CHAPTER – 5

UNIFICATION OF GENERALIZED q-POLYNOMIALS & THEIR PROPERTIES

5.1 INTRODUCTION

In Chapter-4, the ordinary version of the inversion pair ((4.1.1) and (4.1.2)) proved by Gessel and Stanton [1] was given further extension. A general class of polynomials

$$M_n(s,A,\beta;x) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\,\Gamma(A+sk+n\beta)}{(n-sk)!}\,\psi_k\,x^k \qquad \textbf{(5.1.1)}$$

was defined from that extended pair with a view to study integral representations, differential equation (θ-form) and inverse series relations. In this chapter a q-analogue of (5.1.1) denoted here by the symbol $M_n(s,A,\beta;x|q)$, is constructed and the above mentioned properties are given q-extensions through it.

The general class of q-polynomials

$\{M_n(s,A,\beta;x|q),\ n=0,1,2,\ldots\}$ is defined as:

$$M_n(s,A,\beta;x\,|\,q) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\,q^{-snk}\,(Aq^{sk+sk\beta};q^{\beta})_{n-sk}}{[q]_{n-sk}}\,\xi_k x^k, \qquad \textbf{(5.1.2)}$$

where ξ_k is a general sequence (not involving n).

This polynomial unifies several known q-polynomials. In what follows, unless otherwise stated $q_2 \equiv q^{\beta}$.

In fact, through this general polynomial (5.1.2) several known basic polynomials occurring as its special cases, are further extended by giving particular values to the parameters involved.

These extended q-polynomials are listed below.

Extension of Askey-Wilson Polynomial

$$P_{n,s}(x;a,b,c,d\,|\,q)=a^{-n}[q]_n[ab]_n[ac]_n[ad]_n\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[abcdq^{n-1}]_{sk}}{[q]_n[ab]_k}.$$

$$\frac{[ae^{i\theta}]_k[ae^{-i\theta}]_k}{[ac]_k[ad]_k}q^k. \quad \textbf{(5.1.3)}$$

Extension of q-Racah Polynomial

$$R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}[q^{-x}]_k}{[\alpha q]_k[\gamma q]_k[\beta\delta q]_k[q]_k}[\gamma\delta q^{x+1}]_k q^k. \quad \textbf{(5.1.4)}$$

Extension of q-Hahn Polynomial

$$Q_{n,s}(x;\alpha,\beta,N\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}[q^{-x}]_k q^k}{[\alpha q]_k[q^{-N}]_k[q]_k}. \quad \textbf{(5.1.5)}$$

Extended Little q-Jacobi Polynomial

$$p_{n,s}(x;\alpha,\beta;q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}(xq)^k}{[\alpha q]_k[q]_k}. \quad \textbf{(5.1.6)}$$

Extended q-Legendre Polynomial

$$P_{n,s}(x\,|\,q)\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[q^{n+1}]_{sk}(xq)^k}{[q]_k[q]_k}. \quad \textbf{(5.1.7)}$$

Extended q-Konhauser Polynomial

$$Z_{n,s}^{(\alpha)}(x;k\,|\,q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{[n/s]}\frac{q^{kj(n+\alpha+1)+kj(kj-1)/2}(q^{-nk};q^k)_{sj}\,x^{kj}}{(q^k;q^k)_{sj}[\alpha q]_{kj}}. \quad \textbf{(5.1.8)}$$

Extended q-Laguerre Polynomial

$$L_{n,s}^{(\alpha)}(x\,|\,q)=\frac{[\alpha q]_n}{[q]_n}\sum_{j=0}^{[n/s]}\frac{q^{j(n+\alpha+1)+j(j-1)/2}[q^{-n}]_{sj}\,x^j}{[q]_{sj}[\alpha q]_j}. \quad \textbf{(5.1.9)}$$

Extended Wall Polynomial

$$W_{n,s}(x;a,q)=(-1)^n[a]_n q^{n(n+1)/2}\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{k(k-1)/2-snk}}{[a]_k}\begin{bmatrix} n \\ sk \end{bmatrix}x^k \quad \textbf{.(5.1.10)}$$

Extended Stieltjes-Wigert Polynomial

$$S_{n,m}(x,p,q)=(-1)^n q^{-n(2n+1)/2}[p]_n\sum_{k=0}^{[n/m]}(-1)^{mk}q^{k^2+k/2}\begin{bmatrix} n \\ mk \end{bmatrix}\frac{x^k}{[p]_k}.$$

(5.1.11)

The inverse series relations of these extended polynomials is given in section-5.8. For inverting the polynomial (5.1.2) a general inversion formula is also proved. Together with this, a couple of inversion formulas are discussed in sections-5.4 to 5.7. Section-5.2 contains integral representations of $M_n(s,A,\beta;x|q)$ which are derived using some known q-integrals. Whereas section-5.3 deals with q-difference equation (θ-form) of the polynomial (5.1.2).

5.2 q-INTEGRAL REPRESENTATIONS

In this section the q-integral representations of $M_n(s,A,\beta;x|q)$ are obtained using four known basic integral formulas. These q-integrals are as listed below.

q-Beta Integral (Gasper and Rahman[1])

$$\beta_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$$

$$=\int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}d_qt,\quad \mathrm{Re}(x)>0,\ y\neq 0,-1,-2,\ldots\ldots \tag{5.2.1}$$

q-Beta Integral (Askey and Andrews[1])

$$\int_{-c}^{d}\frac{(-qx/c;q)_\infty(qx/d;q)_\infty}{(-xq^\alpha/c;q)_\infty(xq^\beta/d;q)_\infty}d_qx=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}\frac{cd}{c+d}.$$

$$.\frac{(-c/d;q)_\infty(-d/c;q)_\infty}{(-q^\beta c/d;q)_\infty(-q^\alpha d/c;q)_\infty}. \tag{5.2.2}$$

q-Beta Integral (W.Hahn[1, p.9])

$$\int_0^1 x^{\lambda-1}E_q(qx)d(q,x)=(1-q)\frac{(q;q)_\infty}{(q^\lambda,q)_\infty}. \tag{5.2.3}$$

q-Gamma Integral (W.Hahn[1, p.10])

$$\int_0^\infty t^{h-1}e_q(-t)\,d_qt=\frac{(1-q)[q]_\infty}{[q^h]_\infty}q^{-h(h-1)/2}. \tag{5.2.4}$$

(I) On making use of the q-integral (5.2.1), a q-integral for $M_n(s,A,\beta;x|q)$ will now be obtained.

Consider,

$$M_n(s,A,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}}{[q]_{n-sk}}\ \xi_k x^k, \qquad \textbf{(5.2.5)}$$

where

$$(Aq^{sk+sk\beta};q_2)_{n-sk}$$

$$=\frac{(Aq^{sk+sk\beta};q_2)_\infty}{(Aq^{sk+n\beta},q_2)_\infty}\frac{(q_2,q_2)_\infty}{(q_2;q_2)_\infty}$$

$$=\frac{(1-q_2)^{1-A-sk-sk\beta}}{\Gamma_{q_2}(A+sk+sk\beta)}\cdot\frac{\Gamma_{q_2}(A+sk+n\beta)}{(1-q_2)^{1-A-sk-n\beta}}$$

$$=(1-q_2)^{n\beta-sk\beta}\cdot\frac{\Gamma_{q_2}(A+sk+n\beta)}{\Gamma_{q_2}(A+sk+sk\beta)} \qquad \textbf{(5.2.6)}$$

$$=\frac{(1-q_2)^{n\beta-sk\beta}\,\Gamma_{q_2}(A+sk+n\beta)\,\Gamma_{q_2}(A+sk\beta)\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)}{\Gamma_{q_2}(A+sk+sk\beta)\,\Gamma_{q_2}(A+sk\beta)\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)}$$

$$=\frac{(1-q_2)^{n\beta-sk\beta}\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)}{\Gamma_{q_2}(A+sk+sk\beta)\,\Gamma_{q_2}(A+sk\beta)}\int_0^1 t^{A+n\beta+sk-1}\frac{(tq,q_2)_\infty}{(tq^{A+sk\beta},q_2)_\infty}\,d_{q_2}t,$$

Re$(A+n\beta)>0$, $A+sk\beta\neq 0,-1,-2,\ldots$

Therefore, from (5.2.5)

$$M_n(s,A,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}\,\xi_k\,x^k(1-q_2)^{n\beta-sk\beta}\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)}{[q]_{n-sk}\,\Gamma_{q_2}(A+sk+sk\beta)\,\Gamma_{q_2}(A+sk\beta)}.$$

$$\int_0^1 t^{A+n\beta+sk-1}\frac{(tq;q_2)_\infty}{(tq^{A+sk\beta};q_2)_\infty}d_{q_2}t,$$

Re$(A+n\beta)>0$, $A+sk\beta\neq 0,-1,-2,\ldots$

or alternatively

$$M_n(s,A,\beta;x\,|\,q)=\int_0^1 t^{A+n\beta-1}(tq;q_2)_\infty\,\rho_n(s,A,\beta,x,t\,|\,q)\,d_{q_2}t,$$

where

$$\rho_n(s,A,\beta;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(1-q_2)^{n\beta-sk\beta}\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)\xi_k x^k t^{sk}}{[q]_{n-sk}\Gamma_{q_2}(A+sk+sk\beta)\Gamma_{q_2}(A+sk\beta)(tq^{A+sk\beta};q_2)_\infty},$$

and Re$(A+n\beta)>0, A+sk\beta\neq 0,-1,-2,..$ **(5.2.7)**

(II) The q-Beta integral of Askey and Andrews given above in (5.2.2) is now considered with c=d=1, which then reads as:

$$2\int_{-1}^{1}\frac{(-qx;q)_\infty(qx;q)_\infty}{(-xq^\alpha;q)_\infty(xq^\beta;q)_\infty}d_qx=\frac{\Gamma q(\alpha)\Gamma q(\beta)}{\Gamma q(\alpha+\beta)}\frac{(-1;q)_\infty(-1;q)_\infty}{(-q^\beta;q)_\infty(-q^\alpha;q)_\infty}.$$

In (I) it is already shown (in (5.2.6)) that

$$(Aq^{sk+sk\beta};q^\beta)_{n-sk}=(1-q_2)^{n\beta-sk\beta}\frac{\Gamma_{q_2}(A+sk+n\beta)}{\Gamma_{q_2}(A+sk+sk\beta)},$$

therefore it can also be written as:

$$(Aq^{sk+sk\beta};q^\beta)_{n-sk}=\frac{(1-q_2)^{n\beta-sk\beta}\Gamma_{q_2}(A+sk+n\beta)}{\Gamma_{q_2}(A+sk+sk\beta)}\frac{\Gamma_{q_2}(A+sk\beta)}{\Gamma_{q_2}(2A+sk+sk\beta+n\beta)}.$$

$$\frac{\Gamma_{q_2}(2A+sk+sk\beta+n\beta)}{\Gamma_{q_2}(A+sk\beta)}$$

$$=\frac{(1-q_2)^{n\beta-sk\beta}\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)\,\Gamma_{q_2}(A+sk+n\beta)\,\Gamma_{q_2}(A+sk\beta)}{\Gamma_{q_2}(A+sk+sk\beta)\,\Gamma_{q_2}(A+sk\beta)\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)}$$

$$\frac{(-1,q_2)_\infty(-1,q_2)_\infty}{(-q^{A+sk\beta};q_2)_\infty(-q^{A+sk+n\beta},q_2)_\infty}\;\frac{(-q^{A+sk\beta},q_2)_\infty(-q^{A+sk+n\beta},q_2)_\infty}{(-1,q_2)_\infty(-1,q_2)_\infty}$$

$$=\frac{(1-q_2)^{n\beta-sk\beta}\,\Gamma_{q_2}(2A+sk+sk\beta+n\beta)(-q^{A+sk\beta},q_2)_\infty(-q^{A+sk+n\beta};q_2)_\infty}{\Gamma_{q_2}(A+sk+sk\beta)\Gamma_{q_2}(A+sk\beta)(-1;q_2)_\infty(-1,q_2)_\infty}.$$

$$2\int_{-1}^{1}\frac{(-tq;q_2)_\infty(tq;q_2)_\infty}{(-tq^{A+sk+n\beta},q_2)_\infty(-tq^{A+sk\beta},q_2)_\infty}d_{q_2}t$$

but in view of the definition of $\Gamma_q(x)$ *and* $(-q^x;q)_\infty(q^x;q)_\infty=(q^{2x};q^2)_\infty$,

one gets

$$\frac{(-q^{A+sk\beta},q_2)_\infty}{\Gamma_{q_2}(A+sk\beta)(-1;q_2)_\infty}=\frac{(-q^{A+sk\beta};q_2)_\infty(q^{A+sk\beta};q_2)_\infty}{(1-q_2)^{1-A-sk\beta}(q_2,q_2)_\infty(-1;q_2)_\infty}$$

$$=\frac{1}{2}\;\frac{(q^{2A+2sk\beta};q_2^2)_\infty}{(1-q_2)^{1-A-sk\beta}(q_2^2;q_2^2)_\infty}.$$

Thus

$$(Aq^{sk+sk\beta};q^\beta)_{n-sk}=$$

$$\frac{(1-q_2)^{A+n\beta-1}\Gamma_{q_2}(2A+sk+sk\beta+n\beta)(q^{2A+2sk\beta};q_2^2)_\infty(-q^{A+sk+n\beta};q_2)_\infty}{\Gamma_{q_2}(A+sk+sk\beta)(-1;q_2)_\infty(q_2^2;q_2^2)_\infty}.$$

$$.\int_{-1}^{1}\frac{(-tq;q_2)_\infty(tq;q_2)_\infty}{(-tq^{A+sk+n\beta};q_2)_\infty(-tq^{A+sk\beta};q_2)_\infty}d_{q_2}t.$$

Finally substituting the above in $M_n(s,A,\beta;x|q)$ the q-integral is obtained in the form:

$$M_n(s,A,\beta,x\,|\,q)=\int_{-1}^{1}(-tq;q_2)_\infty(tq;q_2)_\infty\,\delta_n(s,A,\beta;x,t\,|\,q)\,d_{q_2}t, \qquad \textbf{(5.2.8)}$$

where

$$\delta_n(s,A,\beta,x,t\,|\,q)=$$

$$\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(1-q_2)^{A+n\beta-1}\Gamma_{q_2}(2A+sk+sk\beta+n\beta)(q^{2A+2sk\beta};q_2^2)_\infty}{[q]_{n-sk}\Gamma_{q_2}(A+sk+sk\beta)(-1,q_2)_\infty(q_2^2,q_2^2)_\infty}.$$

$$\cdot\frac{(-q^{A+sk+n\beta};q_2)_\infty\,\xi_k\,x^k}{(-tq^{A+sk+n\beta};q_2)_\infty(-tq^{A+sk\beta};q_2)_\infty}.$$

(III) Another q-integral representation of the polynomial $M_n(s,A,\beta;x|q)$ is obtained using (5.2.3).

Using

$$(Aq^{sk+sk\beta};q_2)_{n-sk}=\frac{(Aq^{sk+sk\beta},q_2)_\infty}{(Aq^{sk+n\beta};q_2)_\infty}\frac{(q_2;q_2)_\infty}{(q_2;q_2)_\infty},$$

in the defining relation of $M_n(s,A,\beta;x|q)$, one gets in view of (5.2.3),

$$M_n(s,A,\beta:x\bar{t}^s\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_\infty\,\xi_k\,x^k}{[q]_{n-sk}(q_2;q_2)_\infty(1-q_2)}.$$

$$\cdot\int_0^1 t^{A+n\beta-1}E_{q_2}(tq_2)\,d(q_2,t).$$

or

$$M_n(s,A,\beta \cdot x\bar{t}^s \mid q) = \int_0^1 t^{A+n\beta-1} E_{q_2}(tq_2)\Delta_n(s,A,\beta,x \mid q)\ d(q_2,t), \qquad \textbf{(5.2.9)}$$

where

$$\Delta_n(s,A,\beta \cdot x \mid q) = \frac{1}{(q_2;q_2)_\infty} \sum_{k=0}^{[n/s]} \frac{(-1)^{sk} q^{-snk} (Aq^{sk+sk\beta};q_2)_\infty}{[q]_{n-sk}(1-q_2)} \xi_k x^k.$$

(IV) The q-gamma integral defined by Hahn[1, p.10] given above in (5.2.4) is used as a tool to get another q-integral form of $M_n(s,A,\beta;x|q)$.

For that once again using,

$$(Aq^{sk+sk\beta};q_2)_{n-sk} = \frac{(Aq^{sk+sk\beta})_\infty}{(Aq^{sk+n\beta})_\infty} \cdot \frac{(q_2;q_2)_\infty}{(q_2;q_2)_\infty},$$

and (eq. (5.2.4))

$$\frac{[q]_\infty}{[q^h]_\infty} = \frac{q^{h(h-1)/2}}{1-q} \int_0^\infty t^{h-1} e_q(-t) d_q t$$

the polynomial $M_n(s,A,\beta;x|q)$ takes the form:

$$M_n(s,A,\beta : x\bar{t}^s \mid q) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk} q^{-snk} (Aq^{sk+sk\beta};q_2)_\infty \xi_k x^k}{[q]_{n-sk}(q_2,q_2)_\infty}$$

$$\cdot \frac{q^{(A+sk+n\beta)(A+sk+n\beta-1)/2}}{(1-q_2)} \int_0^\infty t^{A+sk+n\beta-1-sk} e_{q_2}(-t) d_{q_2} t$$

which can be written as

$$M_n(s,A,\beta : x\bar{t}^s \mid q) = \int_0^\infty t^{A+n\beta-1} e_{q_2}(-t)\theta_n(s,A,\beta;x \mid q)\ d_{q_2} t, \qquad \textbf{(5.2.10)}$$

where

$$\theta_n(s,A,\beta\ \ x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{(A+sk+n\beta-1)(A+sk+n\beta)/2-snk}\,\xi_k\,x^k}{[q]_{n-sk}(q_2;q_2)_\infty(1-q_2)}$$

$$.(Aq^{sk+sk\beta};q_2)_\infty .$$

5.3 q-DIFFERENCE EQUATION(θ-form)

The θ-formed q-difference equation for the polynomial $M_n(s,A,\beta;x|q)$ will be derived in this section.

As mentioned in section 3.3 the function

$$y={}_r\phi_s\begin{bmatrix}a_1,..,a_r;q,z\\ b_1,..,b_s;\end{bmatrix},$$

where r≤s,0<|q|<1; if r=s+1 then |z|<1, and for |z|=1, Re (Σb_j-Σa_i)>0, satisfies the θ-formed q-difference equation:

$$\{\theta(\theta+q^{1-b_1}-1)(\theta+q^{1-b_2}-1)......(\theta+q^{1-b_s}-1)+zq^{a_1+..+a_r-b_1...-b_s+s}$$

$$.(\theta+q^{-a_1}-1)(\theta+q^{-a_2}-1)....\ (\theta+q^{-a_r}-1)\}y=0. \qquad \textbf{(5.3.1)}$$

From this equation the corresponding q-difference equation for $M_n(s,A,\beta;x|q)$ may be obtained as follows.

Here, in fact a particular case of $M_n(s,A,\beta;x|q)$ is considered for this purpose as the polynomial $M_n(s,A,\beta;x|q)$ is to be expressed in a generalized q-hypergeometric function form.

In the polynomial

$$M_n(s,A,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}}{[q]_{n-sk}}\xi_k\,x^k, \qquad \textbf{(5.3.2)}$$

set $\beta=1$ then the product $(Aq^{sk+sk\beta},q^{\beta})_{n-sk}$ takes the form $(Aq^{2sk},q)_{n-sk}$ but

$$(Aq^{2sk};q)_{n-sk}=\frac{[Aq^{2sk}]_{\infty}}{[Aq^{n+sk}]_{\infty}}\frac{[A]_{\infty}}{[A]_{\infty}}$$

$$=\frac{[A]_{n+sk}}{[A]_{2sk}}$$

$$=\frac{[A]_{n}[Aq^{n}]_{sk}}{[A]_{2sk}}.$$

Thus,

$$M_n(s,A,1;x\mid q)=\frac{[A]_n}{[q]_n}\sum_{k=0}^{[n/s]}\frac{q^{-sk(sk-1)/2}\,[q^{-n}]_{sk}[Aq^{n}]_{sk}}{[A]_{2sk}}\xi_k\,x^k.$$

Next putting $\xi_k=\frac{q^{sk(sk-1)/2}}{[q]_k}$ and denoting this particular case of $M_n(s,A,\beta;x|q)$ by $G_n^s(A,x\mid q)$, one finds

$$G_n^s(A,x\mid q)=\frac{[A]_n}{[q]_n}\sum_{k=0}^{[n/s]}\frac{[\bar{q}^{n}]_{sk}[Aq^{n}]_{sk}}{[q]_k[A]_{2sk}}x^k. \qquad \textbf{(5.3.3)}$$

As mentioned in section-3.3, Chapter-3, the q-factorial functions $[q^{-n}]_{sk},[Aq^{n}]_{sk}$ *and* $[A]_{2sk}$ can be expressed in other q-factorial functions with sk and 2sk replaced by k and with base q, using the formulae (3.3.7) and (3.3.8).

Thus,

$$[q^{-n}]_{sk} = (q^{-n/s};q)_k (q^{-n/s}w;q)_k \quad (q^{-n/s}w^{s-1};q)_k (q^{(-n+1)/s};q)_k \cdots$$

$$(q^{(-n+1)/s}w;q)_k \cdots (q^{(-n+1)/s}w^{s-1};q)_k (q^{(-n+s-1)/s};q)_k \cdots (q^{(-n+s-1)/s}w,q)_k$$

$$\cdots (q^{(-n+s-1)/s}.w^{s-1};q)_k, \qquad \textbf{(5.3.4)}$$

$$[Aq^n]_{sk} = (A^{1/s}q^{n/s};q)_k (A^{1/s}q^{n/s}\,w,q)_k \cdots (A^{1/s}q^{n/s}\,w^{s-1},q)_k$$

$$.(A^{1/s}q^{(n+1)/s};q)_k (A^{1/s}q^{(n+1)/s}\,w,q)_k \cdots .(A^{1/s}q^{(n+1)/s}w^{s-1};q)_k \cdots$$

$$\cdots (A^{1/s}q^{(n+s-1)/s};q)_k (A^{1/s}q^{(n+s-1)/s}w;q)_k \cdots (A^{1/s}q^{(n+s-1)/s}w^{s-1};q)_k$$

(5.3.5)

and

$$[A]_{2sk} = (A^{1/2s};q)_k (A^{1/2s}\,\sigma;q)_k \cdots (A^{1/2s}\,\sigma^{2s-1};q)_k \,.$$

$$.(A^{1/2s}q^{1/2s},q)_k (A^{1/2s}q^{1/2s}\sigma;q)_k \quad .(A^{1/2s}q^{1/2s}\sigma^{2s-1};q)_k \cdots$$

$$\cdots (A^{1/2s}q^{(2s-1)/2s};q)_k (A^{1/2s}q^{(2s-1)/s}\sigma;q)_k \cdots (A^{1/2s}q^{(2s-1)/s}\sigma^{2s-1},q)_k$$

(5.3.6)

Now, in the light of (5.3.3) to (5.3.6) the general q-difference equation (5.3.1) takes the form:

$$\left\{\theta\left[\prod_{j=0}^{2s-1}\prod_{m=0}^{2s-1}(\theta+\sigma^m q^{(2s-A-j)/2s}-1)\right]+xq^{2s^2+3s-As}.\right.$$

$$\left.\cdot\left[\prod_{p=0}^{s-1}\prod_{v=0}^{s-1}(\theta+w^v q^{(n-p)/s}-1)(\theta+w^v q^{-(A+n+p)/s}-1)\right]\right\}G_n^s(A,x\,|\,q)=0,$$

(5.3.7)

where σ is $(2s)^{th}$ and w is s^{th} roots of unity.

5.4 q-INVERSE SERIES RELATIONS

If a polynomial in x is known in the explicit form then it will be interesting to know its inverse series. With that intention the following results are proved.

The inversion pair

$$\phi(n) = \sum_{k=0}^{n} \frac{q^{-nk}(Aq^{k+k\beta};q^{\beta})_{n-k}}{[q]_{n-k}} \psi(k) \tag{5.4.1}$$

$\Leftrightarrow$

$$\psi(n) = \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k+1)/2+nk} (Aq^{n+(n-1)\beta}, q^{-\beta})_{n-k-1}}{[q]_{n-k}} \cdot$$

$$\cdot (1 - Aq^{k+k\beta})\phi(k) \tag{5.4.2}$$

proved by Gessel and Stanton[1] will be further extended in this section.

In fact, the polynomial

$$M_n(s,A,\beta;x\,|\,q) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk} q^{-snk} (Aq^{sk+sk\beta};q^{\beta})_{n-sk}}{[q]_{n-sk}} \xi_k\, x^k, \tag{5.4.3}$$

is defined through the proposed (in this section) extension of (5.4.1). With a view to invert the polynomial $M_n(s,A,\beta;x|q)$ the series (5.4.2) is extended.

The proposed extension of the above pair (5.4.1) and (5.4.2) is stated here as

Theorem-9. For s=1, 2, 3, ...,

if

$$U(n)=\sum_{k=0}^{[n/s]} a(n,k,s)\ \ V(k) \tag{5.4.4}$$

and

$$V(n)=\sum_{k=0}^{sn} b(n,k,s)\ \ U(k) \tag{5.4.5}$$

then

$$a(n,k,s)=\frac{(-1)^{sk}\, q^{-snk}\,(Aq^{sk+sk\beta},q^{\beta})_{n-sk}}{[q]_{n-sk}} \tag{5.4.6}$$

implies and is implied by

$$b(n,k,s)=\frac{(-1)^{k}\, q^{(sn-k)(sn-k+1)/2+snk}\,(1-Aq^{k+k\beta})}{[q]_{sn-k}}.$$

$$.(Aq^{sn+(sn-1)\beta};q^{-\beta})_{sn-k-1} \tag{5.4.7}$$

and

$$\sum_{k=0}^{n} b(n,k,1)U(k)=0 \text{ if } n\neq sj,\ j\in N. \tag{5.4.8}$$

For s=1 theorem-9 reduces to the pair (5.4.1) and (5.4.2) given above. This theorem is proved in section-5.6, with the help of a lemma which will be proved in section-5.5. It also suggests another general inverse series relation, which is stated here as:

Theorem-10. For s=1, 2, 3...,

if

$$A(n)=\sum_{k=0}^{[n/s]} c(n,k,s)B(k) \tag{5.4.9}$$

and

$$B(n)=\sum_{k=0}^{sn} d(n,k,s)A(k) \tag{5.4.10}$$

then

$$c(n,k,s)=\frac{(-1)^{n-sk}q^{(n-sk)(n-sk+1)/2+snk}(1-Aq^{sk+sk\beta})}{[q]_{n-sk}}.$$

$$.(Aq^{n+sk\beta+\beta};q^{\beta})_{n-sk-1} \tag{5.4.11}$$

if and only if

$$d(n,k,s)=\frac{q^{-snk}(Aq^{k+k\beta};q^{\beta})_{sn-k}}{[q]_{sn-k}} \tag{5.4.12}$$

and

$$\sum_{k=0}^{n} d(n,k,1)A(k)=0, \text{ if } n\neq sj,\ j\in N. \tag{5.4.13}$$

The proof of this theorem is given in section 5.7.

5.5 LEMMA-2

For n=0, 1, 2,... (Gould and Hsu [1])

$$\phi(n)=\sum_{k=0}^{n} q^{k(k+1)/2-nk}\left\{\prod_{i=1}^{n}(a_i+q^k b_i)\right\}\frac{\psi(k)}{[q]_{n-k}} \tag{5.5.1}$$

$\Leftrightarrow$

$$\psi(n)=\sum_{k=0}^{n}(-1)^{n-k}q^{k(k-1)/2}\ \frac{(a_{k+1}+q^k b_{k+1})\phi(k)}{\prod_{i=1}^{k+1}(a_i+q^n b_i)[q]_{n-k}}. \tag{5.5.2}$$

Proof

The proof of this lemma uses the orthogonality relation

$$\sum_{k=0}^{N}(-1)^k\begin{bmatrix}N\\k\end{bmatrix}q^{k(k-1)/2}\frac{(a_{k+j+1}+q^{k+j}b_{k+j+1})\prod_{i=1}^{k+j}(a_i+q^jb_i)}{\prod_{i=1}^{k+j+1}(a_i+q^nb_i)}=\begin{bmatrix}O\\N\end{bmatrix} \quad \textbf{(5.5.3)}$$

where N=n-j.

In order to prove the first part, it should be proved that (5.5.1)⇒(5.5.2).

Denoting the right hand side of (5.5.2) by T(n) one gets,

$$T(n)=\sum_{k=0}^{n}(-1)^{n-k}q^{k(k-1)/2}\frac{(a_{k+1}+q^kb_{k+1})\phi(k)}{\prod_{i=1}^{k+1}(a_i+q^nb_i)\ [q]_{n-k}}.$$

This in the light of (5.5.1) gets reduced to

$$T(n)=\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(-1)^{n-k}q^{k(k-1)/2+j(j+1)/2-kj}\ (a_{k+1}+q^kb_{k+1})}{\prod_{i=1}^{k+1}(a_i+q^nb_i)}.$$

$$\cdot\frac{\prod_{i=1}^{k}(a_i+q^jb_i)\ \ \psi(j)}{[q]_{n-k}\ [q]_{k-j}}.$$

Now using the double series relation:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}A(k,j)=\sum_{j=0}^{n}\sum_{k=0}^{n-j}A(k+j,j),$$

T(n) becomes

$$T(n)=\sum_{j=0}^{n}\sum_{k=0}^{n-j}\frac{(-1)^{n-j-k}q^{(k+j)(k+j-1)/2+j(j+1)/2-j(k+j)}}{[q]_{n-j-k}\,[q]_k}\cdot$$

$$\cdot\frac{(a_{k+j+1}+q^{k+j}\,b_{k+j+1})\prod_{i=1}^{k+j}(a_i+q^{j}b_i)\ \psi(j)}{\prod_{i=1}^{k+j+1}(a_i+q^{n}\,b_i)}$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{n-j}\frac{(-1)^{n-j-k}q^{k(k-1)/2}(a_{k+j+1}+q^{k+j}b_{k+j+1})}{[q]_{n-j-k}[q]_k}\cdot$$

$$\frac{\prod_{i=1}^{k+j}(a_i+q^{j}b_i)}{\prod_{i=1}^{k+j+1}(a_i+q^{n}b_i)}\psi(j)$$

$$=\psi(n)+\sum_{j=0}^{n-1}\frac{(-1)^{n-j}\psi(j)}{[q]_{n-j}}\sum_{k=0}^{n-j}(-1)^{k}\begin{bmatrix}n-j\\k\end{bmatrix}_q q^{k(k-1)/2}\cdot$$

$$\cdot\frac{(a_{k+j+1}+q^{k+j}\,b_{k+j+1})\prod_{i=1}^{k+j}(a_i+q^{j}b_i)}{\prod_{i=1}^{k+j+1}(a_i+q^{n}b_i)}$$

$$=\psi(n)+\sum_{j=0}^{n-1}\frac{(-1)^{n-j}\psi(j)}{[q]_{n-j}}\begin{bmatrix}o\\n-j\end{bmatrix},$$

in view of (5.5.3).

Therefore, $T(n)=\psi(n)$, which proves that (5.5.1)⇒(5.5.2).

To prove the converse part that (5.5.2)⇒(5.5.1), it is just sufficient to show that the diagonal elements of the coefficient matrix of each of (5.5.1) and (5.5.2) are non-zero.

Let the diagonal elements of these coefficient matrices be ϕ_{nn} and ψ_{nn} respectively.

Then

$$\phi_{nn} = q^{(n(n+1)/2)-n^2} \prod_{i=1}^{n}(a_i + q^n b_i)$$

$$= q^{(-n^2/2)+n/2} \prod_{i=1}^{n}(a_i + q^n b_i),$$

$$\psi_{nn} = q^{n(n-1)/2} \frac{a_{n+1} + q^n b_{n+1}}{\prod_{i=1}^{n+1} (a_i + q^n b_i)}$$

$$= q^{n(n-1)/2} \frac{1}{\prod_{i=1}^{n} (a_i + q^n b_i)}.$$

Clearly $\phi_{nn} \neq 0$ and $\psi_{nn} \neq 0$, which implies that the inverse is unique. Therefore (5.5.2)⇒(5.5.1). Thus (5.5.1)⇔(5.5.2), which completes the proof of Lemma-2.

5.6 PROOF OF THER

OREM-9

With a view to prove the first part, that is (5.4.6)⇒(5.4.7) and (5.4.6)⇒(5.4.8), let the right hand side of (5.4.5) be denoted by f_n, then

$$f_n = \sum_{k=0}^{sn} \frac{(-1)^k q^{(sn-k)(sn-k+1)/2+snk} (1 - Aq^{k+k\beta})}{[q]_{sn-k}} .$$

$$. (Aq^{sn+(sn-1)\beta}; q^{-\beta})_{sn-k-1} \cup(k)$$

which in view of (5.4.4) takes the form

$$f_n = \sum_{k=0}^{sn} \sum_{r=0}^{[k/s]} \frac{(-1)^{k+sr} q^{(sn-k)(sn-k+1)/2+(sn-sr)k}(1-Aq^{k+k\beta})}{[q]_{sn-k}} .$$

$$\cdot \frac{(Aq^{sn+(sn-1)\beta};q^{-\beta})_{sn-k-1}(Aq^{sr+sr\beta};q^{\beta})_{k-sr}}{[q]_{k-sr}} V(r).$$

Next using the double series

$$\sum_{k=0}^{sn} \sum_{r=0}^{[k/s]} A(k,r) = \sum_{r=0}^{n} \sum_{k=0}^{sn-sr} A(k+sr,r)$$

it becomes

$$f_n = \sum_{r=0}^{n} \sum_{k=0}^{sn-sr} \frac{(-1)^k q^{(sn-sr-k)(sn-sr-k+1)/2+(sn-sr)(k+sr)}}{[q]_{sn-sr-k}} V(r) \cdot$$

$$\cdot \frac{(1-Aq^{(k+sr)(\beta+1)})(Aq^{sn+(sn-1)\beta};q^{-\beta})_{sn-sr-k-1}\ (Aq^{sr+sr\beta},q^{\beta})_k}{[q]_k}$$

Taking n-r=N, sn-sr=sN, one gets, using the relation,

$$(Aq^{sn+(sn-1)\beta},q^{-\beta})_{sN-k-1}\frac{(1-Aq^{sn+sn\beta})}{(1-Aq^{sn+sn\beta})} = \frac{(Aq^{sn+sn\beta},q^{-\beta})_{sN-k}}{(1-Aq^{sn+sn\beta})},$$

$$f_n = \sum_{r=0}^{n} \sum_{k=0}^{sn} \frac{(-1)^k q^{(sN-k)(sN-k+1)/2+sN(k+sr)}(1-Aq^{(k+sr)(\beta+1)})}{[q]_{sN-k}} .$$

$$\cdot \frac{(Aq^{sn+sn\beta};q^{-\beta})_{sN-k}\ (Aq^{sr+sr\beta};q^{\beta})_k}{(1-Aq^{sn+sn\beta})[q]_k} V(r)$$

$$= V(n) + \sum_{r=0}^{n-1} \frac{V(r)}{(1-Aq^{sn+sn\beta})[q]_{sN}} \sum_{k=0}^{sN} (-1)^k q^{(sN-k)(sN-k+1)/2+sN(k+sr)} .$$

$$\begin{bmatrix} sN \\ k \end{bmatrix} (1-Aq^{(k+sr)(\beta+1)})(Aq^{sn+sn\beta};q^{-\beta})_{sN-k}(Aq^{sr+sr\beta};q^{\beta})_k .$$

But,

$$\frac{(Aq^{sn+sn\beta};q^{-\beta})_{sN-k}}{(1-Aq^{sn+sn\beta})}=(Aq^{sn+sn\beta-\beta};q^{-\beta})_{sN-k-1}.$$

The q-factorial function on the right hand side can also be written with base q^{β}, thus,

$$(Aq^{sn+(sn-1)\beta},q^{-\beta})_{sN-k-1}=(Aq^{sn+(sr+k+1)\beta};q^{\beta})_{sN-k-1}.$$

Therefore,

$$f_n=V(n)+\sum_{r=0}^{n-1}\frac{q^{sN(sN+1)/2+sNsr}V(r)}{[q]_{sN}}\sum_{k=0}^{sN}(-1)^k q^{k(k-1)/2}\begin{bmatrix}sN\\k\end{bmatrix}.$$

$$(1-Aq^{(k+sr)(\beta+1)})(Aq^{sn+(sr+k+1)\beta};q^{\beta})_{sN-k-1}(Aq^{sr+sr\beta};q^{\beta})_k$$

$$=V(n)+\sum_{r=0}^{n-1}\frac{q^{sN(sN+1)/2+sNsr}(Aq^{sn+sr\beta};q^{\beta})_{sN}V(r)}{[q]_{sN}}.$$

$$\cdot\sum_{k=0}^{sN}(-1)^k q^{k(k-1)/2}\begin{bmatrix}sN\\k\end{bmatrix}\frac{(1-Aq^{(k+sr)(\beta+1)})(Aq^{sr+sr\beta};q^{\beta})_k}{(Aq^{sn+sr\beta};q^{\beta})_{k+1}} \quad \textbf{(5.6.1)}$$

If the inner series here is denoted by $\psi(M)$ where sN=M then it can be obtained from (5.5.2) of Lemma-2. In fact choosing $a_i=1$ for i=1, 2,..., k+1, $b_i=-Aq^{sr(\beta+1)+(i-1)\beta}$, i=1, 2,....,k+1 in Lemma-2 it reduces to

$$U(n)=\sum_{k=0}^{n}\frac{q^{k(k+1)/2-nk}(Aq^{k+(\beta+1)sr};q^{\beta})_n T(k)}{[q]_{n-k}} \quad \textbf{(5.6.2)}$$

if and only if

$$T(n)=\sum_{k=0}^{n}\frac{(-1)^{n-k}q^{k(k-1)/2}(1-Aq^{(k+sr)(\beta+1)})}{(Aq^{n+sr\beta+sr};q^{\beta})_{k+1}}\begin{bmatrix}n\\k\end{bmatrix}\mathrm{U}(k). \qquad \textbf{(5.6.3)}$$

Thus the inner series above in (5.6.1) is actually

$$T(M)=\sum_{k=0}^{M}(-1)^{k}q^{k(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}\frac{(1-Aq^{(k+sr)(\beta+1)})}{(Aq^{M+sr+sr\beta};q^{\beta})_{k+1}}\mathrm{U}(k), \qquad \textbf{(5.6.4)}$$

where $(Aq^{sr+sr\beta},q^{\beta})_k$ is taken as U(k).

The inverse of this follows from (5.6.2) in the form:

$$\mathrm{U}(M)=\sum_{k=0}^{M}q^{k(k+1)/2-Mk}\begin{bmatrix}M\\k\end{bmatrix}(Aq^{k+(\beta+1)sr};q^{\beta})_{M}\,T(k). \qquad \textbf{(5.6.5)}$$

Thus (5.6.4)⇔(5.6.5) in view of the inverse series relation (5.6.2) and (5.6.3).

Now consider $T(k)=\begin{bmatrix}0\\k\end{bmatrix}$ in (5.6.5) then

$$\mathrm{U}(M)=\sum_{k=0}^{M}q^{k(k+1)/2-Mk}\begin{bmatrix}M\\k\end{bmatrix}(Aq^{k+(\beta+1)sr};q^{\beta})_{M}\begin{bmatrix}0\\k\end{bmatrix}$$

$$=(Aq^{sr+sr\beta};q^{\beta})_{M}.$$

Thus with $T(k)=\begin{bmatrix}0\\k\end{bmatrix}$ the choice $\mathrm{U}(k)=(Aq^{sr+sr\beta};q^{\beta})_k$ is restored.

Now with these choices of T(k) and U(k) in (5.6.1), one gets,

$$f_n=V(n)+\sum_{r=0}^{n-1}\frac{q^{sN(sN+1)/2+sNsr}(Aq^{sn+sr\beta};q^{\beta})_{sN}\,V(r)}{[q]_{sN}}\begin{bmatrix}0\\sN\end{bmatrix}$$

thus, f(n)=V(n), which proves that (5.4.6)⇒(5.4.7).

In order to prove that (5.4.6) also implies (5.4.8) consider the left member of (5.4.8) and denote it by A(n), then

$$A(n)=\sum_{k=0}^{n}\frac{(-1)^k q^{(n-k)(n-k+1)/2+nk}(1-Aq^{k+k\beta})}{[q]_{n-k}}.$$

$$(Aq^{n+(n-1)\beta},q^{-\beta})_{n-k-1}\,U(k) \qquad \textbf{(5.6.6)}$$

but

$$(Aq^{n+n\beta-\beta};q^{-\beta})_{n-k-1}=(Aq^{n+k\beta+\beta};q^{\beta})_{n-k-1},$$

thus

$$A(n)=\sum_{k=0}^{n}\frac{(-1)^k q^{(n-k)(n-k+1)/2+nk}(1-Aq^{k+k\beta})}{[q]_{n-k}}.$$

$$.(Aq^{n+k\beta+\beta},q^{\beta})_{n-k-1}\,U(k).$$

This in view of (5.4.4) takes the form:

$$A(n)=\sum_{k=0}^{n}\sum_{j=0}^{[k/s]}\frac{(-1)^{k+sj}q^{(n-k)(n-k+1)/2+(n-sj)k}(1-Aq^{k+k\beta})}{[q]_{n-k}\,[q]_{k-sj}}.$$

$$.(Aq^{n+k\beta+\beta};q^{\beta})_{n-k-1}\,(Aq^{sj+sj\beta};q^{\beta})_{k-sj}\,V(j).$$

Using the double series relation

$$\sum_{k=0}^{n}\sum_{j=0}^{[k/s]}A(k,j)=\sum_{j=0}^{[n/s]}\sum_{k=0}^{n-sj}A(k+sj,j)$$

it becomes

$$A(n)=\sum_{j=0}^{[n/s]}\frac{V(j)}{[q]_{n-sj}}\sum_{k=0}^{n-sj}(-1)^k q^{(n-sj-k)(n-sj-k+1)/2+nk+(n-k)sj}.$$

$$\begin{bmatrix} n-sj \\ k \end{bmatrix}(1-Aq^{(k+sj)(1+\beta)})(Aq^{n+(k+sj+1)\beta};q^{\beta})_{n-sj-k-1}(Aq^{sj+sj\beta};q^{\beta})_k$$

$$=\sum_{j=0}^{[n/s]}\frac{q^{(n-sj)(n-sj+1)/2+nsj}\ V(j)}{[q]_{n-sj}}\sum_{k=0}^{n-sj}(-1)^k q^{k(k-1)/2}\begin{bmatrix}n-sj\\k\end{bmatrix}.$$

$$.(1-Aq^{(k+sj)(1+\beta)})(Aq^{n+(k+sj+1)\beta};q^{\beta})_{n-sj-k-1}(Aq^{sj+sj\beta},q^{\beta})_k.$$

Taking n-sj=R, one obtains,

$$A(n)=\sum_{j=0}^{[n/s]}\frac{q^{R(R+1)/2+nsj}\ V(j)}{[q]_R}\sum_{k=0}^{R}(-1)^k q^{k(k-1)/2}\begin{bmatrix}R\\k\end{bmatrix}(1-Aq^{(k+sj)(\beta+1)}).$$

$$.(Aq^{n+(k+sj+1)\beta};q^{\beta})_{R-k-1}\ (Aq^{sj+sj\beta};q^{\beta})_k,$$

but

$$(Aq^{n+(k+sj+1)\beta};q^{\beta})_{R-k-1}=\frac{(Aq^{n+sj\beta};q^{\beta})_R}{(Aq^{n+sj\beta};q^{\beta})_{k+1}}$$

therefore

$$A(n)=\sum_{j=0}^{[n/s]}\frac{q^{R(R+1)/2+nsj}(Aq^{n+sj\beta};q^{\beta})_R V(j)}{[q]_R}\sum_{k=0}^{R}(-1)^k q^{k(k-1)/2}\begin{bmatrix}R\\k\end{bmatrix}.$$

$$.\frac{(1-Aq^{(k+sj)(1+\beta)})(Aq^{sj+sj\beta};q^{\beta})_k}{(Aq^{R+sj\beta+sj};q^{\beta})_{k+1}}$$

The inner series in k, above is the same as (5.6.3), whose sum

T(n)= $\begin{bmatrix}0\\n\end{bmatrix}$ as seen above.

Thus

$$A(n)=\sum_{j=0}^{[n/s]}\frac{q^{R(R+1)/2+nsj}(Aq^{n+sj\beta};q^{\beta})_R\ V(j)}{[q]_R}\begin{bmatrix}0\\R\end{bmatrix}$$

=0 if R≠0, i.e. if n≠sj, j∈N,

which proves that (5.4.6)⇒(5.4.8), completing the proof of the first part.

To prove the converse part, it is to be proved that (5.4.7) and (5.4.8) together imply (5.4.6). Since (5.4.8) holds true, in view of (5.6.6) above A(n)=0 if n≠sj, j∈N, holds true. Also it is to be noted that A(ns)=A(sn)=V(n).

Thus in view of the inversion pair (5.4.1) and (5.4.2) it is proved that

$$\psi(sn) = \sum_{k=0}^{sn} \frac{(-1)^{sn-k} q^{(sn-k)(sn-k+1)/2+snk} (1-Aq^{k+k\beta})}{[q]_{sn-k}} \cdot$$

$$\cdot (Aq^{sn+(sn-1)\beta}; q^{-\beta})_{sn-k-1}\, \phi(k)$$

⇒

$$\phi(n) = \sum_{k=0}^{[n/s]} \frac{q^{-snk} (Aq^{sk+sk\beta}; q^{\beta})_{n-sk}}{[q]_{n-sk}}\, \psi(k),$$

subject to ψ(n)=0 if n≠sj, j∈N. This in fact proves that (5.4.7)⇒(5.4.6) subject to (5.4.8), which completes the proof of the converse part and that of the theorem.

5.7 PROOF OF THEOREM-10

To prove that (5.4.11) implies (5.4.12) let the right hand side of (5.4.10) be denoted by F(n), then in view of (5.4.12),

$$F(n)=\sum_{k=0}^{sn}\frac{q^{-snk}(Aq^{k+k\beta};q^{\beta})_{sn-k}}{[q]_{sn-k}}A(k)$$

$$=\sum_{k=0}^{sn}\sum_{j=0}^{[k/s]}\frac{(-1)^{k-sj}q^{(k-sj)(k-sj+1)/2+skj-snk}(1-Aq^{sj+sj\beta})}{[q]_{k-sj}\,[q]_{sn-k}}.$$

$$.(Aq^{k+k\beta},q^{\beta})_{sn-k}(Aq^{k+sj\beta+\beta};q^{\beta})_{k-sj-1}\,B(j).$$

Using the double series

$$\sum_{k=0}^{sn}\sum_{j=0}^{[k/s]}A(k,j)=\sum_{j=0}^{n}\sum_{k=0}^{sn-sj}A(k+sj,j)$$

one arrives at,

$$F(n)=\sum_{j=0}^{n}\sum_{k=0}^{sn-sj}\frac{(-1)^{k}q^{k(k+1)/2+sj(k+sj)-sn(k+sj)}(1-Aq^{sj+sj\beta})}{[q]_{k}\,[q]_{sn-sj-k}}.$$

$$.(Aq^{k+sj+k\beta+sj\beta};q^{\beta})_{sn-sj-k}\,(Aq^{k+sj+sj\beta+\beta};q^{\beta})_{k-1}\,B(j)$$

but

$$(Aq^{k+sj+(sj+1)\beta};q^{\beta})_{k-1}\,\frac{(1-Aq^{k+sj+sj\beta})}{(1-Aq^{k+sj+sj\beta})}=\frac{(Aq^{k+sj+sj\beta};q^{\beta})_{k}}{(1-Aq^{k+sj+sj\beta})}$$

therefore

$$F(n)=\sum_{j=0}^{n}\sum_{k=0}^{sn-sj}\frac{(-1)^{k}q^{k(k+1)/2-(sn-sj)(k+sj)}(1-Aq^{sj+sj\beta})}{[q]_{k}\,[q]_{sn-sj-k}}.$$

$$\frac{(Aq^{k+sj+k\beta+sj\beta};q^{\beta})_{sn-sj-k}(Aq^{k+sj+sj\beta};q^{\beta})_{k}}{(1-Aq^{k+sj+sj\beta})}\,B(j)$$

$$= B(n) + \sum_{j=0}^{n-1} \sum_{k=0}^{sn-sj} \frac{(-1)^k q^{k(k+1)/2-(sn-sj)(k+sj)} (1-Aq^{sj+sj\beta})}{[q]_k [q]_{sn-sj-k}}.$$

$$.(Aq^{k+sj+k\beta+sj\beta};q^\beta)_{sn-sj-k} (Aq^{k+sj+sj\beta+\beta};q^\beta)_{k-1} B(j).$$

Taking n-j=N (⇒sn-sj=sN), one gets

$$F(n) = B(n) + \sum_{j=0}^{n-1} \frac{q^{-Njs^2} (1-Aq^{sj+sj\beta}) B(j)}{[q]_{sN}} \sum_{k=0}^{sn} (-1)^k q^{k(k-1)/2} \begin{bmatrix} sN \\ k \end{bmatrix}$$

$$.(Aq^{(k+sj)(\beta+1)};q^\beta)_{sN-k} (Aq^{k+sj+sj\beta+\beta};q^\beta)_{k-1}\ q^{(1-sN)k}$$

but

$$(Aq^{k+sj+sj\beta+\beta};q^\beta)_{k-1} (Aq^{(k+sj)(\beta+1)};q^\beta)_{sN-k}$$

$$= (Aq^{k+sj+(sj+1)\beta};q^\beta)_{sN-1}$$

$$= \sum_{r=0}^{sN-1} C_r q^{kr}.$$

Therefore,

$$F(n) = B(n) + \sum_{j=0}^{n-1} \frac{q^{-Njs^2} (1-Aq^{sj+sj\beta}) B(j)}{[q]_{sN}} \sum_{k=0}^{sN} (-1)^k q^{k(k-1)/2} \begin{bmatrix} sN \\ k \end{bmatrix}.$$

$$.q^{(1-sN)k} \sum_{r=0}^{sN-1} C_r q^{kr}$$

$$= B(n) + \sum_{j=0}^{n-1} \frac{q^{-Njs^2} (1-Aq^{sj+sj\beta}) B(j)}{[q]_{sN}} \sum_{r=0}^{sN-1} C_r.$$

$$\sum_{k=0}^{sN} (-1)^k q^{k(k-1)/2} \begin{bmatrix} sN \\ k \end{bmatrix} q^{-(sN-r-1)k}.$$

Here, the inner series in k vanishes because

$$\sum_{k=0}^{sN} (-1)^k q^{k(k-1)/2} \begin{bmatrix} sN \\ k \end{bmatrix} q^{-(sN-r-1)k}$$

$$=(1-q^{1+r-sN})(1-q^{1+r-sN+1}).........(1-q^r)$$

$$=0,$$

for r=0,1,2,......,sN-1. Hence, F(n)=B(n), which proves that (5.4.11)⇒(5.4.12).

To complete the proof of the first part it is left to prove that (5.4.11)⇒(5.4.13).

Consider the left hand side of (5.4.13), and denote it by G(n), then

$$G(n)=\sum_{k=0}^{n} \frac{q^{-nk}(Aq^{k+k\beta};q^{\beta})_{n-k}A(k)}{[q]_{n-k}} \tag{5.7.1}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{[k/s]} \frac{(-1)^{k-sj} q^{(k-sj)(k-sj+1)/2+skj-nk}(1-Aq^{sj+sj\beta})B(j)}{[q]_{k-sj}[q]_{n-k}}.$$

$$.(Aq^{k+sj\beta+\beta};q^{\beta})_{k-sj-1}\,(Aq^{k+k\beta},q^{\beta})_{n-k}$$

Using the double series relation

$$\sum_{k=0}^{n}\sum_{j=0}^{[k/s]} A(k,j)=\sum_{j=0}^{[n/s]}\sum_{k=0}^{n-sj} A(k+sj,j),$$

and putting n-sj=N, one gets,

$$G(n)=\sum_{j=0}^{[n/s]} \frac{(1-Aq^{sj+sj\beta})B(j)}{[q]_N} \sum_{k=0}^{N} (-1)^k q^{k(k-1)/2} q^{k-N(k+sj)} \begin{bmatrix} N \\ k \end{bmatrix}$$

$$.(Aq^{k+sj+sj\beta+\beta};q^{\beta})_{k-1}(Aq^{k+sj+k\beta+sj\beta};q^{\beta})_{N-k}$$

Here,

$$(Aq^{k+sj+sj\beta+\beta};q^{\beta})_{k-1}(Aq^{k+sj+k\beta+sj\beta};q^{\beta})_{N-k}$$

$$=(Aq^{k+sj+sj\beta+\beta};q^{\beta})_{N-1}$$

$$=\sum_{r=0}^{N-1} C_r q^{kr}.$$

Therefore,

$$G(n)=\sum_{j=0}^{[n/s]}\frac{(1-Aq^{sj+sj\beta})B(j)}{[q]_N}\sum_{k=0}^{N}(-1)^k q^{k(k-1)/2}\begin{bmatrix}N\\k\end{bmatrix}.$$

$$.\left\{\sum_{r=0}^{N-1}C_r\, q^{kr}\right\}q^{k-N(k+sj)}.$$

$$=\sum_{j=0}^{[n/s]}\frac{q^{-Nsj}(1-Aq^{sj+sj\beta})\,B(j)}{[q]_N}\sum_{r=0}^{N-1}C_r\sum_{k=0}^{N}(-1)^k.q^{k(k-1)/2}\begin{bmatrix}N\\k\end{bmatrix}q^{-(N-r-1)k}.$$

Here, the inner series in k vanishes because

$$\sum_{k=0}^{N}(-1)^k q^{k(k-1)/2}\begin{bmatrix}N\\k\end{bmatrix}q^{-(N-r-1)k}$$

$$=(1-q^{1+r-N})(1-q^{2+r-N})\ldots\ldots\ldots(1-q^{r})$$

$$=0,$$

for $r=0,1,2,\ldots\ldots,N-1$. Therefore $G(n)=0$, if $N=n-sj\neq 0$, i.e. $n\neq sj$, $s=0,1,2,\ldots\ldots$. This proves that (5.4.11)⇒(5.4.13), which completes the proof of the first part.

For proving the converse part, it will be proved that (5.4.12) and (5.4.13) together imply (5.4.11).

This means that

$$B(n)=\sum_{k=0}^{sn}\frac{q^{-snk}(Aq^{k+k\beta},q^{\beta})_{sn-k}A(k)}{[q]_{sn-k}}$$

and

$$\sum_{k=0}^{n}\frac{q^{-nk}(Aq^{k+k\beta};q^{\beta})_{n-k}A(k)}{[q]_{n-k}}=0,\ \text{if } n\neq sj,\ j=0,1,2,\ldots\ldots$$

hold true.

This in the notation of (5.7.1), means that

G(n)=0 if n≠sj, j=0,1,2,...... **(5.7.2)**

and

G(sn) = G(ns) = B(n). **(5.7.3)**

In fact, in view of the pair of inverse series relations (5.4.1) and (5.4.2), we have

$$G(n)=\sum_{k=0}^{n}\frac{q^{-nk}(Aq^{k+k\beta},q^{\beta})_{n-k}A(k)}{[q]_{n-k}}$$

⇒

$$A(n)=\sum_{k=0}^{n}\frac{(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}(1-Aq^{k+k\beta})(Aq^{n+k\beta+\beta})_{n-k-1}G(k)}{[q]_{n-k}}.$$

This proves that (5.7.2)⇒(5.4.11).

But in view of (5.7.2) and (5.7.3) above, one has

$$B(n)=\sum_{k=0}^{sn}\frac{q^{-nsk}(Aq^{k+k\beta};q^{\beta})_{sn-k}A(k)}{[q]_{sn-k}}$$

implies

$$A(n)=\sum_{sk=0}^{n}\frac{(-1)^{n-sk}q^{(n-sk)(n-sk+1)/2+snk}(1-Aq^{sk+sk\beta})}{[q]_{n-sk}}.$$

$$.(Aq^{n+sk\beta+\beta};q^{\beta})_{n-sk-1}G(sk)$$

which proves in fact that (5.4.12)⇒(5.4.11), subject to the condition (5.4.13).

This completes the proof of the converse, and hence that of the theorem.

5.8 SPECIAL CASES

The polynomial special cases (in extended form) of the general class of polynomials $M_n(s,A,\beta;x|q)$ defined in (5.1.2) are given in this section where the corresponding particular values of the parameters are also accompanying. Also the properties of this general polynomial, studied in sections 5.2 to 5.7 are particularized.

To begin with, first the extended versions of the various known polynomials occurring as the special cases of

$$M_n(s,A,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q^{\beta})_{n-sk}}{[q]_{n-sk}}\xi_k\,x^k \qquad \textbf{(5.8.1)}$$

are described together with the specializations. The extensions of the Askey-Wilson polynomial, basic Hahn polynomial, q-Racah polynomial, little q-Jacobi polynomial, q-Legendre polynomial, q-Konhauser polynomial, q-Laguerre polynomial, Wall polynomial and the Stieltjes-

Wigert polynomial can be obtained from a slightly modified version of (5.8.1) by taking β=1.

Thus taking β=1 and $M_n^*(s,A;x\,|\,q)=\dfrac{M_n(s,A,1;x\,|\,q)}{[Aq^n]_\infty}$ in (5.8.1), after a little simplification it takes the form:

$$M_n^*(s,A;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{-sk(sk+1)/2}[q^{-n}]_{sk}[Aq^n]_\infty[Aq^{2sk}]_\infty\,\xi_k\,x^k}{[q]_n[Aq^{n+sk}]_\infty}$$

but

$\dfrac{[Aq^n]_\infty}{[Aq^{n+sk}]_\infty}=[Aq^n]_{sk}$, and denoting $[Aq^{2sk}]_\infty\,\xi_k$ by the symbol σ_k one arrives at

$$M_n^*(s,A;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{-sk(sk+1)/2}\,[q^{-n}]_{sk}\,[Aq^n]_{sk}\,\sigma_k\,x^k}{[q]_n}. \tag{5.8.2}$$

Now, specializing A and σ_k suitably the polynomial special cases are obtained as follows.

Putting x=1, $A=abcdq^{-1}$ and

$$\sigma_k=\frac{q^{sk(sk+1)/2+k}\,[ae^{-i\theta}]_k\,[ae^{i\theta}]_k}{[ab]_k[ac]_k[ad]_k[q]_k},$$

in (5.8.2) one gets, an extension of the Askey-Wilson polynomial; denoted here by $P_{n,s}(x;a,b,c,d|q)$:

$$\frac{P_{n,s}(x;a,b,c,d\,|\,q)}{[ab]_n[ac]_n[ad]_n a^{-n}}=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,[abcdq^{n-1}]_{sk}\,[ae^{-i\theta}]_k\,[ae^{i\theta}]_k\,q^k}{[q]_k[ab]_k[ac]_k[ad]_k}. \tag{5.8.3}$$

An extension of the q-Racah polynomial with the notation $R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;x)$, where $\mu(x)=q^{-x}+\gamma\delta q^{x+1}$, can be obtained from (5.8.2) by taking x=1, $A=\alpha\beta q$ and

$$\sigma_k = \frac{q^{sk(sk+1)/2+k}[q^{-x}]_k[\gamma\delta q^{x+1}]_k}{[\alpha q]_k[\beta\delta q]_k[\gamma q]_k[q]_k}.$$

Thus,

$$\frac{R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q)}{[q]_n} = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}[q^{-x}]_k}{[q]_n[\alpha q]_k[\beta\delta q]_k} \frac{[\gamma\delta q^{x+1}]_k q^k}{[\gamma q]_k[q]_k},$$

therefore

$$R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}[q^{-x}]_k [\gamma\delta q^{x+1}]_k q^k}{[\alpha q]_k[\beta\delta q]_k[\gamma q]_k[q]_k}. \quad \textbf{(5.8.4)}$$

Setting x=1, $A=\alpha\beta q$ and $\sigma_k = \frac{q^{(sk(sk+1)/2)+k}[q^{-x}]_k}{[\alpha q]_k[q^{-N}]_k[q]_k}$, (5.8.2) yields an extension of basic Hahn polynomial, which is denoted here by $Q_{n,s}(x;\alpha,\beta,N|q)$.

Therefore,

$$\frac{Q_{n,s}(x;\alpha,\beta,N|q)}{[q]_n} = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}[q^{-x}]_k q^k}{[q]_n[\alpha q]_k[q^{-N}]_k[q]_k}.$$

or

$$Q_{n,s}(x;\alpha,\beta,N|q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}[q^{-x}]_k q^k}{[\alpha q]_k[q^{-N}]_k[q]_k}. \quad \textbf{(5.8.5)}$$

For obtaining an extended little q-Jacobi polynomial the same choice for 'A' as above is required, together with

$$\sigma_k = \frac{q^{(sk(sk+1)/2)+k}}{[\alpha q]_k [q]_k}.$$

It is expressed using the symbol $p_{n,s}(x;\alpha,\beta;q)$.

Thus

$$\frac{p_{n,s}(x;\alpha,\beta;q)}{[q]_n} = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}(xq)^k}{[q]_n[\alpha q]_k[q]_k},$$

therefore

$$\frac{p_{n,s}(x;\alpha,\beta;q)}{[q]_n} = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[\alpha\beta q^{n+1}]_{sk}(xq)^k}{[\alpha q]_k[q]_k}. \quad \textbf{(5.8.6)}$$

An extension of the q-Legendre polynomial denoted here by $P_{n,s}(x|q)$ can be obtained very easily from (5.8.6) just by taking $q^{\alpha}=q^{\beta}=1$, which is:

$$P_{n,s}(x|q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[q^{n+1}]_{sk}(xq)^k}{[q]_k[q]_k}. \quad \textbf{(5.8.7)}$$

The q-Konhauser polynomial $Z_n^{(\alpha)}(x;k|q)$ is also a special case (s=1) of (5.8.2). An extension denoted by $Z_{n,s}^{(\alpha)}(x;k|q)$, of this polynomial is obtained by setting

$A(=q^A)=0$, $0<q<1$, $\sigma_j = \frac{q^{ksj(sj+1)/2+kj(\alpha+1)+kj(kj-1)/2}}{(q^k;q^k)_{sj}[\alpha q]_{kj}}$.

Also q is to be replaced by q^k and x by $x^k q^{nk}$, $k \in N$.

Then one obtains,

$$Z_{n,s}^{(\alpha)}(x,k\,|\,q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{[n/s]}\frac{q^{kj(n+\alpha+1)+kj(kj-1)/2}(q^{-nk};q^k)_{sj}}{(q^k;q^k)_{sj}[\alpha q]_{kj}\,x^{-kj}}$$

Thus,

$$Z_{n,s}^{(\alpha)}(x;k\,|\,q)=\frac{[\alpha q]_n}{(q^k;q^k)_n}\sum_{j=0}^{[n/s]}\frac{q^{kj(n+\alpha+1)+kj(kj-1)/2}(q^{-nk},q^k)_{sj}\,x^{kj}}{(q^k;q^k)_{sj}\,[\alpha q]_{kj}}. \quad \textbf{(5.8.8)}$$

Extended q-Laguerre polynomial $L_{n,s}^{(\alpha)}(x\,|\,q)$ is obtained from (5.8.8) just by taking k=1. Therefore,

$$L_{n,s}^{(\alpha)}(x\,|\,q)=\frac{[\alpha q]_n}{[q]_n}\sum_{j=0}^{[n/s]}\frac{q^{j(n+\alpha+1)+j(j-1)/2}[q^{-n}]_{sj}\,x^j}{[q]_{sj}\,[\alpha q]_j} \quad \textbf{(5.8.9)}$$

The above polynomials given in (5.8.3) to (5.8.9) are called extended versions because if s=1, then all of them reduce to their corresponding original forms.

The remaining part of this section is divided into three parts according to the properties discussed in sections 5.2 to 5.7. They are:

(I) Special cases of q-integral representations.

(II) Special cases of q-difference equation.

(III) Special cases of q-inverse series relations.

The part (III) is further divided as follows.

- (III-a) Extensions of inversion pairs of Gessel and Stanton.

- (III-b) Extensions of pairs of inverse series relations of polynomial special cases.

The polynomials occurring as special cases of the general polynomial $M_n(s,A,\beta;x|q)$ are described above, mentioning the particular values of the parameters. Taking into consideration these same particular values, the properties derived for the polynomial $M_n(s,A,\beta;x|q)$ can be particularized for the special polynomials. To avoid repeatation these particular values are not described again in (I), (II) and (III).

(I) Special cases of q-integral representations

The integral representations of the above mentioned polynomials in view of (5.2.7) are listed below.

$$\frac{P_{n,s}(x;a,b,c,d\,|\,q)}{[ab]_n[ac]_n[ad]_n a^{-n}} = \int_0^1 t^{a+b+c+d+n-2}[tq]_\infty \rho_n(s,a,b,c,d;x,t\,|\,q)d_q t,$$

where

$$\rho_n(s,a,b,c,d;x,t\,|\,q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}[ae^{-i\theta}]_k[ae^{i\theta}]_k}{[q]_n \Gamma_q(a+b+c+d+sk-1)[ab]_k[ac]_k[ad]_k}.$$

$$\cdot\frac{\Gamma_q(2(a+b+c+d-1)+2sk+n)(1-q)^{n-sk}q^k t^{sk}[abcdq^{n-1}]_\infty}{\Gamma_q(a+b+c+d+2sk-1)[tabcdq^{sk-1}]_\infty[abcdq^{2sk-1}]_\infty[q]_k},$$

and Re(a+b+c+d+n-1)>0, a+b+c+d+sk-1≠0, -1, -2,......

$$R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q) = \int_0^1 t^{\alpha+\beta+n}[tq]_\infty \sigma_n(s,\alpha,\beta,x,t\,|\,q)d_q t,$$

where

$$\sigma_n(s,\alpha,\beta,x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[q^{-x}]_k[\gamma\delta q^{x+1}]_k[\alpha\beta q^{n+1}]_\infty}{[\alpha\beta q^{2sk+1}]_\infty[\alpha q]_k[\beta\delta q]_k[\gamma q]_k[q]_k}.$$

$$\cdot\frac{\Gamma_q(2\alpha+2\beta+2sk+n+2)(1-q)^{n-sk}q^k t^{sk}}{\Gamma_q(\alpha+\beta+2sk+1)\Gamma_q(\alpha+\beta+sk+1)[t\alpha\beta q^{sk+1}]_\infty},$$

and Re(α+β+n+1)>0, α+β+sk+1≠0, -1, -2,……, $\mu(x)=q^{-x}+\gamma\delta q^{x+1}$.

$$Q_{n,s}(x;\alpha,\beta,N\,|\,q)=\int_0^1 t^{\alpha+\beta+n}[tq]_\infty\lambda_n(s,\alpha,\beta;x,t\,|\,q)d_qt,$$

where

$$\lambda_n(s,\alpha,\beta;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[q^{-x}]_k\Gamma_q(2\alpha+2\beta+2sk+n+2)}{[q]_k[\alpha q]_k[q^{-N}]_k[tq^{A+sk}]_\infty}.$$

$$\cdot\frac{(1-q)^{n-sk}t^{sk}}{\Gamma_q(\alpha+\beta+2sk+1)\Gamma_q(\alpha+\beta+sk+1)},$$

and Re(α+β+n+1)>0, α+β+sk+1≠0, -1, -2,……,

$$p_{n,s}(x;\alpha,\beta;q)=\int_0^1 t^{\alpha+\beta+n}[tq]_\infty\gamma_n(s,\alpha,\beta;x,t\,|\,q)d_qt,$$

where

$$\gamma_n(s,\alpha,\beta;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\Gamma_q(2\alpha+2\beta+2sk+n+2)(1-q)^{n-sk}}{\Gamma_q(\alpha+\beta+2sk+1)\,\Gamma_q(\alpha+\beta+sk+1)[t\alpha\beta q^{sk+1}]_\infty}.$$

$$\cdot\frac{(xq)^k t^{sk}}{[\alpha\beta q^{2sk+1}]_\infty[\alpha q]_k[q]_k},$$

and Re(α+β+n+1)>0, α+β+sk+1≠0, -1, -2,.......

$$P_{n,s}(x\,|\,q)=\int_0^1 t^n\,[tq]_\infty\,\Delta_n(s,x,t\,|\,q)\,d_q t,$$

where

$$\Delta_n(s;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,\Gamma_q(2sk+n+2)(1-q)^{n-sk}\,(xq)^k\,t^{sk}}{\Gamma_q(2sk+1)\Gamma_q(sk+1)[tq^{sk+1}]_\infty\,[q^{2sk+1}]_\infty\,[q]_k\,[q]_k}.$$

When the parameters are specialized suitably, the integral (5.2.8) gets particularized as given in the list below.

$$\frac{P_{n,s}(x;a,b,c,d\,|\,q)}{[ab]_n\,[ac]_n\,[ad]_n\,[q]_n\,a^{-n}}=\int_{-1}^{1}[-tq]_\infty\,[tq]_\infty\,\rho_n(s,a,b,c,d;x,t\,|\,q)\,d_q t,$$

where

$$\rho_n(s;a,b,c,d,x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,[ae^{-i\theta}]_k\,[ae^{i\theta}]_k\,[abcdq^{n+1}]_\infty\,q^k}{[abcdq^{2sk-1}]_\infty\,[ab]_k\,[ac]_k\,[ad]_k\,[q]_k\,[q]_n}.$$

$$\frac{(2abcdq^{2sk-2},q^2)_\infty\,[-abcdq^{sk+n-1}]_\infty\,(1-q)^{a+b+c+d+n-2}}{[-1]_\infty(q^2;q^2)_\infty\,[-tabcdq^{sk+n-1}]_\infty\,[-tabcdq^{sk-1}]_\infty}.$$

$$\cdot\frac{\Gamma_q(2(a+b+c+d)+2sk+n-2)}{\Gamma_q(a+b+c+d+2sk-1)}.$$

$$R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q)=\int_{-1}^{1}[-tq]_\infty\,[tq]_\infty\,\lambda_n(s,\alpha,\beta;x,t\,|\,q)\,d_q t,$$

where

$$\lambda_n(s,\alpha,\beta;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,[q^{-x}]_k\,[\gamma\delta q^{x+1}]_k\,(1-q)^{\alpha+\beta+n}}{[\alpha\beta q^{2sk+1}]_\infty\,[\alpha q]_k\,[\beta\delta q]_k\,[\gamma q]_k\,[q]_k}.$$

$$\cdot\frac{\Gamma_q(2\alpha+2\beta+2sk+n+2)(2\alpha\beta q^{2sk+2};q^2)_\infty\,[-\alpha\beta q^{sk+n+1}]_\infty\,q^k}{\Gamma_q(\alpha+\beta+2sk+1)\,[-1]_\infty\,(q^2;q^2)_\infty\,[-t\alpha\beta q^{sk+n+1}]_\infty\,[-t\alpha\beta q^{sk+1}]_\infty}.$$

$$Q_{n,s}(x;\alpha,\beta,N\,|\,q)=\int_{-1}^{1}[-tq]_{\infty}[tq]_{\infty}\,\varepsilon_n(s,\alpha,\beta;x,t\,|\,q)\,d_q t,$$

where

$$\varepsilon_n(s,\alpha,\beta;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}[q^{-x}]_k(1-q)^{\alpha+\beta+n}}{[q]_n[\alpha q]_k[q^{-N}]_k[q]_k[\alpha\beta q^{2sk+1}]_{\infty}[-t\alpha\beta q^{sk+n+1}]_{\infty}}\cdot$$

$$\cdot\frac{\Gamma_q(2\alpha+2\beta+2sk+n+2)(2\alpha\beta q^{2sk+2};q^2)_{\infty}[-\alpha\beta q^{sk+n+1}]_{\infty}[\alpha\beta q^{n+1}]_{\infty}q^k}{[-t\alpha\beta q^{sk+1}]_{\infty}\Gamma_q(\alpha+\beta+2sk+1)[-1]_{\infty}[\gamma]_{\infty}(q^2;q^2)_{\infty}}.$$

$$p_{n,s}(x;\alpha,\beta;q)=\int_{-1}^{1}[-tq]_{\infty}[tq]_{\infty}\,\Delta_n(s,\alpha,\beta,x,t\,|\,q)\,d_q t,$$

where

$$\Delta_n(s,\alpha,\beta;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}(1-q)^{\alpha+\beta+n}\,\Gamma_q(2\alpha+2\beta+2sk+n+2)}{[\alpha\beta q^{2sk+1}]_{\infty}[\alpha q]_k[q]_k\Gamma_q(\alpha+\beta+2sk+1)}\cdot$$

$$\cdot\frac{(2\alpha\beta q^{2sk+2};q^2)_{\infty}[-\alpha\beta q^{sk+n+1}]_{\infty}(xq)^k}{[-1]_{\infty}(q^2;q^2)_{\infty}[-t\alpha\beta q^{sk+n+1}]_{\infty}[-t\alpha\beta q^{sk+1}]_{\infty}}.$$

$$P_{n,s}(x\,|\,q)=\int_{-1}^{1}[-tq]_{\infty}[tq]_{\infty}\,\gamma_n(s;x,t\,|\,q)\,d_q t,$$

where

$$\gamma_n(s;x,t\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}(1-q)^n\,\Gamma_q(2sk+n+2)(q^{2sk+2};q^2)_{\infty}}{[q^{2sk+1}]_{\infty}[q]_k[q]_k\Gamma_q(2sk+1)[-1]_{\infty}}\cdot$$

$$\cdot\frac{[-q^{sk+n+1}]_{\infty}q^k x^k}{(q^2,q^2)_{\infty}[-tq^{sk+n+1}]_{\infty}[-tq^{sk+1}]_{\infty}}.$$

From the general q-integrals (5.2.7) and (5.2.8) the reducibilities for the family of polynomials $\left\{Z_n^{(\alpha)}(x;k\,|\,q), n=0,1,2,\,..\right\}$ are not obtainable as the q-gamma function $\Gamma_q(A+sk+n\beta)$ and other such q-gamma functions fail to exist when A and n are vanishing.

The special cases of the general integral (5.2.9) are as given below.

$$\frac{P_{n,s}(x;a,b,c,d\,|\,q)}{[ab]_n[ac]_n[ad]_n[q]_n a^{-n}} = \int_0^1 t^{a+b+c+d+n-2}\,\mathrm{E}_q(tq)\,\delta_n(s,a,b,c,d;x\,|\,q)\,d(q,t),$$

where

$$\delta_n(s,a,b,c,d;x\,|\,q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}\,[ae^{-i\theta}]_k\,[ae^{i\theta}]_k\,[abcdq^{n+1}]_\infty\,q^k}{(1-q)[ab]_k[ac]_k[ad]_k[q]_k[q]_n[q]_\infty}.$$

$$R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q) = \int_0^1 t^{\alpha+\beta+n}\,E_q(tq)\,\rho_n(s,\alpha,\beta,\gamma,\delta;x\,|\,q)\,d(q,t),$$

where

$$\rho_n(s,\alpha,\beta,\gamma,\delta;x\,|\,q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}\,[q^{-x}]_k\,[\gamma\delta q^{x+1}]_k\,[\alpha\beta q^{n+1}]_\infty\,q^k}{[\alpha q]_k[\beta\delta q]_k[\gamma q]_k[q]_k[q]_n(1-q)[q]_\infty}.$$

$$Q_{n,s}(x;\alpha,\beta,N\,|\,q) = \int_0^1 t^{\alpha+\beta+n}\,E_q(tq)\,\lambda_n(s,\alpha,\beta,N;x\,|\,q)\,d(q,t),$$

where

$$\lambda_n(s,\alpha,\beta,N;x\,|\,q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk}\,[q^{-x}]_k\,[\alpha\beta q^{n+1}]_\infty\,q^k}{[\alpha q]_k[q^{-N}]_k[q]_k[q]_n(1-q)[q]_\infty}.$$

$$p_{n,s}(x;\alpha,\beta;q) = \int_0^1 t^{\alpha+\beta+n}\,E_q(tq)\,\mu_n(s,\alpha,\beta;x\,|\,q)\,d(q,t),$$

where

$$\mu_n(s,\alpha,\beta,x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,(xq)^k}{[\alpha q]_k\,[q]_k\,(1-q)[q]_\infty}.$$

$$P_{n,s}(x\,|\,q)=\int_0^1 t^n\,E_q(tq)\,\upsilon_n(s;x\,|\,q)\,d(q,t),$$

where

$$\upsilon_n(s;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}(xq)^k}{[q]_k[q]_k(1-q)[q]_\infty}.$$

$$Z_{n,s}^{(\alpha)}(x;j\,|\,q)=\int_0^1 t^{n-1}\,E_q(tq)\,\delta_n(s,\alpha;x\,|\,q)\,d(q,t),$$

where

$$\delta_n(s,\alpha,x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{kj(\alpha+n+1)+kj(kj-1)/2}\,(q^{-nj};q^j)_{sk}\,[\alpha q]_{nj}\,x^{kj}}{(q^j;q^j)_{sk}\,[\alpha q]_{kj}\,(q^j;q^j)_n\,(1-q^j)(q^j;q^j)_\infty}.$$

$$L_{n,s}^{(\alpha)}(x\,|\,q)=\int_0^1 t^{n-1}\,E_q(tq)\,\gamma_n(s,\alpha;x\,|\,q)\,d(q,t),$$

where

$$\gamma_n(s,\alpha;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{k(\alpha+n+1)+k(k-1)/2}\,[q^{-n}]_{sk}\,x^k}{[q]_{sk}\,[\alpha q]_k\,[q]_n\,(1-q)[q]_\infty}.$$

$$W_{n,s}(x;a,q)=\int_0^1 t^{n-1}\,E_q(tq)\,\phi_n(s,a;x\,|\,q)\,d(q,t),$$

where

$$\phi_n(s,a;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{n+sk}q^{k(k-1)/2-snk+n(n+1)/2}[q^{2sk}]_\infty [a]_n [q]_n x^k}{[q]_{n-sk}[a]_k [q]_{sk}(1-q)[q]_\infty}.$$

$$S_{n,m}(x;p,q)=\int_0^1 t^{n-1}\,E_q(tq)\,\psi_n(m,p;x\,|\,q)\,d(q,t),$$

where

$$\psi_n(m,p;x\,|\,q)=\sum_{k=0}^{[n/m]}\frac{(-1)^{n+km}q^{k(2k+1)/2}[q^{2mk}]_\infty [p]_n [q]_n x^k}{[q]_{n-mk}(1-q)[q]_{mk}[p]_k [q]_\infty\, q^{n(2n+1)/2}}.$$

The particular cases of the q-integral (5.2.10) are enlisted as follows.

$$\frac{P_{n,s}(x;a,b,c,d\,|\,q)}{[ab]_n[ac]_n[ad]_n[q]_n a^{-n}}=\int_0^\infty t^{a+b+c+d+n-2}\,e_q(-t).$$

$$\cdot\mu_n(s,a,b,c,d;x\,|\,q)\,d_q t,$$

where

$$\mu_n(s;a,b,c,d;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{(a+b+c+d+sk+n-2)(a+b+c+d+sk+n-1)/2}}{[ab]_k[ac]_k[ad]_k[q]_k[q]_n[q]_\infty(1-q)}.$$

$$\cdot[q^{-n}]_{sk}[ae^{-i\theta}]_k[ae^{i\theta}]_k q^k.$$

$$R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q)=\int_0^\infty t^{\alpha+\beta+n}\,e_q(-t)\,\Delta_n(s,\alpha,\beta,\gamma,\delta;x\,|\,q)\,d_q t,$$

where

$$\Delta_n(s,\alpha,\beta,\gamma,\delta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{(\alpha+\beta+sk+n)(\alpha+\beta+sk+n+1)/2}[q^{-n}]_{sk}}{(1-q)[\alpha q]_k[\beta\delta q]_k[\gamma q]_k[q]_k[q]_n[q]_\infty}.$$

$$\cdot[q^{-x}]_k[\gamma\delta q^{x+1}]_k[\alpha\beta q^{n+1}]_\infty.$$

$$Q_{n,s}(x;\alpha,\beta,N\,|\,q)=\int_0^\infty t^{\alpha+\beta+n}\,e_q(-t)\,\upsilon_n(s,\alpha,\beta,N,x\,|\,q)\,d_q t,$$

where

$$\upsilon_n(s,\alpha,\beta,N;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{(\alpha+\beta+sk+n)(\alpha+\beta+sk+n+1)/2}[q^{-n}]_{sk}}{(1-q)[\alpha q]_k[q^{-N}]_k[q]_k[q]_n[q]_\infty}.$$

$$[q^{-x}]_k[\alpha\beta q^{n+1}]_\infty q^k.$$

$$p_{n,s}(x,\alpha,\beta,q)=\int_0^\infty t^{\alpha+\beta+n}e_q(-t)\mu_n(s,\alpha,\beta;x\,|\,q)\,d_qt,$$

where

$$\mu_n(s,\alpha,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{(\alpha+\beta+sk+n)(\alpha+\beta+sk+n+1)/2}[q^{-n}]_{sk}q^k}{(1-q)[\alpha q]_k[q]_k[q]_\infty}.$$

$$P_{n,s}(x\,|\,q)=\int_0^\infty t^n\,e_q(-t)\;\lambda_n(s;x\,|\,q)\;d_qt,$$

where

$$\lambda_n(s,x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{(n+sk)(n+sk+1)/2}[q^{-n}]_{sk}q^k}{[q]_k[q]_k[q]_\infty(1-q)}.$$

$$Z_{n,s}^{(\alpha)}(x,j\,|\,q)=\int_0^\infty t^{n-1}e_q(-t)\gamma_n(s,\alpha,x\,|\,q)\;d_qt,$$

where

$$\gamma_n(s,\alpha;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{j(sk+n-1)(sk+n)/2+jk(\alpha+n+1)+jk(jk-1)/2}}{(q^j;q^j)_{sk}[\alpha q]_{kj}(q^j;q^j)_n(q^j;q^j)_\infty}.$$

$$\frac{[\alpha q]_{nj}(q^{-nj};q^j)_{sk}x^{jk}}{(1-q^j)}.$$

$$L_{n,s}^{(\alpha)}(x\,|\,q) = \int_0^\infty t^{n-1}\, e_q(-t)\, \Delta_n(s,\alpha;x\,|\,q)\; d_q t,$$

where

$$\Delta_n(s,\alpha;x\,|\,q) = \sum_{k=0}^{[n/s]} \frac{q^{(sk+n-1)(sk+n)/2+k(\alpha+n+1)+k(k-1)/2}\,[\alpha q]_n\,[q^{-n}]_{sk}}{[q]_{sk}\,[\alpha q]_k\,[q]_n\,[q]_\infty\,(1-q)}\,x^k$$

$$W_{n,s}(x;a,q) = \int_0^\infty t^{n-1}\; e_q(-t)\;\; \delta_n(s,a;x\,|\,q)\;\; d_q t,$$

where

$$\delta_n(s,a;x\,|\,q) = \sum_{k=0}^{[n/s]} \frac{(-1)^{n+sk}\, q^{(sk+n-1)(sk+n)/2-snk+k(k-1)/2+n(n+1)/2}}{(1-q)[q]_\infty\,[q]_{n-sk}\,[a]_k\,[q]_{sk}}\,.$$

$$\cdot [q]_n\,[a]_n\,[q^{2sk}]_\infty\, x^k.$$

$$S_{n,m}(x;p,q) = \int_0^\infty t^{n-1}\, e_q(-t)\,\rho_n(m,p;x\,|\,q)\;\; d_q t,$$

where

$$\rho_n(m,p;x\,|\,q) = \sum_{k=0}^{[n/m]} \frac{(-1)^{n+km}\, q^{(mk+n-1)(mk+n)/2-n(2n+1)/2+k^2+k/2}}{[q]_{mk}\,[p]_k\,[q]_\infty\,(1-q)[q]_{n-mk}}\,.$$

$$\cdot [p]_n\,[q]_n\,[q^{2mk}]_\infty\, x^k.$$

(II) Special cases of q-difference equation(θ-form)

The θ-form q-difference equation (equation (5.3.7)) derived for the polynomial $M_n(s,A,\beta;x|q)$ in section 5.3 will now be particularized for the special polynomials. These particular q-difference equations for the specialized polynomials are given in the following list.

$$\{\theta(\theta+q^{1-a-b}-1)(\theta+q^{1-a-c}-1)(\theta+q^{1-a-d}-1)+q^{4+s(a+b+c+d+1)+ae^{-i\theta}}.$$

$$.q^{ae^{i\theta}}-3a-b-c-d\left[\prod_{p=0}^{s-1}\prod_{v=0}^{s-1}(\theta+w^{v}q^{\frac{n}{s}-\frac{p}{s}}-1)(\theta+w^{v}q^{\frac{-(a+b+c+d+n-1)}{s}-\frac{p}{s}}-1)\right].$$

$$.(\theta+q^{-ae^{-i\theta}}-1)(\theta+q^{-ae^{i\theta}}-1)\}P_{n,s}(x;a,b,c,d\,|\,q)=0.$$

$$\{\theta(\theta+q^{-\alpha}-1)(\theta+q^{-\beta-\delta}-1)(\theta+q^{-\gamma}-1)+q^{3(s+1)+(s-1)(\alpha+\beta)}.$$

$$\left[\prod_{p=0}^{s-1}\prod_{v=0}^{s-1}(\theta+w^{v}q^{\frac{n}{s}-\frac{p}{s}}-1)(\theta+w^{v}q^{\frac{-(\alpha+\beta+n+1)}{s}-\frac{p}{s}}-1)\right].$$

$$.(\theta+q^{x}-1)(\theta+q^{-(\gamma+\delta+x+1)}-1)\}R_{n,s}(\mu(x),\alpha,\beta,\gamma,\delta;x)=0.$$

$$\{\theta(\theta+q^{-\alpha}-1)(\theta+q^{N}-1)+q^{2+s(\alpha+\beta+3)-x-\alpha+N}.$$

$$\left[\prod_{p=0}^{s-1}\prod_{v=0}^{s-1}(\theta+w^{v}q^{\frac{n}{s}-\frac{p}{s}}-1)(\theta+w^{v}q^{\frac{-(\alpha+\beta+n+1)}{s}-\frac{p}{s}}-1)\right].$$

$$.(\theta+q^{x}-1)\}Q_{n,s}(x;\alpha,\beta,N\,|\,q)=0.$$

$$\{\theta(\theta+q^{-\alpha}-1)+xq^{2+s(\alpha+\beta+3)-\alpha}.$$

$$.\left[\prod_{p=0}^{s-1}\prod_{v=0}^{s-1}(\theta+w^{v}q^{\frac{n}{s}-\frac{p}{s}}-1)(\theta+w^{v}q^{\frac{-(\alpha+\beta+n+1)}{s}-\frac{p}{s}}-1)\right].$$

$$.(\theta+q^{-x-1}-1)\}p_{n,s}(x;\alpha,\beta;q)=0.$$

$\{\theta^2 + xq^{3s+2}.$

$$\cdot\left[\prod_{p=0}^{s-1}\prod_{v=0}^{s-1}(\theta+w^{v}q^{\frac{n}{s}-\frac{p}{s}}-1)(\theta+w^{v}q^{\frac{-(n+1)}{s}-\frac{p}{s}}-1)\right]\}P_{n,s}(x|q)=0.$$

In all of the above q-difference equations w is the s[th] roots of unity.

(III) <u>Special cases of q-inverse series relations</u>

This set of special cases is further divided into two subsets, which are

(III-a) Extensions of inversion pairs of Gessel and Stanton.

(III-b) Extensions of pairs of inverse series relations of polynomial special cases.

We now start with

(III-a) <u>Extensions of inversion pairs of Gessel and Stanton</u>

The various inverse series relations proved by Gessel and Stanton ([1], theorems 3.1 to 3.6) admit extensions under therorem-9, which is proved in section 5.6. In fact when the parameter β is assigned the values β=1/2,1,-1/2,-1/3, and 2 in theorem-9, one obtains the following extended inversion pairs.

(i) β=1/2

$$A_n = \sum_{k=0}^{[n/s]} \frac{q^{-snk}(Aq^{3sk/2};q^{1/2})_{n-sk}}{[q]_{n-sk}} B_k$$

⇔

$$B_n = \sum_{k=0}^{sn} \frac{(-1)^{sn-k} q^{(sn-k)(sn-k+1)/2+snk}(Aq^{sn+(k+1)/2};q^{1/2})_{sn-k-1} A_k}{[q]_{sn-k}(1-Aq^{3k/2})^{-1}}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k+1)/2+nk} (Aq^{n+(k+1)/2}; q^{1/2})_{n-k-1} A_k}{[q]_{n-k}(1-Aq^{3k/2})^{-1}} = 0, \text{ if } n \neq sj, j \in N.$$

(ii) β=1

$$A_n = \sum_{k=0}^{[n/s]} \frac{q^{-snk}(Aq^{2sk};q)_{n-sk} B_k}{[q]_{n-sk}}$$

$\Leftrightarrow$

$$B_n = \sum_{k=0}^{sn} \frac{(-1)^{sn-k} q^{(sn-k)(sn-k+1)/2+snk} (Aq^{sn+k+1}, q)_{sn-k-1} A_k}{[q]_{sn-k}(1-Aq^{2k})^{-1}}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k+1)/2+nk} (Aq^{n+k+1}; q)_{n-k-1} A_k}{[q]_{n-k}(1-Aq^{2k})^{-1}} = 0, \text{ if } n \neq sj, j \in N.$$

(iii) β=-1/2

$$A_n = \sum_{k=0}^{[n/s]} \frac{q^{-snk}(Aq^{sk/2};q^{-1/2})_{n-sk} B_k}{[q]_{n-sk}}$$

$\Leftrightarrow$

$$B_n = \sum_{k=0}^{sn} \frac{(-1)^{sn-k} q^{(sn-k)(sn-k+1)/2+snk} (Aq^{sn-(k+1)/2}; q^{-1/2})_{sn-k-1} A_k}{[q]_{sn-k}(1-Aq^{k/2})^{-1}}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k+1)/2+nk} (Aq^{n-(k+1)/2}; q^{-1/2})_{n-k-1} A_k}{[q]_{n-k}(1-Aq^{k/2})^{-1}} = 0, \text{ if } n \neq sj, j \in N.$$

(iv) β=-1/3

$$A_n = \sum_{k=0}^{[n/s]} \frac{q^{-snk}(Aq^{2sk/3};q^{-1/3})_{n-sk} B_k}{[q]_{n-sk}}$$

$\Leftrightarrow$

$$B_n = \sum_{k=0}^{sn} \frac{(-1)^{sn-k} q^{(sn-k)(sn-k+1)/2+snk}(Aq^{sn-(k+1)/3};q^{-1/3})_{sn-k-1} A_k}{[q]_{sn-k}(1-Aq^{2k/3})^{-1}}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k+1)/2+nk}(Aq^{n-(k+1)/3};q^{-1/3})_{n-k-1}}{[q]_{n-k}(1-Aq^{2k/3})^{-1} A_k^{-1}} = 0, \text{ if } n \neq sj, j \in N.$$

(v) β=2

$$A_n = \sum_{k=0}^{[n/s]} \frac{q^{-snk}(Aq^{3sk};q^{2})_{n-sk} B_k}{[q]_{n-sk}}$$

$\Leftrightarrow$

$$B_n = \sum_{k=0}^{sn} \frac{(-1)^{sn-k} q^{(sn-k)(sn-k+1)/2+snk}(Aq^{sn+2(k+1)};q^{2})_{sn-k-1}}{[q]_{sn-k}(1-Aq^{3k})^{-1} A_k^{-1}}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k+1)/2+nk}(Aq^{n+2(k+1)};q^{2})_{n-k-1}(1-Aq^{3k})A_k}{[q]_{n-k}} = 0,$$

$$\text{if } n \neq sj, j \in N.$$

(III-b) Extension of pairs of inverse series relations of polynomial special cases

The general class of polynomial $\{M_n(s,A,\beta;x\,|\,q), n=0,1,2,\ldots\}$ defined in section-5.1, given by

$$M_n(s,A,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q^{\beta})_{n-sk}\,\xi_k\,x^k}{[q]_{n-sk}} \quad \textbf{(5.8.10)}$$

is inverted through theorem-9. From the second series (5.4.5) of theorem-9, the inverse series of $M_n(s,A,\beta;x\,|\,q)$ can be obtained as follows.

In (5.4.4) taking V(k)=$\xi_k x^k$, one obtains U(n)= $M_n(s,A,\beta;x\,|\,q)$.

Therefore from (5.4.5) the inverse series of (5.8.10) is given by

$$\xi_n\, x^n = \sum_{k=0}^{sn}\frac{(-1)^k q^{(sn-k)(sn-k+1)/2+snk}(1-Aq^{k+k\beta})}{[q]_{sn-k}}.$$

$$.(Aq^{sn+(sn-1)\beta};q^{-\beta})_{sn-k-1}\; M_k(s,A,\beta;x\,|\,q). \quad \textbf{(5.8.11)}$$

For obtaining the polynomial special cases a slightly modified version (given in (5.8.2)) of (5.8.10) is used, which is denoted by $M_n^*(s,A;x\,|\,q)$, as it was shown in the beginning of this section.

Therefore,

$$M_n^*(s,A,x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{q^{-sk(sk-1)/2}[q^{-n}]_{sk}[Aq^n]_{sk}\,\lambda_k\,x^k}{[q]_n} \quad \textbf{(5.8.12)}$$

where

$$M_n^*(s,A,x|q)=\frac{M_n(s,A,1,x|q)}{[Aq^n]_\infty},\quad \lambda_k=[Aq^{2sk}]_\infty\ \xi_k,$$ and β=1.

Making these replacements in (5.8.11) one arrives at:

$$\frac{\lambda_n\, x^n}{[Aq^{2sn}]_\infty}=\sum_{k=0}^{sn}\frac{(-1)^k q^{(sn-k)(sn-k+1)/2+snk}\,(Aq^{2sn-1};q^{-1})_{sn-k-1}}{[q]_{sn-k}[Aq^k]_\infty}\,.$$

$$.(1-Aq^{2k})\,M_k^*(s,A;x|q)\,.$$

But

$$[Aq^{2sn}]_\infty\,(Aq^{2sn-1},q^{-1})_{sn-k-1}\ =[Aq^{sn+k+1}]_{sn-k-1}\,[Aq^{2sn}]_\infty$$

$$=[Aq^{sn+k+1}]_\infty$$

therefore

$$\lambda_n\, x^n=\sum_{k=0}^{sn}\frac{(-1)^k q^{(sn-k)(sn-k+1)/2+snk}(1-Aq^{2k})}{[q]_{sn-k}[Aq^k]_\infty}[Aq^{sn+k+1}]_\infty\,.$$

$$.\,M_k^*(s,A;x|q)\,,$$

which is the inverse series of (5.8.12) and it can be written in a simplified form as:

$$\lambda_n\, x^n=\sum_{k=0}^{sn}\frac{(-1)^k q^{sn(sn+1)/2+k(k-1)/2}(1-Aq^{2k})}{[q]_{sn-k}[Aq^k]_{sn+1}}\ M_k^*(s,A;x|q)\,. \qquad \textbf{(5.8.13)}$$

Thus the general pair of inverse series relation (5.8.12) and (5.8.13) contains number of polynomials as special cases in extended form together with their inverse series. These pairs are given below.

Taking $x=1, A=abcdq^{-1}$ and $\lambda_n = \dfrac{[ae^{-i\theta}]_n [ae^{i\theta}]_n q^{sn(sn+1)/2+n}}{[ab]_n [ac]_n [ad]_n [q]_n}$, one obtains the pair of inverse series relations of an extended form of Askey-Wilson polynomial given by:

$$\frac{P_{n,s}(x;a,b,c,d\,|\,q)}{[ab]_n [ac]_n [ad]_n a^{-n}} = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk} [abcdq^{n-1}]_{sk} [ae^{-i\theta}]_k [ae^{i\theta}]_k}{[q]_k [ab]_k [ac]_k [ad]_k} q^k$$

if and only if **(5.8.14)**

$$\frac{[ae^{-i\theta}]_n [ae^{i\theta}]_n q^n}{[ab]_n [ac]_n [ad]_n [q]_n} = \sum_{k=0}^{sn} \frac{(-1)^k q^{k(k-1)/2} (1-abcdq^{2k-1})}{[q]_{sn-k} [abcdq^{k-1}]_{sn+1}} P_{k,s}(x;a,b,c,d\,|\,q).$$

The inversion pair of Extended q-Racah polynomial can be obtained from the pair (5.8.12) and (5.8.13), if x=1, A=αβq and $\lambda_n = \dfrac{q^{sn(sn+1)/2+n} [q^{-x}]_n [\gamma\delta q^{x+1}]_n}{[\alpha q]_n [\beta\delta q]_n [\gamma q]_n [q]_n}$ is chosen.

Therefore

$$R_{n,s}(\mu(x);\alpha,\beta,\gamma,\delta;q) = \sum_{k=0}^{[n/s]} \frac{[q^{-n}]_{sk} [\alpha\beta q^{n+1}]_{sk} [q^{-x}]_k [\gamma\delta q^{x+1}]_k q^k}{[q]_k [\alpha q]_k [\beta\delta q]_k [\gamma q]_k}$$

if and only if **(5.8.15)**

$$\frac{q^n [q^{-x}]_n [\gamma\delta q^{x+1}]_n}{[\alpha q]_n [\beta\delta q]_n [\gamma q]_n [q]_n} = \sum_{k=0}^{sn} \frac{(-1)^k q^{k(k-1)/2} (1-\alpha\beta q^{2k+1})}{[q]_k [q]_{sn-k} [\alpha\beta q^{k+1}]_{sn+1}} R_{k,s}(\mu(x);\alpha,\beta,\gamma,\delta;q),$$

where $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$

By choosing the same values for x and A as above and setting

$$\lambda_n = \frac{q^{sn(sn+1)/2+n} [q^{-x}]_n}{[\alpha q]_n [q^{-N}]_n [q]_n},$$

one can obtain the pair of inverse series relation of extended q-Hahn polynomial in the form:

$$Q_{n,s}(x,\alpha,\beta,N\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,[\alpha\beta q^{n+1}]_{sk}\,[q^{-x}]_k\,q^k}{[\alpha q]_k\,[q^{-N}]_k\,[q]_k}$$

$\Leftrightarrow$ **(5.8.16)**

$$\frac{q^n[q^{-x}]_n}{[\alpha q]_n\,[q^{-N}]_n\,[q]_n}=\sum_{k=0}^{sn}\frac{(-1)^k\,q^{k(k-1)/2}\,(1-\alpha\beta q^{2k+1})}{[q]_{sn-k}\,[\alpha\beta q^{k+1}]_{sn+1}}\,Q_{k,s}(x;\alpha,\beta,N\,|\,q).$$

Next for deriving the inversion pair of an extended form of the little q-Jacobi polynomial one should set A=αβq and $\lambda_n=\frac{q^{sn(sn+1)/2}}{[\alpha q]_n[q]_n}$, then

$$p_{n,s}(x;\alpha,\beta;q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,[\alpha\beta q^{n+1}]_{sk}\,(xq)^k}{[\alpha q]_k\,[q]_k}$$

if and only if **(5.8.17)**

$$\frac{(xq)^n}{[\alpha q]_n\,[q]_n}=\sum_{k=0}^{sn}\frac{(-1)^k\,q^{k(k-1)/2}\,(1-\alpha\beta q^{2k+1})}{[q]_{sn-k}\,[\alpha\beta q^{k+1}]_{sn+1}}\,p_{k,s}(x,\alpha,\beta;q).$$

From the above pair (5.8.17) the pair of inverse series relation of an extended q-Legendre polynomial can be obtained just by setting α=β=0 (i.e. $q^\alpha=q^\beta=1,|q|<1$). Thus,

$$P_{n,s}(x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{[q^{-n}]_{sk}\,[q^{n+1}]_{sk}\,(xq)^k}{[q]_k\,[q]_k}$$

$\Leftrightarrow$ **(5.8.18)**

$$\frac{(xq)^n}{[q]_n\,[q]_n}=\sum_{k=0}^{sn}\frac{(-1)^k\,q^{k(k-1)/2}\,(1-q^{2k+1})}{[q]_{sn-k}\,[q^{k+1}]_{sn+1}}\,P_{k,s}(x\,|\,q).$$

The extended q-Konhauser polynomial and its inverse series can be obtained from $M_n^*(s,A;x\,|\,q)$ and its inverse series, which are given in (5.8.12) and (5.8.13) respectively.

In fact, taking $A(=q^A)=0$, $\lambda_n = \dfrac{q^{ksn(sn+1)/2+kn(\alpha+1)+kn(kn-1)/2}}{(q^k;q^k)_{sn}\,[\alpha q]_{kn}}$,

and replacing q by q^k, x by $x^k q^{nk}$, $k \in N$, and $M_j^*(s,A;x\,|\,q)$ by $\dfrac{M_j^*(s,A;x\,|\,q)}{[\alpha q]_j}$,

one obtains

$$Z_{n,s}^{(\alpha)}(x;k\,|\,q) = \frac{[\alpha q]_n}{(q^k;q^k)_n} \sum_{j=0}^{[n/s]} \frac{q^{kj(n+\alpha+1)+kj(kj-1)/2}}{(q^k;q^k)_{sj}\,[\alpha q]_{kj}} (q^{-nk};q^k)_{sj}\, x^{kj}$$

if and only if **(5.8.19)**

$$\frac{q^{kn(\alpha+1)+kn(kn-1)/2-snk}\, x^{kn}}{(q^k;q^k)_{sn}\,[\alpha q]_{kn}} = \sum_{j=0}^{sn} \frac{(-1)^j\, q^{kj(j-1)/2}}{(q^k;q^k)_{sn}} Z_{j,s}^{(\alpha)}(x;k\,|\,q).$$

Just by putting k=1 in the above pair (5.8.19), one can straight away get the inversion pair of the extended q-Laguerre polynomial.

Thus,

$$L_{n,s}^{(\alpha)}(x\,|\,q) = \frac{[\alpha q]_n}{[q]_n} \sum_{j=0}^{[n/s]} \frac{q^{j(n+\alpha+1)+j(j-1)/2}\,[q^{-n}]_{sj}\, x^j}{[q]_{sj}\,[\alpha q]_j}$$

$\Leftrightarrow$ **(5.8.20)**

$$\frac{q^{n(\alpha+1)+n(3n-1)/2}\, x^n}{[q]_{sn}\,[\alpha q]_n} = \sum_{j=0}^{sn} \frac{(-1)^j\, q^{j(j-1)/2}}{[q]_{sn-j}\,[\alpha q]_j} L_{j,s}^{(\alpha)}(x\,|\,q).$$

5.9 EXPANSION OF GENERAL q-POLYNOMIALS

The pairs of inverse series relations of the polynomials $M_n^*(s,A;x|q)$ and $S_n(l,m,\alpha,\beta;x|p)$ are given by ((5.8.12) and (5.8.13))

$$M_n^*(s,A;x|q)=\sum_{k=0}^{[n/s]} q^{-sk(sk-1)/2}\frac{[q^{-n}]_{sk}}{[q]_n}[Aq^n]_{sk}\lambda_k x^k \quad \textbf{(5.9.1)}$$

$$\lambda_n x^n=\sum_{k=0}^{sn}(-1)^k q^{snk+(sn-k)(sn-k+1)/2}\frac{(1-Aq^{2k})}{[q]_{sn-k}[Aq^k]_{sn+1}}M_k^*(s,A;x|q) \quad \textbf{(5.9.2)}$$

and ((3.9.21) and (3.9.22))

$$S_n(l,m,\alpha,\beta;x|q)=\sum_{k=0}^{[n/m]} p^{mk}\frac{(p^{-n};p)_{mk}}{(\beta q^{1-n\alpha};p)_{lk}}\sigma_k x^k, \quad \textbf{(5.9.3)}$$

$$\sigma_n x^n=\sum_{k=0}^{mn}(-1)^k p^{k(k-1)/2}\frac{(1-\beta q^{1-\alpha})(\beta q^{1-n\alpha};p)_\infty S_k(l,m,\alpha,\beta;x|p)}{(p;p)_{mn-k}(\beta q^{1-k\alpha+ln-\alpha};p)_\infty (p;p)_k}, \quad \textbf{(5.9.4)}$$

wherein $l=m\alpha$ and $p=q^\alpha, \alpha\neq 0$, as before,

Now on substituting the inverse series (5.9.4) in (5.9.1) for x^k, one gets

$$M_n^*(s,A;x|q)=\sum_{k=0}^{[n/s]}\sum_{j=0}^{mk}(-1)^j q^{\alpha j(j-1)/2-sk(sk-1)/2}\frac{[q^{-n}]_{sk}[Aq^n]_{sk}\lambda_k}{[q]_n (p;p)_{mk-j}(p;p)_j\sigma_k}$$

$$\cdot\frac{(1-\beta q^{1-\alpha})(\beta q^{1-n\alpha};p)_\infty}{(\beta q^{1-j\alpha+lk-\alpha};p)_\infty}S_j(l,m,\alpha,\beta;x|p). \quad \textbf{(5.9.5)}$$

On the other hand, combining the inverse series (5.9.2) with the polynomial (5.9.3), the expression occurs in the form

$$S_n(l,m,\alpha,\beta,x|q)=\sum_{k=0}^{[n/m]}\sum_{j=0}^{sk}(-1)^j\, q^{\alpha mk+skj+(sk-j)(sk-j+1)/2}\,.$$

$$\cdot\frac{(p^{-n};p)_{mk}\,(1-Aq^{2j})\,\sigma_k}{\lambda_k\,(\beta q^{1-n\alpha};p)_{lk}[Aq^j]_{sk+1}[q]_{sk-j}}\,M_j^*(s,A,x|q)\,.\textbf{(5.9.6)}$$

The expansion formula (5.9.5) may be illustrated in particular by taking $A=\alpha'\beta'q,\lambda_k=q^{sk(sk+1)/2}/[\alpha'q]_k[q]_k, s=1,$ and $\sigma_k=\dfrac{[a_1]_k\cdots[a_s]_k}{[b_1]_k\cdot[b_r]_k\,(p;p)_k}$

in which case the polynomial in (5.9.1) defines little q-Jacobi polynomial $\dfrac{p_n(x;\alpha',\beta',q)}{[\alpha'q]_n}$ and the polynomial in (5.9.3) reduces to the q-extended Jacobi polynomial $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b):x|q]$; and (5.9.5) would yield

$$p_n(x,\alpha',\beta',q)=\sum_{k=0}^{n}\sum_{j=0}^{mk}\frac{[\alpha q]_n[q^{-n}]_k[\alpha'\beta'q^{n+1}]_k[b_1]_k\cdots[b_r]_k\,(p;p)_k\,(1-\beta q^{1-\alpha})}{[q]_n[\alpha'q]_k[q]_k[a_1]_k\cdots[a_s]_k\,(\beta q^{1-j\alpha+lk-\alpha},p)_\infty\,(p;p)_{mk-j}}$$

$$\cdot(\beta q^{1-n\alpha};p)_\infty\,(-1)^j\,q^{k+\alpha j(j-1)/2}\,\mathcal{H}_{j,l,m}^{(\alpha,\beta)}[(a);(b):xp^m|p]\,.\ \textbf{(5.9.7)}$$

Similarly, the expansion formula (5.9.6) gives

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b):xp^m|p]=\sum_{k=0}^{[n/m]}\sum_{j=0}^{k}\frac{(p^{-n};p)_{mk}[a_1]_k\cdots[a_s]_k[\alpha'q]_k[q]_k}{(\beta q^{1-n\alpha};p)_{lk}[b_1]_k\cdot[b_r]_k\,(p;p)_k}\,.$$

$$\cdot\frac{(1-\alpha'\beta'q^{2j+1})}{[q]_{k-j}[\alpha'\beta'q^{j+1}]_{k+1}}(-1)^j\,q^{\alpha mk-k+j(j-1)/2}\,p_j(x,\alpha',\beta';q)\,.\ \textbf{(5.9.8)}$$