}.$$

Now using the basic binomial theorem (3.6.1) for the last series we get

$$\sum_{n=0}^{\infty} g(n,1)\,t^n=\frac{1}{(\beta q,p)_\infty}\sum_{k=0}^{\infty}(-1)^k\;p^{k(k-1)/2}\;G(k,1)\,t^k\,\frac{(t\beta q^{1-\alpha},p)_\infty}{(t;p)_\infty}.$$

Therefore, we have

$$\frac{(t,p)_\infty(\beta q,p)_\infty}{(t\beta q^{1-\alpha};p)_\infty}\sum_{k=0}^{\infty} g(k,1)\,t^k=\sum_{k=0}^{\infty}(-1)^k\;p^{k(k-1)/2}\;G(k,1)\,t^k\;, \qquad \textbf{(3.6.6)}$$

but

$$\frac{(t;p)_\infty}{(t\beta q^{1-\alpha};p)_\infty}={}_1\phi_0\left[\begin{matrix} q^{\alpha-1-\beta};\,p,\,t\beta q^{1-\alpha}\\ ;\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(q^{\alpha-1-\beta};p)_n\,(t\beta q^{1-\alpha})^n}{(p;p)_n}$$

from the basic binomial theorem. Hence, left hand side L of (3.6.6) gets simplified to

$$L=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{n\beta+n-n\alpha}(q^{\alpha+\beta};q_1)_n(\beta q,p)_\infty g(k,1)}{(p,p)_n}t^{n+k}$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{q^{(n-k)(\beta-\alpha+1)}(q^{\alpha+\beta},p)_{n-k}(\beta q;p)_\infty g(k,1)}{(p,p)_{n-k}}\right\}t^n$$

Here,

$$(q^{\alpha+\beta},p)_{n-k}=(q^{\alpha+\beta};p)_{-(k-n)}=\frac{(-1)^{n-k}p^{k(k-1)/2+n(n+1)/2-nk}q^{-(n-k)(1+\beta)}}{(q^{1+\beta},p)_{k-n}}.$$

Thus, L reduces to

$$L=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(-1)^k p^{k(k+1)/2-nk}(\beta q,p)_\infty g(k,1)}{(p,p)_{n-k}(q^{1+\beta},p)_{k-n}}\right\}(-1)^n t^n p^{n(n-1)/2}$$

but since

$$\frac{(\beta q,p)_\infty}{(q^{1+\beta};p)_{k-n}}=\frac{(\beta q;p)_\infty}{(\beta q,p)_{k-n}}=\frac{(\beta q;p)_\infty(\beta q\cdot p^{k-n};p)_\infty}{(\beta q;p)_\infty},$$

$$L=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(-1)^k p^{k(k+1)/2-nk}(\beta^{k\alpha-n\alpha+1};p)_\infty g(k,1)}{(p,p)_{n-k}}\right\}(-1)^n t^n p^{n(n-1)/2},$$

Therefore (3.6.6) can now be written as;

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(-1)^k p^{k(k+1)/2-nk}(\beta q^{k\alpha-n\alpha+1};p)_\infty g(k,1)}{(p;p)_{n-k}}\right\}(-1)^n p^{n(n-1)/2}t^n$$

$$=\sum_{n=0}^{\infty}(-1)^n p^{n(n-1)/2}G(n,1)t^n.$$

Comparing the coefficients of $t^n$ on both the sides we get,

$$G(n,1)=\sum_{k=0}^{n}\frac{(-1)^k p^{k(k+1)/2-nk}(\beta q^{k\alpha-n\alpha+1},p)_\infty g(k,1)}{(p,p)_{n-k}}.$$

Thus, it is proved that

$$g(n,1)=\sum_{k=0}^{n}\frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})}{(p;p)_{n-k}(\beta q^{n\alpha-k\alpha+1-\alpha},p)_\infty}G(k,1) \tag{3.6.7}$$

$\Leftrightarrow$

$$G(n,1)=\sum_{k=0}^{n}\frac{(-1)^k p^{k(k+1)/2-nk}(\beta q^{k\alpha-n\alpha+1},p)_\infty}{(p;p)_{n-k}}g(k,1).$$

But, in the hypothesis of the second part (converse)

$$\sum_{k=0}^{n}\frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})G(k,1)}{(p;p)_{n-k}(\beta q^{n\alpha-k\alpha+1-\alpha},p)_\infty}=0 \text{ if } n\neq ms,\ s=1,2,3,\ldots\ldots..$$

Thus, from (3.6.7) it is clear that

$g(n,1)=0$ for $n\neq ms$, $s=1,2,3,\ldots\ldots\ldots$ .

And for $n=ms$,

$$g(n)=\sum_{k=0}^{mn}\frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})G(k)}{(p;p)_{mn-k}(\beta q^{mn\alpha-k\alpha+1-\alpha};p)_\infty}$$

$\Leftrightarrow$

$$G(n)=\sum_{mk=0}^{n}\frac{(-1)^{mk} p^{mk(mk+1)/2-mnk}(\beta q^{mk\alpha-n\alpha+1},p)_\infty\, g(k)}{(p;p)_{n-mk}}.$$

Thus, it is proved that (3.4.8) and (3.4.9) together imply (3.4.7) which completes the proof of the theorem.

## 3.7 AN AUXILIARY q-INVERSION PAIR

We shall require the following inverse pair in this chapter. This pair is actually a q-analogue of the series relations stated in (2.3.1) and (2.3.2).

$$T(n)=\sum_{k=0}^{n} p^{k(k+1)/2-nk}\frac{(\beta q^{k\alpha-n\alpha+1},p)_\infty}{(p,p)_{n-k}}S(k) \qquad \textbf{(3.7.1)}$$

$\Leftrightarrow$

$$S(n)=\sum_{k=0}^{n}\frac{(-1)^{n-k}p^{k(k-1)/2}(1-\beta q^{1-\alpha})T(k)}{(p;p)_{n-k}(\beta q^{n\alpha-k\alpha+1-\alpha};p)_\infty} \qquad \textbf{(3.7.2)}$$

**Proof**

In order to prove the first part that is (3.7.1)$\Rightarrow$(3.7.2), let the right hand member of (3.7.2) be denoted by $\phi(n)$, then substituting for T(k) from (3.7.1) one gets

$$\phi(n)=\sum_{k=0}^{n}\sum_{r=0}^{k}\frac{(-1)^{n-k}p^{k(k-1)/2+r(r+1)/2-rk}(\beta q^{r\alpha-k\alpha+1};p)_\infty(1-\beta q^{1-\alpha})S(r)}{(p;p)_{n-k}(p,p)_{k-r}(\beta q^{n\alpha-k\alpha+1-\alpha};p)_\infty}.$$

In view of the double series (identity) relation

$\sum_{k=0}^{n}\sum_{j=0}^{k}A(k,j)=\sum_{j=0}^{n}\sum_{k=0}^{n-j}A(k+j,j)$ with N=n-r,

this further simplifies to

$$\phi(n)=\sum_{r=0}^{n}\sum_{k=0}^{N}\frac{(-1)^{N-k}p^{k(k-1)/2}(1-\beta q^{1-\alpha})(\beta q^{1-k\alpha};p)_\infty S(r)}{(p;p)_{N-k}(p;p)_k(\beta q^{N\alpha-k\alpha+1-\alpha};p)_\infty}$$

$$=S(n)+\sum_{r=0}^{n-1}\frac{(-1)^N(1-\beta q^{1-\alpha})S(r)}{(p;p)_N}\sum_{k=0}^{N}(-1)^k\begin{bmatrix}N\\k\end{bmatrix}_p p^{k(k-1)/2}\frac{(\beta q^{1-k\alpha};p)_\infty}{(\beta q^{N\alpha-k\alpha+1-\alpha};p)_\infty}.$$

Once again writing

$$\frac{(\beta q^{1-k\alpha};p)_\infty}{(\beta q^{N\alpha-k\alpha+1-\alpha};p)_\infty}=(\beta q^{1-k\alpha},p)_{N-1}=\sum_{j=0}^{N-1}c_j\,p^{-jk}$$

one finds

$$\phi(n)=S(n)+\sum_{r=0}^{n-1}\frac{(-1)^{N}(1-\beta q^{1-\alpha})S(r)}{(p;p)_N}\sum_{k=0}^{N}(-1)^k\begin{bmatrix}N\\k\end{bmatrix}p^{k(k-1)/2}\sum_{j=0}^{N-1}c_j\,p^{-jk}.$$

Now, observing that

$$\sum_{k=0}^{N}(-1)^k\begin{bmatrix}N\\k\end{bmatrix}p^{k(k-1)/2}\sum_{j=0}^{N-1}c_j\,p^{-jk}$$

$$=\sum_{k=0}^{N}(-1)^k\begin{bmatrix}N\\k\end{bmatrix}p^{k(k-1)/2}c_s\,q^{-sk}\qquad (s=0,1,2,\ldots\ldots,N-1)$$

$= 0,$

for s=0,1,2,....N-1 and hence $\phi(n)=S(n)$. Thus, it is proved that (3.7.1)⇒(3.7.2). To prove the converse, it is sufficient to show that the diagonal elements, say $t_{nn}$ and $s_{nn}$ of the matrix representations of the series (3.7.1) and (3.7.2) respectively in the above pair are non-zero. Clearly,

$$t_{nn}=p^{-n(n-1)/2}(\beta q;p)_\infty\neq 0$$

and

$$s_{nn}=\frac{p^{n(n-1)/2}}{(\beta q;p)_\infty}\neq 0.$$

Thus, (3.7.2)⇒(3.7.1) as the inverse series is unique.

Hence, (3.7.1)⇔(3.7.2) which completes the proof of the auxiliary q-inversion pair.

### 3.8 PROOF OF THEOREM-6

The proof of theorem-6 will be given in this section. To prove the 'only if' part, consider the right hand side of (3.4.11) denoted by say f(n) then in view of (3.4.10) and (3.4.13) one gets

$$f(n)=\sum_{k=0}^{mn}\sum_{r=0}^{[k/m]} p^{k(k+1)/2-mnk}\frac{(\beta q^{k\alpha-mn\alpha+1};p)_\infty\, c(k,r,m;q)}{(p;p)_{mn-k}}B(r)$$

$$=\sum_{k=0}^{mn}\sum_{r=0}^{[k/m]}(-1)^{k-mr}p^{mr(mr-1)/2+k(k+1)/2-mnk}$$

$$\cdot\frac{(1-\beta q^{1-\alpha})(\beta q^{k\alpha-mn\alpha+1},p)_\infty B(r)}{(\beta q^{k\alpha-mr\alpha+1-\alpha},p)_\infty(p,p)_{mn-k}(p,p)_{k-mr}}.$$

Here on making use of the double series identity,

$$\sum_{k=0}^{mn}\sum_{j=0}^{[k/m]}A(k,j)=\sum_{j=0}^{n}\sum_{k=0}^{mn-mj}A(k+mj,j)$$

and putting mn−mr=M after some simplification, one arrives at

$$f(n)=\sum_{r=0}^{n}\sum_{k=0}^{M}\frac{(-1)^k p^{k(k+1)/2-Mk-mrM}(1-\beta q^{1-\alpha})B(r)(\beta q^{k\alpha-M\alpha+1};p)_\infty}{(p;p)_{M-k}(p;p)_k(\beta q^{k\alpha+1-\alpha};p)_\infty}$$

$$=B(n)+\sum_{r=0}^{n-1}p^{-mrM}(1-\beta q^{1-\alpha})B(r)\sum_{k=0}^{M}\frac{(-1)^k p^{k(k+1)/2-Mk}(\beta q^{k\alpha-M\alpha+1};p)_\infty}{(p;p)_{M-k}(p;p)_k(\beta q^{k\alpha+1-\alpha};p)_\infty}.$$

As before,

$$\frac{(\beta q^{k\alpha-M\alpha+1};p)_\infty}{(\beta q^{k\alpha+1-\alpha};p)_\infty}=(\beta q^{k\alpha-M\alpha+1};p)_{M-1}$$

$$=\sum_{j=0}^{M-1}c_j\,p^{jk},\quad \text{say.}$$

$$f(n)=B(n)+\sum_{r=0}^{n-1}\frac{p^{-mrM}(1-\beta q^{1-\alpha})\,B(r)}{(p;p)_M}\sum_{k=0}^{M}(-1)^k p^{k(k+1)/2-Mk}\begin{bmatrix}M\\k\end{bmatrix}_\alpha\sum_{j=0}^{M-1}c_j\,p^{jk}$$

$$=B(n)+\sum_{r=0}^{n-1}\frac{p^{-mrM}(1-\beta q^{1-\alpha})\,B(r)}{(p;p)_M}\sum_{j=0}^{M-1}\sum_{k=0}^{M}(-1)^k p^{k(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}_\alpha c_j\,p^{-(M-j-1)k}.$$

Here we see that the inner series

$$\sum_{k=0}^{M}(-1)^k p^{k(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}_\alpha c_j\,p^{-(M-j-1)k}=0$$

because of the fact that

$$\sum_{k=0}^{M}(-1)^k p^{k(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}_\alpha c_j\,p^{-(M-j-1)k}=(1-p^{j(M-1)})(1-p^{j-(m-2)})....(1-p^{j})$$

$$=0,\quad j=0,1,2,....M-1.$$

Thus, F(n)=B(n) which proves that (3.4.12)⇒(3.4.13).

Now, it will be proved that (3.4.12) also implies (3.4.14). For this consider the left member say F(n) of (3.4.14) and use (3.4.13) with m=1, in place of the coefficient, then

$$F(n)=\sum_{k=0}^{n}d(n,k,1;q)A(k) \qquad \textbf{(3.8.1)}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{[k/m]}\frac{(-1)^{k-mj}\,p^{mj(mj-1)/2+k(k+1)/2-nk}(1-\beta q^{1-\alpha})\,B(j)}{(p;p)_{m-k}(p;p)_{k-mj}}\cdot\frac{(\beta q^{k\alpha-n\alpha+1};p)_\infty}{(\beta q^{k\alpha+mj\alpha+1-\alpha};p)_\infty}.$$

Now using the double series relation:

$$\sum_{k=0}^{n}\sum_{j=0}^{[k/m]}A(k,j)=\sum_{j=0}^{[n/m]}\sum_{k=0}^{n-mj}A(k+mj,j)$$

one obtains after some simplifications,

$$F(n)=\sum_{j=0}^{[n/m]}\sum_{k=0}^{n-mj}\frac{(-1)^k p^{k(k+1)/2-(k+mj)(n-mj)}(1-\beta q^{1-\alpha})\,B(j)}{(p;p)_{n-mj-k}(p,p)_k}.$$

$$\cdot\frac{(\beta q^{k\alpha-n\alpha+mj\alpha+1};p)_\infty}{(\beta q^{k\alpha+mj\alpha+1-\alpha};p)_\infty}.$$

A routine observation with n-mj=M, is

$$\frac{(\beta q^{k\alpha-n\alpha+mj\alpha+1};p)_\infty}{(\beta q^{k\alpha+mj\alpha+1-\alpha};p)_\infty}=(\beta q^{1+k\alpha-M\alpha};p)_{M-1}$$

$$=\sum_{r=0}^{M-1} c_r\, p^{rk},\quad \text{say.}$$

Therefore,

$$F(n)=\sum_{j=0}^{[n/m]}\frac{p^{-mjM}(1-\beta q^{1-\alpha})\,B(j)}{(p;p)_M}\sum_{k=0}^{M}(-1)^k p^{k(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}\sum_{r=0}^{M-1}c_r\, p^{-(M-r-1)k}.$$

Here the series

$$\sum_{k=0}^{M}(-1)^k p^{k(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}\sum_{r=0}^{M-1}c_r\, p^{-(M-r-1)k}$$

$$=\sum_{k=0}^{M-1}(-1)^k p^{k(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}p^{-sk}c_s$$

$$=0$$

for s=0,1,2, ......M-1, hence F(n)=0, showing that (3.4.12)⇒(3.4.14), which completes the proof of the first part.

To prove the converse part it is assumed that the relations (3.4.13) and (3.4.14) namely,

$$d(n,k,m,q)=\frac{p^{k(k+1)/2-mnk}(\beta q^{k\alpha-mn\alpha+1},p)_{\infty}}{(p,p)_{mn-k}}$$

and

$\sum_{k=0}^{n} d(n,k,1;q)A(k)=0$, when $n \neq ms$, $s \in N$,

respectively hold true.

Now, in view of (3.8.1) and (3.4.14) one readily gets

$$F(n)=0,\ n \neq sm,\ s=1,2,3,.... \qquad \textbf{(3.8.2)}$$

and also by comparing (3.8.1) and (3.4.13) one finds that

$$F(nm)=F(mn)=B(n). \qquad \textbf{(3.8.3)}$$

Thus, in view of (3.8.3) the auxiliary q–inversion pair (3.7.1) and (3.7.2) takes the form (where T(n)=F(n) and S(n)=G(n)):

$$F(n)=\sum_{k=0}^{mn} p^{k(k+1)/2-mnk}\frac{(\beta q^{k\alpha-mn\alpha+1},p)_{\infty}}{(p,p)_{mn-k}}G(k)$$

implies

$$G(n)=\sum_{k=0}^{[n/m]}\frac{(-1)^{n-mk}p^{mk(mk-1)/2}(1-\beta q^{1-\alpha})}{(p;p)_{n-mk}(\beta q^{n\alpha-mk\alpha+1-\alpha};p)_{\infty}}F(k)$$

which completes the proof of the 'if' part and hence that of the theorem.

## 3.9 PARTICULAR CASES

In this section the q-polynomials contained in the general class $\{S_n(l,m,\alpha,\beta,x\mid p),$n=0,1,2,.....$\}$ are obtained. In particular q-analogues of the extended Jacobi polynomial $\mathcal{H}^{(\alpha,\beta)}_{n,l,m}[(a);(b):x]$, the Brafman polynomial $B_n^m[(a);(b).x]$, the well-known Laguerre and Jacobi polynomials, as well as those of the biorthogonal polynomials $Z_n^{(\alpha)}(x;k)$ and $W_n^{(\alpha,\beta)}(x;k)$ are obtained from $S_n(l,m,\alpha,\beta,x\,|\,p)$ by choosing the parameters suitably. Next, the properties namely the q-integrals discussed in section-3.2, the q-difference equation derived in section-3.3 and the q-inverse series relations proved in the subsequent sections will be particularized for the above mentioned polynomials.

In the beginning, the q-polynomials mentioned above are now obtained from

$$S_n(l,m,\alpha,\beta;x\,|\,p)=\sum_{k=0}^{[n/m]}(-1)^{mk}p^{mk(mk+1)/2-mnk}\,.$$

$$\cdot\frac{(\beta q^{1-n\alpha+lk};p)_\infty}{(p;p)_{n-mk}}\sigma_k\,x^k, \qquad \textbf{(3.9.1)}$$

by proper choice of the parameters involved.

For that, a slightly modified form of (3.9.1) is used, which is obtained as follows.

On making use of the formula,

$$(q;q)_{n-mk}=\frac{(-1)^{mk}q^{mk(mk+1)/2-mnk}[q]_n}{[q^{-n}]_{mk}}$$

and replacing $S_n(l,m,\alpha,\beta;x\,|\,p)$ by $S_n(l,m,\alpha,\beta;x\,|\,p)\,(\beta q^{1-n\alpha};p)_\infty$ in (3.9.1) one gets,

$$S_n(l,m,\alpha,\beta;x\mid p)=\sum_{k=0}^{[n/m]}\frac{p^{mk}(p^{-n};p)_{mk}\,\sigma_k\,x^k}{(\beta q^{1-n\alpha};p)_{lk}\,(p,p)_n}. \quad \textbf{(3.9.2)}$$

The q-extended Jacobi polynomial can be obtained from (3.9.2) just by choosing

$\sigma_k=\dfrac{[\alpha_1]_k\cdots[\alpha_s]_k}{[\beta_1]_k \quad [\beta_r]_k(p,p)_k}$, and replacing $S_n(l,m,\alpha,\beta,x\mid p)$ by $S_n(l,m,\alpha,\beta,x\mid p)/(p;p)_n$.

Thus,

$$\mathcal{H}^{(\alpha,\beta)}_{n,l,m}[(\alpha),(\beta).\,xp^m\mid p]=\sum_{k=0}^{[n/m]}\frac{(p^{-n},p)_{mk}[\alpha_1]_k \quad [\alpha_s]_k(xp^m)^k}{(\beta q^{1-n\alpha};p)_{lk}[\beta_1]_k\cdots[\beta_r]_k(p;p)_k}. \quad \textbf{(3.9.3)}$$

By taking $l=0$, $\alpha=1$ and $\sigma_k=\dfrac{[\alpha_1]_k\cdots\cdots[\alpha_s]_k}{[\beta_1]_k\cdots[\beta_r]_k[q]_k}$, and using the same replacement for $S_n(l,m,\alpha,\beta;x\mid p)$, one gets

$$B^m_n[\alpha_1,\ldots,\alpha_s;\beta_1,\ldots,\beta_r \quad xq^m\mid q]=\sum_{k=0}^{[n/m]}\frac{[q^{-n}]_{mk}\,[\alpha_1]_k\cdots[\alpha_s]_k(xq^m)^k}{[\beta_1]_k\cdots[\beta_r]_k\,[q]_k}. \quad \textbf{(3.9.4)}$$

If $l=0$, $\alpha=k$, $k\in N$, $x$ is replaced by $(xq^n)^k$, and then $S_n(l,m,\alpha,\beta;x\mid q)$ is replaced by $S_n(l,m,\alpha,\beta;x\mid q)/[\alpha q]_n$ and

$\sigma_k=\dfrac{q^{(\alpha+1)kj+kj(kj-1)/2-mkj}}{(q^k;q^k)_{mj}\,[\alpha q]_{kj}}$ then (3.9.2) yields,

$$Z^{(\alpha)}_{n,m}(x;k\mid q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{[n/m]}\frac{q^{kj(n+\alpha+1)+kj(kj-1)/2}\,(q^{-kn};q^k)_{mj}\,x^{kj}}{(q^k;q^k)_{mj}\,[\alpha q]_{kj}}. \quad \textbf{(3.9.5)}$$

This is an extension of the q-Konhauser polynomial $Z_n^{(\alpha)}(x,k\,|\,q)$ which occurs when m is unity (Al-salam and Verma [1]),

$$Z_n^{(\alpha)}(x,k\,|\,q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{q^{kj(n+\alpha+1)+kj(kj-1)/2}\,(q^{-nk};q^k)_j\;x^{kj}}{(q^k;q^k)_j\,[\alpha q]_{kj}}. \tag{3.9.6}$$

Setting k=1 in (3.9.5), an extension of the q-Laguerre polynomial is obtained in the form:

$$L_{n,m}^{(\alpha)}(x\,|\,q)=\frac{[\alpha q]_n}{[q]_n}\sum_{j=0}^{[n/m]}\frac{q^{j(n+\alpha+1)+j(j-1)/2}\,(q^{-n};q)_{mj}\;x^{j}}{(q;q)_{mj}\,(\alpha q,q)_j} \tag{3.9.7}$$

and the well-known q-Laguerre polynomial $L_n^{(\alpha)}(x\,|\,q)$ can be obtained from (3.9.7) by choosing m=1.

Thus,

$$L_n^{(\alpha)}(x\,|\,q)=\frac{[\alpha q]_n}{[q]_n}\sum_{j=0}^{n}\frac{q^{j(n+\alpha+1)+j(j-1)/2}(q^{-n};q)_j\;x^{j}}{(q;q)_j(\alpha q;q)_j}. \tag{3.9.8}$$

The polynomial (3.9.2) again with the substitution $l$=0, α=k, k∈N yields an extension of the biorthogonal polynomial $W_n^{(\alpha,\beta)}(x;k\,|\,q)$ when x is replaced by $\left(\frac{1-x}{2}\right)^k$, $S_n(l,m,\alpha,\beta,x\,|\,q)$ is replaced by $S_n(l,m,\alpha,\beta;x\,|\,q)/[\alpha q]_n$ and $\sigma_j=\frac{(\alpha\beta q;q)_{kj}}{(\alpha q;q)_{kj}(q^k;q^k)_j}$.

Thus,

$$W_{n,m}^{(\alpha,\beta-n)}(x,k\mid q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{[n/m]}\frac{q^{mkj}(q^{-nk},q^k)_{mj}(\alpha\beta q,q)_{kj}}{(q^k;q^k)_j(\alpha q,q)_{kj}}\left(\frac{1-x}{2}\right)^{kj}$$

**(3.9.9)**

Taking m=1, this polynomial gives the q-biorthogonal polynomial,

$$W_{n}^{(\alpha,\beta-n)}(x,k\mid q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{q^{kj}(q^{-nk};q^k)_j(\alpha\beta q,q)_{kj}}{(q^k,q^k)_j(\alpha q;q)_{kj}}\left(\frac{1-x}{2}\right)^{kj}. \quad \textbf{(3.9.10)}$$

The polynomial in (3.9.9) for k=1 reduces to

$$W_{n,m}^{(\alpha,\beta-n)}(x,1\mid q)=\frac{[\alpha q]_n}{[q]_n}\sum_{j=0}^{[n/m]}\frac{q^{mj}(q^{-n},q)_{mj}\,(\alpha\beta q,q)_j}{(q;q)_j\,(\alpha q;q)_j}\left(\frac{1-x}{2}\right)^{j}$$

$$=\frac{[\alpha q]_n}{[q]_n}\,p_{n,m}\left(q^m\left(\frac{1-x}{2}\right),\alpha,\beta-n,q\right)$$

in which the q-polynomial

$$p_{n,m}\left(q^m\left(\frac{1-x}{2}\right),\alpha,\beta-n,q\right)=\sum_{j=0}^{[n/m]}\frac{[q^{-n}]_{mj}[\alpha\beta q]_j}{[q]_j[\alpha q]_j}\left\{q^m\left(\frac{1-x}{2}\right)\right\}^j, \quad \textbf{(3.9.11)}$$

is an extension of the little q-Jacobi polynomial in the sense that setting m=1 it yields the little q-Jacobi polynomial:

$$p_n\left(q\left(\frac{1-x}{2}\right);\alpha,\beta-n;q\right)=\sum_{j=0}^{n}\frac{[q^{-n}]_j[\alpha\beta q]_j}{[q]_j[\alpha q]_j}\left\{q\left(\frac{1-x}{2}\right)\right\}^j \quad \textbf{(3.9.12)}$$

The remaining portion of this section is further divided into three parts for convenience. These three parts are:

(A) Special cases of q-integrals

(B) Special cases of the general q-difference equation (θ-form)

(C) Special cases of the general inverse series relations.

Now to begin with

(A) Special cases of q-integrals.

The q-integrals (3.2.7) and (3.2.10) derived in section-3.2, will now be particularized for the above mentioned specialized polynomials.

First the special cases of the q-integral (3.2.7) will be illustrated. In the light of the reducibility of $S_n(l,m,\alpha,\beta,x \mid p)$ to the q-extended Jacobi polynomial given by (3.9.3), the corresponding particularization of (3.2.7) follows in the form

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(\alpha),(\beta)\,.\,xp^{m} \mid p] = \int_{0}^{1} t^{\beta-n\alpha}\,(tp,p)_{\infty}\,\rho_n(m,\alpha,\beta;xp^{m},t,p)\,d_p t\,, \quad \textbf{(3.9.13)}$$

where

$$\rho_n(m,\alpha,\beta,xp^{m},t,p) = \sum_{k=0}^{[n/m]} \frac{(p^{-n};p)_{mk}}{(tp^{lk};p)_{\infty}(p;p)_k}\,\frac{[\alpha_1]_k \cdots [\alpha_s]_k (xp^{m})^k}{(1-p)^{lk}\,[\beta_1]_k \cdots [\beta_r]_k\,\Gamma_p(lk)}$$

and Re( 1+β-nα)>0, $lk \neq 0,-1,-2,\ldots$ .

In the light of the q-integral (3.2.10) some of the particular cases of $S_n(l,m,\alpha,\beta;x \mid q)$ are written in q-integral forms as follows.

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(\alpha),(\beta):xp^{m} \mid p] = \int_{-1}^{1} \frac{(-tp,p)_{\infty}(tq;p)_{\infty}}{(-tp^{a+B-n};p)_{\infty}}\,\xi_n(l,m,\alpha,\beta;x,t,p)\ d_p t \quad \textbf{(3.9.14)}$$

where

$$\xi_n(l,m,\alpha,\beta;x,t,p)=\frac{(-p^{a+B-n},p)_\infty}{(1-p)(p^2;p^2)_\infty(-1;p)_\infty}\sum_{k=0}^{[n/m]}\frac{(p^{-n};p)_{mk}(p^{2vk},p^2)_\infty}{(p;p)_k}.$$

$$\cdot\frac{[\alpha_1]_k\cdots[\alpha_s]_k(xp^m)^k}{(tp^{vk};p)_\infty[\beta_1]_k\cdots[\beta_r]_k}.$$

(B) <u>Special cases of the general q-difference equation (θ-form)</u>

In section-3.3 a general q-difference equation in θ-form is derived for a particular case $\sigma_k=\frac{1}{(p,p)_k}$ of $S_n(l,m,\alpha,\beta,x|p)$. This particular case of $S_n(l,m,\alpha,\beta,x|p)$ is denoted by $R_n^m(x|p)$. When the parameters involved in this q-difference equation, (3.3.13) are specialized, it gets particularized to the respective polynomials. These particular q-difference equations are listed below.

$$\Bigg[\theta\left\{\prod_{j=0}^{l-1}\prod_{k=0}^{l-1}(\theta+p^{1-\frac{1+\beta-n\alpha+j\alpha}{l}}\delta^k-1)\right\}\prod_{i=1}^{r}(\theta+p^{1-\beta_i}-1)$$

$$+x\,p^{-mn+\sum_{a=1}^{s}\alpha_a+2m+s-l(1+\beta-n\alpha)+l(l+1)/2-\sum_{i=1}^{r}\beta_i+r}$$

$$\left\{\prod_{u=0}^{m-1}\prod_{v=0}^{m-1}(\theta+p^{\frac{n-u}{m}}w^v-1)\right\}\prod_{a=1}^{s}(\theta+p^{-\alpha_a}-1)\Bigg]\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(\alpha);(\beta).xp^m|p]=0,$$

**(3.9.15)**

where δ is $l^{th}$ and ω is the $m^{th}$ roots of unity.

$$\left[\theta \prod_{i=0}^{r}(\theta+p^{1-\beta_i}-1)+x\,q^{2m-mn+\sum_{a=1}^{s}\alpha_a-\sum_{i=1}^{r}\beta_i+r}\right.$$

$$\left.\left\{\prod_{u=0}^{m-1}\prod_{v=0}^{m-1}(\theta+p^{\frac{n-u}{m}}w^{v}-1)\right\}\prod_{a=1}^{s}(\theta+p^{-\alpha_a}-1)\right]B_n^m[(\alpha);(\beta):x^m\mid p]=0\,,$$

**(3.9.16)**

where $\omega$ is the $m^{th}$ root of unity.

$$\left[\theta\left\{\prod_{u=0}^{m-1}\prod_{v=0}^{m-1}(\theta+p^{1-\frac{k+u}{m}}\omega^{v}-1)\right\}\left\{\prod_{i=0}^{k-1}\prod_{l=0}^{k-1}(\theta+q^{1-\frac{1+\alpha+i}{k}}\lambda^{l}-1)\right\}+\right.$$

$$\left.+x\,p^{-mn+2m-mk-\alpha k}\left\{\prod_{s=0}^{m-1}\prod_{r=0}^{m-1}(\theta+p^{\frac{n}{m}-\frac{s}{m}}w^{r}-1)\right\}\right]Z_{n,m}^{(\alpha)}(x;k\mid q)=0\,, \quad \textbf{(3.9.17)}$$

where $\lambda$ is the $k^{th}$ root of unity, and $p \equiv q^k$.

$$\left\{\theta(\theta+q^{-\alpha}-1)+xq^{-n-\alpha}(\theta+q^{-n}-1)\right\}L_n^{(\alpha)}(x\mid q)=0\,. \quad \textbf{(3.9.18)}$$

$$\left[\theta\left\{\prod_{i=0}^{k-1}\prod_{j=0}^{k-1}(\theta+q^{1-\frac{1+\alpha+i}{k}}\lambda^{j}-1)\right\}+\left(\frac{1-x}{2}\right)^{k}p^{2m+2-mn+\beta}\right.$$

$$\left.\left\{\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}(\theta+p^{\frac{n-r}{m}}\omega^{s}-1)\right\}\left\{\prod_{u=0}^{k-1}\prod_{v=0}^{k-1}(\theta+q^{-\frac{1+\alpha+\beta+v}{k}}\lambda^{v}-1)\right\}\right]W_{n,m}^{(\alpha,\beta-n)}(x;k\mid q)=0$$

**(3.9.19)**

where $p=q^k$.

$$\left\{\theta(\theta+q^{-\alpha}-1)+\left(\frac{1-x}{2}\right)q^{2+\beta-n}(\theta+q^{-1-\alpha}-1)\right\}p_n\left(q\left(\frac{1-x}{2}\right),\alpha,\beta-n,q\right)=0$$

**(3.9.20)**

(C) <u>Special cases of the general inverse series relations</u>

For deriving the inverse series relations of the polynomial special cases of the general class of polynomials $S_n(l,m,\alpha,\beta,x\,|\,q)$, first take $g(k)=\sigma_k\, x^k$ in (3.4.5) so that G(n)= $S_n(l,m,\alpha,\beta;x\,|\,p)$; and there by (3.4.6) will give the inverse series of $S_n(l,m,\alpha,\beta,x\,|\,p)$.

For convenience the polynomial $S_n(l,m,\alpha,\beta;x\,|\,p)$ is replaced by $\dfrac{S_n(l,m,\alpha,\beta;x\,|\,p)(\beta q^{1-n\alpha};p)_\infty}{(p;p)_n}$ and the formula

$$(p,p)_{n-mk}=\frac{(-1)^{mk}\,p^{mk(mk-1)/2-mnk}[p]_n}{[p^{-n}]_{mk}}$$

is applied, then one gets

$$S_n(l,m,\alpha,\beta;x\,|\,p)=\sum_{k=0}^{[n/m]}\frac{p^{mk}\,(p^{-n};p)_{mk}\,\sigma_k\,x^k}{(\beta q^{1-n\alpha};p)_{lk}}. \qquad \textbf{(3.9.21)}$$

The inverse series (3.4.6) then takes the form

$$\sigma_n\,x^n=\sum_{k=0}^{mn}\frac{(-1)^k\,p^{k(k-1)/2}\,(1-\beta q^{1-\alpha})(\beta q^{1-n\alpha};p)_\infty\,S_k(l,m,\alpha,\beta;x\,|\,p)}{(p;p)_{mn-k}\,(\beta q^{1-k\alpha+ln-\alpha},p)_\infty\,(p,p)_k}.$$

**(3.9.22)**

The inversion pair (3.9.21) and (3.9.22) is utilized now for obtaining various polynomials and their respective inverse series. It is to be noted here that $m\alpha=l$, $l\in N$ for the property of inverse series relation.

Choosing $\sigma_k = \frac{[\alpha_1]_k \cdots\cdots [\alpha_s]_k}{[\beta_1]_k \cdots [\beta_r]_k (p,p)_k}$ in (3.9.21) it gets reduced to the extended Jacobi polynomial, where as (3.9.22) gives its inverse series relation.

The pair is given by

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(\alpha);(\beta):xp^m \mid p] = \sum_{k=0}^{[n/m]} \frac{(p^{-n};p)_{mk}\,[\alpha_1]_k \cdots .[\alpha_s]_k\,(xp^m)^k}{(\beta q^{1-n\alpha};p)_{lk}\,[\beta_1]_k \cdots [\beta_r]_k\,(p;p)_k} \qquad \textbf{(3.9.23)}$$

$\Leftrightarrow$

$$x^n = \frac{[\beta_1]_n \; .[\beta_r]_n (p,p)_n}{[\alpha_1]_n \cdots ..[\alpha_s]_n} \sum_{k=0}^{mn} \frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})(\beta q^{1-n\alpha};p)_\infty}{(\beta q^{1-k\alpha+ln-\alpha};p)_\infty (p,p)_{mn-k}} \cdot$$

$$.\mathcal{H}_{k,l,m}^{(\alpha,\beta)}[(\alpha);(\beta)\cdot xp^m \mid p] \qquad \textbf{(3.9.24)}$$

subject to

$$\sum_{k=0}^{n} \frac{(-1)^k p^{k(k-1)/2}(1-\beta q^{1-\alpha})(\beta q^{1-n\alpha};p)_\infty}{(\beta q^{1-k\alpha+n\alpha-\alpha};p)_\infty (p;p)_{n-k}} \mathcal{H}_{k,l,1}^{(\alpha,\beta)}[(\alpha);(\beta):xp \mid p]=0$$

if $n\neq ms$, $s\in N$. **(3.9.25)**

In order to obtain the q-Brafman polynomial, and the extended versions of the biorthogonal polynomials $Z_n^{(\alpha)}(x;k \mid q)$ and $W_n^{(\alpha,\beta)}(x,k \mid q)$

together with their inverses, the parameter β→∞ is considered so that the product $(\beta q^{1-n\alpha};p)_{lk} \to 1$. Then α is taken to be 1 for the q-Brafman polynomial and α=k, k∈N, for the rest of the two cases.

Since 0<q<1, letting β→∞ (i.e. $q^{\beta} \to 0$) (3.9.21) and (3.9.22) take the form:

$$\lim_{\beta\to\infty} S_n(l,m,\alpha,\beta;x\,|\,p) = \sum_{k=0}^{[n/m]} p^{mk}(p^{-n};p)_{mk}\,\sigma_k\,x^k \tag{3.9.26}$$

and its inverse is

$$\sigma_n\,x^n = \sum_{k=0}^{mn} \frac{(-1)^k\,p^{k(k-1)/2}\,S_k(l,m,\alpha,\beta;x\,|\,p)}{(p;p)_{mn-k}\,(p,p)_k}. \tag{3.9.27}$$

This pair is used now to obtain the polynomials $B_n^m[(\alpha);(\beta);xq^m\,|\,q]$, $Z_{n,m}^{(\alpha)}(x;k\,|\,q)$ and $W_{n,m}^{(\alpha,\beta-n)}(x,k\,|\,q)$ together with their inverse as follows. Putting α=1 and $\sigma_k = \frac{[\alpha_1]_k \quad [\alpha_s]_k}{[\beta_1]_k \cdots [\beta_r]_k (q)_k}$ in the above pair, one gets

$$B_n^m[\alpha_1,\ldots\alpha_s;\beta_1,\ldots\beta_r\;\; xq^m\,|\,q] = \sum_{k=0}^{[n/m]} \frac{[q^{-n}]_{mk}[\alpha_1]_k \cdots [\alpha_s]_k (xq^m)^k}{[q]_k[\beta_1]_k \cdots [\beta_r]_k} \tag{3.9.28}$$

if and only if

$$x^n = \frac{[\beta_1]_n \cdots [\beta_r]_n [q]_n}{[\alpha_1]_n \cdots [\alpha_s]_n} \sum_{k=0}^{mn} \frac{(-1)^k q^{k(k-1)/2} B_k^m[\alpha_1,\ldots\ldots\alpha_s;\beta_1\ldots\beta_r : xq^m\,|\,q]}{(q;q)_k (q;q)_{mn-k}}.$$

**(3.9.29)**

Taking $\alpha=k$, $k\in N$, and $\sigma_j = \dfrac{q^{(\alpha+1)kj+kj(kj-1)/2-mkj}}{(q^k,q^k)_{mj}[\alpha q]_{kj}}$ and replacing x

by $(xq^n)^k$ and $S_n(l,m,\alpha,\beta;x|q)$, by $\dfrac{S_n(l,m,\alpha,\beta,x|q)(q^k,q^k)_n}{[\alpha q]_{nk}}$ in (3.9.26)

and (3.9.27), the pair of inverse series relations of $Z_{n,m}^{(\alpha)}(x;k|q)$ is obtained in the form:

$$Z_{n,m}^{(\alpha)}(x,k|q)=\frac{[\alpha q]_n}{(q^k,q^k)_n}\sum_{j=0}^{[n/m]}\frac{q^{kj(\alpha+n+1)+kj(kj-1)/2}(q^{-nk};q^k)_{mj}\,x^{kj}}{(q^k;q^k)_{mj}\,[\alpha q]_{kj}} \quad \textbf{(3.9.30)}$$

$\Leftrightarrow$

$$x^{kn}=\frac{(q^k;q^k)_{mn}\,[\alpha q]_{kn}}{q^{kn(\alpha+1)+kn(kn-1)/2-mnk}}\sum_{j=0}^{mn}\frac{(-1)^j q^{kj(j-1)/2}\,Z_{j,m}^{(\alpha)}(x;k|q)}{(q^k;q^k)_{mn-j}\,[\alpha q]_j}. \quad \textbf{(3.9.31)}$$

Setting m=1 in this pair one gets

$$Z_n^{(\alpha)}(x;k|q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{q^{kj(\alpha+n+1)+kj(kj-1)/2}(q^{-nk},q^k)_j\,x^{kj}}{(q^{-nk};q^k)_j\,[\alpha q]_{kj}} \quad \textbf{(3.9.32)}$$

$\Leftrightarrow$

$$x^{kn}=\frac{(q^k,q^k)_n\,[\alpha q]_{kn}}{q^{kn(\alpha+1)+kn(kn-1)/2-kn}}\sum_{j=0}^{n}\frac{(-1)^j q^{kj(j-1)/2}\,Z_j^{(\alpha)}(x;k|q)}{(q^k;q^k)_{n-j}\,[\alpha q]_j} \quad \textbf{(3.9.33)}$$

where (3.9.32) is the q-Konhauser polynomial (Al-Salam and Verma [1]).

The inversion pair of an extension $L_{n,m}^{(\alpha)}(x|q)$ of the q-Laguerre polynomial is obtained from (3.9.30) and (3.9.31) by choosing k=1.

$$L_{n,m}^{(\alpha)}(x|q)=\frac{[\alpha q]_n}{[q]_n}\sum_{j=0}^{[n/m]}\frac{q^{j(\alpha+n+1)+j(j-1)/2}(q^{-n};q)_{mj}\,x^j}{(q,q)_{mj}\,[\alpha q]_j} \tag{3.9.34}$$

$\Leftrightarrow$

$$x^n=\frac{[q]_{mn}[\alpha q]_n}{q^{n(\alpha+1)+n(n-1)/2-mn}}\sum_{j=0}^{mn}\frac{(-1)^j q^{j(j-1)/2}\,L_{k,m}^{(\alpha)}(x|q)}{(q,q)_{mn-j}\,[\alpha q]_j}. \tag{3.9.35}$$

Further taking m=1 in this pair it reduces to the well known pair of inverse series relation of the q-Laguerre polynomial (Gasper and Rehman [1]).

$$L_n^{(\alpha)}(x|q)=\frac{[\alpha q]_n}{[q]_n}\sum_{j=0}^{n}\frac{q^{j(\alpha+n+1)+j(j-1)/2}(q^{-n},q)_j\,x^j}{(q;q)_j\,[\alpha q]_j} \tag{3.9.36}$$

$\Leftrightarrow$

$$x^n=\frac{[q]_n\,[\alpha q]_n}{q^{n(\alpha+1)+n(n-1)/2-n}}\sum_{j=0}^{n}\frac{(-1)^j q^{j(j-1)/2}\,L_k^{(\alpha)}(x|q)}{(q;q)_{n-j}\,[\alpha q]_j}. \tag{3.9.37}$$

Now if one puts α=k, k∈N, $\sigma_j=\frac{(\alpha\beta q,q)_{kj}}{(\alpha q;q)_{kj}\,(q^k;q^k)_j}$ and replaces $x$ by $\left(\frac{1-x}{2}\right)^k$ and $S_n(l,m,\alpha,\beta;x|q)$ by $\frac{S_n(l,m,\alpha,\beta;x|q)\,(q^k;q^k)_n}{[\alpha q]_n}$ in (3.9.26) and (3.9.27), then one gets the pair of inverse series relations of an extension $W_{n,m}^{(\alpha,\beta-n)}(x;k|q)$ of the biorthogonal polynomial

$W_n^{(\alpha,\beta-n)}(x,k|q)$. As above taking m=1 it gives the inversion pair of $W_n^{(\alpha,\beta-n)}(x,k|q)$, whereas with k=1 it reduces to the pair of inverse series relations of an extension $p_{n,m}\left(q^m\left(\frac{1-x}{2}\right);\alpha,\beta-n,q\right)$ of the little q-Jacobi polynomial. And next with m=1 this new pair reduces to the inversion pair of $p_n\left(q\left(\frac{1-x}{2}\right);\alpha,\beta-n,q\right)$. All these inverse series relations are listed below.

$$W_{n,m}^{(\alpha,\beta-n)}(x;k|q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{[n/m]}\frac{q^{kmj}(q^{-nk},q^k)_{mj}\,[\alpha\beta q]_{kj}}{[\alpha q]_{kj}\,(q^k,q^k)_j}\left(\frac{1-x}{2}\right)^{kj} \quad \textbf{(3.9.38)}$$

$\Leftrightarrow$

$$\left(\frac{1-x}{2}\right)^{nk}=\frac{[\alpha q]_{kn}(q^k;q^k)_n}{[\alpha\beta q]_{kn}}\sum_{j=0}^{mn}\frac{(-1)^j\,q^{kj(j-1)/2}\,W_{j,m}^{(\alpha,\beta-j)}(x,k|q)}{(q^k,q^k)_{mn-j}\,[\alpha q]_j}, \quad \textbf{(3.9.39)}$$

$$W_n^{(\alpha,\beta-n)}(x,k|q)=\frac{[\alpha q]_n}{(q^k,q^k)_n}\sum_{j=0}^{n}\frac{q^{kj}(q^{-nk};q^k)_j\,[\alpha\beta q]_{kj}}{[\alpha q]_{kj}\,(q^k,q^k)_j}\left(\frac{1-x}{2}\right)^{kj} \quad \textbf{(3.9.40)}$$

$\Leftrightarrow$

$$\left(\frac{1-x}{2}\right)^{nk}=\frac{[\alpha q]_{kn}(q^k;q^k)_n}{[\alpha\beta q]_{kn}}\sum_{j=0}^{n}\frac{(-1)^j\,q^{kj(j-1)/2}\,W_j^{(\alpha,\beta-j)}(x;k|q)}{(q^k,q^k)_{n-j}\,[\alpha q]_j}, \quad \textbf{(3.9.41)}$$

$$p_{n,m}\left(q^m\left(\frac{1-x}{2}\right);\alpha,\beta-n,q\right)=\sum_{j=0}^{[n/m]}\frac{[q^{-n}]_{mj}\,[\alpha\beta q]_j}{[\alpha q]_j\,[q]_j}\left\{q^m\left(\frac{1-x}{2}\right)\right\}^j \quad \textbf{(3.9.42)}$$

$\Leftrightarrow$

$$\left(\frac{1-x}{2}\right)^n = \frac{[\alpha q]_n\,[q]_n}{[\alpha\beta q]_n} \sum_{j=0}^{mn} \frac{(-1)^j\, q^{j(j-1)/2}}{[q]_{mn-j}\,[q]_j}\, p_{j,m}\left(q^m\left(\frac{1-x}{2}\right);\alpha,\beta-j,q\right), \quad \textbf{(3.9.43)}$$

$$p_n\left(q\left(\frac{1-x}{2}\right),\alpha,\beta-n;q\right) = \sum_{j=0}^{n} \frac{[q^{-n}]_j\,[\alpha\beta q]_j}{[\alpha q]_j\,[q]_j} \left\{q\left(\frac{1-x}{2}\right)\right\}^j \quad \textbf{(3.9.44)}$$

$\Leftrightarrow$

$$\left(\frac{1-x}{2}\right)^n = \frac{[\alpha q]_n\,[q]_n}{[\alpha\beta q]_n} \sum_{j=0}^{n} \frac{(-1)^j\, q^{j(j-1)/2}}{[q]_{n-j}\,[q]_j}\, p_j\left(q\left(\frac{1-x}{2}\right);\alpha,\beta-j;q\right). \quad \textbf{(3.9.45)}$$

## 3.10 LIMITING CASE

When q→1 is taken then the polynomial $S_n(l,m,\alpha,\beta;x\,|\,q)$ reduces to $S_n(l,m,\alpha,\beta;x)$, which is already considered in chapter-2.

For doing this, replacing $S_n(l,m,\alpha,\beta;x\,|\,q)$ by $S_n(l,m,\alpha,\beta;x\,|\,q)\,(\beta q^{1-n\alpha},p)_\infty$ one gets

$$S_n(l,m,\alpha,\beta;x\,|\,q) = \sum_{k=0}^{[n/m]} \frac{(-1)^{mk}\, p^{mk(mk+1)/2-mnk}\,(\beta q^{1-n\alpha+lk};p)_\infty\, \sigma_k\, x^k}{(p;p)_{n-mk}\,(\beta q^{1-n\alpha};p)_\infty},$$

so that

$$\lim_{q\to 1} S_n(l,m,\alpha,\beta,x\,|\,q)$$

$$= \lim_{q\to 1} \sum_{k=0}^{[n/m]} \frac{(-1)^{mk}\, p^{mk(mk+1)/2-mnk}\,(\beta q^{1-n\alpha+lk};p)_\infty\, \sigma_k\, x^k}{(p;p)_{n-mk}\,(\beta q^{1-n\alpha};p)_\infty}.$$

Since

$$\lim_{q\to 1} \frac{(\beta q^{1-n\alpha+lk},p)_\infty}{(p;p)_\infty}\, \frac{(p,p)_\infty}{(\beta q^{1-n\alpha};p)_\infty}\, \frac{(1-p)^{n-mk}}{(p;p)_{n-mk}}$$

$$= \lim_{q \to 1} \frac{\Gamma_p(1+\beta-n\alpha)(1-p)^{\beta-n\alpha}}{\Gamma_p(1+\beta-n\alpha+lk)(1-p)^{\beta-n\alpha+lk}} \frac{1}{(n-mk)!}$$

$$= \frac{1}{(n-mk)!} \lim_{q \to 1} \frac{\Gamma_p(1+\beta-n\alpha)}{\Gamma_p(1+\beta-n\alpha+lk)} \frac{p^{lk}}{(1-p)^{lk}}$$

$$= \frac{\Gamma(1+\beta-n\alpha)}{\Gamma(1+\beta-n\alpha+lk)},$$

as $\Gamma_q(x) \to \Gamma(x)$ when q→1. Therefore

$$\lim_{q \to 1} S_n(l,m,\alpha,\beta,x \mid q) = \sum_{k=0}^{[n/m]} \frac{(-1)^{mk}\, \Gamma(1+\beta-n\alpha)\, \sigma_k\, x^k}{(n-mk)!\; \Gamma(1+\beta-n\alpha+lk)}$$

$$= \Gamma(1+\beta-n\alpha) \sum_{k=0}^{[n/m]} \frac{(-1)^{mk}\, \sigma_k\, x^k}{\Gamma(1+\beta-n\alpha+lk)\,(n-mk)!}$$

$$= \Gamma(1+\beta-n\alpha)\; S_n(l,m,\alpha,\beta;x).$$