Chapter 3

Fixed Point Results on F-contraction in Rectangular b-metric Space

In this chapter, some new fixed point theorems are provided for F-contraction on rectangular b-metric spaces in which maps need not be continuous. Especially, we derive a common fixed point theorem for two pairs of weakly compatible mappings for a new type of F-contraction on rectangular b-metric spaces (not necessarily continuous). Our results not only generalize many known results in the literature but also improve some of the results therein. In addition, the results are justified by appropriate examples and deployed to examine the existence and uniqueness of solutions for a system of Volterra integral equations.

3.1 Introduction

The famous result of Banach (1922), celebrated as the Banach Contraction Principle. Many researchers generalized this fixed point result in different metric spaces like b-metric space (Bakhtin (1989), Goswami et al. (2019)) and rectangular metric space (Branciari (2000), Azam and Arshad (2008)).

The study of common fixed points was initiated by Jungck (1976) and Jungck (1986) and this concept attracted many researchers to prove the existence of fixed points by using various metrical contractions. Some authors proved the existence and uniqueness of common fixed points for two or more than two maps (Rhoades et al. (1987)).

George et al. (2015) introduced the concept of rectangular b-metric space, which was not necessarily Hausdorff and which generalized the concept of metric space, RMS, and b-metric space. He also proved Banach and Kannan's fixed point theorems for rectangular b-metric space. Then Mitrović and Radenović (2017) established a common fixed point for two maps in such space.

3.2 Rectangular b- metric Space (RbMS)

Definition 3.2.1. *(George et al. (2015)) Let X be a nonempty set with the coefficient* $s \geq 1$, *and the mapping* $\hat{d} : X \times X \to [0, \infty)$ *satisfies the following:*

(1) $\hat{d}(x, y) = 0$ *if and only if* $x = y$,

(2) $\hat{d}(x, y) = \hat{d}(y, x)$, *for all* $x, y \in X$,

(3) $\hat{d}(x, y) \leq s[\hat{d}(x, w) + \hat{d}(w, z) + \hat{d}(z, y)]$, *for all* $x,\ y \in X$ *and for all distinct points w,z* $\in X - \{x, y\}$.

Then $(X, \hat{d})$ *is called a rectangular b- metric space(in short RbMS).*

Remark 3.2.1.1. *Every metric space is a RMS and every RMS is a RbMS(with coefficient* $s = 1$*), however, the converse is not always true. [see Example 1.4.2.1 and next Example 3.2.2]*

The notions of Cauchy sequence, convergence, and completeness in a RbMS are defined the same as the metric space.

Remark 3.2.1.2. *(1) Limit of a sequence in a RbMS is not always unique and also every convergent sequence in RbMS is not essentially a Cauchy sequence. Further, RbMS is not a continuous map.*

(2) In RbMs, the open balls are not always open and $(X, \hat{d})$ is not Hausdorff.

(George et al., 2015) defined the following example using the $\frac{\alpha}{2n}$ term in place of $\frac{\alpha}{2ny}$. As a result, we alter his example as follows, which justifies the preceding statements.

Example 3.2.2. *Let $X = U \cup V$, where $U = \{\frac{1}{n} : n \in N\}$ and V denotes the collection of all positive integers. Define $\hat{d} : X \times X \to [0, \infty)$ with $\hat{d}(x, y) = \hat{d}(y, x)$ for all $x, y \in X$ and*

$$\hat{d}(x,y) = \begin{cases} 2\alpha & ; x, y \in A \\ \frac{\alpha}{2ny} & ; x \in A \quad \text{and} \quad y \in \{2,3\} \\ \alpha & ; \text{otherwise} \\ 0 & ; x = y, \end{cases} \tag{3.1}$$

where $\alpha > 0$ is a constant. Then $(X, \hat{d})$ is a RbMS with $s = 2 > 1$. Now we have the following:

(i) $\hat{d}(\frac{1}{2}, \frac{1}{3}) = 2\alpha \geq \frac{83}{72}\alpha = \hat{d}(\frac{1}{2}, 2) + \hat{d}(2, 3) + \hat{d}(3, \frac{1}{3})$ and hence $(X, \hat{d})$ is not a rectangular metric space.

(ii) Here $\hat{d}(\frac{1}{n}, 2)$ and $\hat{d}(\frac{1}{n}, 3) \to 0$ as $n \to \infty$. So, the limit is not unique. Also, $\hat{d}(\frac{1}{n}, \frac{1}{n+1}) = 2\alpha \nrightarrow 0$ as $n \to \infty$. Therefore, $\{\frac{1}{n}\}$ is not a Cauchy sequence in RbMS.

(iii) $\lim_{n\to\infty} \hat{d}(\frac{1}{n}, 2) = 0 \neq \hat{d}(0, 2) = \alpha$. So, $\hat{d}$ is not a continuous function.

(iv) $B_{\frac{\alpha}{2}}(\frac{1}{2}) = \{2, 3, \frac{1}{2}\}$, *which is not an open set. Also, there does not exist any* $r_1, r_2 > 0$ *such that* $B_{r_1} \cap B_{r_2} = \phi$*. That means,* $(X, \hat{d})$ *is not Hausdorff.*

As we know, a sequence in a RbMS may have two limits. But, there is a special case where this is not possible, and this will be incorporated into our results, as shown by the following result.

Theorem 3.2.3. *(Roshan et al. (2016)) Let* $(X, \hat{d})$ *be a RbMS with* $s \geq 1$ *and let* $\{x_n\}$ *be a b-rectangular-Cauchy sequence in* X *such that* $x_n \neq x_m$ *whenever* $n \neq m$*. Then* $\{x_n\}$ *can converge to at most one point.*

3.3 Wardowski F-Contraction

Wardowski (2012) introduced a new type of contraction called F-contraction.

Definition 3.3.1. *Let* $(X, \hat{d})$ *be a metric space, then a mapping* $T : X \to X$ *is said to be a Wardowski F-contraction if there exists* $\tau > 0$ *such that* $\hat{d}(Tx, Ty) > 0$ *implies*

$$\tau + F(\hat{d}(Tx, Ty)) \leq F(\hat{d}(x, y)) \qquad ;\forall\ x, y \in X, \tag{3.2}$$

where, $F : (0, \infty) \to R$ *is a continuous mapping satisfying the following conditions:*

(F_1) *F is strictly increasing.*

(F_2) *For each sequence* $\{\alpha_n\}_{n \in N}$ *of positive numbers* $\lim_{n\to\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

(F_3) *There exists* $k \in (0, 1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Examples of such type of functions that satisfy all the properties from (F_1) to (F_3) are as follows:

Example 3.3.2. *1.* $F(x) = ln(x); x > 0.$

2. $F(x) = ln(x^2 + x); x > 0.$

3. $F(x) = x + ln(x); x > 0.$

4. $F(x) = \frac{-1}{x^p}; p > 0.$

5. $F(x) = \frac{1}{1-e^x}; x > 0.$

6. $F(x) = x - \frac{1}{x}; x > 0.$

Khan et al. (2016) derived fixed point results for F-contraction involving rational expressions in metric space. In the recent year, Sholastica et al. (2019) and Wangwe and Kumar (2021) obtained fixed point results on F-contraction. Vujaković et al. (2020) derived Wardowski type F-contraction fixed point results for the setting of four continuous maps and Fabiano et al. (2020) obtained fixed point theorems on W-contraction (F-contraction) of Jungck-Ciric-Wardowski type in metric space. Recently, Radenović et al. (2021) proved fixed point theorems on F-contraction with only one condition. i.e. F be strictly increasing.
Since F is an increasing function, it is easily seen that every Wardoski's F-contraction mapping is a contraction mapping and hence continuous (Goswami et al. (2019)). However, the mappings which have been found here need not be continuous. Also, the continuity of the metric space is not necessary.

3.4 F-contraction Fixed Point Results in RbMS

The next result generalizes Wardoski's theorem for the setting of four maps in RbMS.

Theorem 3.4.1. *Suppose that A, B, S, and T are self-maps on a complete rectangular b-metrics space (RbMS) X with coefficient* $s > 1$ *such that* $AX \subset TX, BX \subset SX$

and if there exists $\tau > 0$ such that $\hat{d}(Ax, By) > 0$ implies

$$\tau + F(\hat{d}(Ax, By)) \leq F(\alpha(x, y)) \quad ; \forall x, y \in X, \tag{3.3}$$

where,

$$\alpha(x, y) = max\{\hat{d}(Sx, Ty), \hat{d}(Ax, Sx), \hat{d}(By, Ty), \hat{d}(Ax, Ty)\}.$$

If one of the ranges AX, BX, TX and SX is a closed subset of $(X, \hat{d})$, then

(i) A and S have a coincidence point.

(ii) B and T have a coincidence point.

Moreover, if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, T and S have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $AX \subset TX$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$, and $BX \subset SX$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, \qquad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \qquad ; \forall n \in \mathbb{N}.$$

First of all it is shown that $\{y_n\}$ is a Cauchy sequence in the RbMS.
From the equation (3.3) with $x = x_{2k}$ and $y = x_{2k+1}$, one obtains

$$\tau + F(\hat{d}(Ax_{2k}, Bx_{2k+1})) \leq F(\alpha(x_{2k}, x_{2k+1})),$$

where,

$$\alpha(x_{2k}, x_{2k+1}) = max\{\hat{d}(y_{2k-1}, y_{2k}), \hat{d}(y_{2k+1}, y_{2k})\}.$$

If $\alpha(x_{2k}, x_{2k+1}) = \hat{d}(y_{2k+1}, y_{2k})$, then $\tau \leq 0$, which contradicts with $\tau > 0$.

This means $\alpha(x_{2k}, x_{2k+1}) = \hat{d}(y_{2k-1}, y_{2k})$. Therefore, one finds that

$$\tau + F(\hat{d}(y_{2k}, y_{2k+1})) \leq F(\hat{d}(y_{2k-1}, y_{2k})). \tag{3.4}$$

Similarly, one obtains

$$\tau + F(\hat{d}(y_{2k+1}, y_{2k+2})) \leq F(\hat{d}(y_{2k}, y_{2k+1})). \tag{3.5}$$

Therefore, from (3.4) and (3.5),

$$F(\hat{d}(y_n, y_{n+1})) \leq F(\hat{d}(y_n, y_{n-1})) - \tau \qquad ;\forall \ \ n \geq 1. \tag{3.6}$$

Likewise,

$$F(\hat{d}(y_n, y_{n+1})) \leq F(\hat{d}(y_{n-1}, y_{n-2})) - 2\tau.$$

Continuing this process, one arrives at

$$F(\hat{d}(y_n, y_{n+1})) \leq F(\hat{d}(y_0, y_1)) - n\tau. \tag{3.7}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} F(\hat{d}(y_n, y_{n+1})) = -\infty,$$

which together with condition (F2) of definition 3.3.1 gives

$$\lim_{n\to\infty} \hat{d}(y_n, y_{n+1}) = 0. \tag{3.8}$$

By using the condition (F3) of definition 3.3.1, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (\hat{d}(y_n, y_{n+1}))^k F(\hat{d}(y_n, y_{n+1})) = 0. \tag{3.9}$$

From (3.7), one infers the following for all $n \in \mathbb{N}$.

$$(\hat{d}(y_n, y_{n+1}))^k (F(\hat{d}(y_n, y_{n+1})) - F(\hat{d}(y_0, y_1))) \leq -(\hat{d}(y_n, y_{n+1}))^k n\tau \leq 0. \tag{3.10}$$

From (3.8),(3.9) and letting $n \to \infty$ in (3.10), one gets

$$\lim_{n\to\infty} (n(\hat{d}(y_n, y_{n+1}))^k) = 0. \tag{3.11}$$

This implies that, there exists $n_0 \in \mathbb{N}$, such that $n(\hat{d}(y_n, y_{n+1}))^k \leq 1$ for all $n \geq n_0$

$$\hat{d}(y_n, y_{n+1}) \leq \frac{1}{n^{1/k}} \qquad ; \forall n \geq n_0. \tag{3.12}$$

Since $(X, \hat{d})$ is a RbMS, one gets

$$\hat{d}(y_n, y_{n+p}) \leq s[\hat{d}(y_{n+p}, y_{n+p-1}) + \hat{d}(y_{n+p-1}, y_{n-1}) + \hat{d}(y_{n-1}, y_n)] \ \ ; p > 0.$$

Again using the same property of RbMS and from (3.12), one arrives at

$$\hat{d}(y_{n+p}, y_n) \leq \frac{s}{1-s}\{\frac{1}{(n+p-1)^{1/k}} + \frac{1}{(n-1)^{1/k}}\}.$$

Thus, $\{y_n\}$ is a b-rectangular Cauchy sequence. Since X is complete, there exists $w \in X$ such that

$$\lim_{n\to\infty} y_n = w, \tag{3.13}$$

which yields

$$\hat{d}(w, w) = \lim_{n\to\infty} \hat{d}(y_n, w) = 0.$$

Thus, one finds that

$$\lim_{n\to\infty} \hat{d}(Ax_{2n}, w) = \lim_{n\to\infty} \hat{d}(Tx_{2n+1}, w) = 0.$$

and

$$\lim_{n\to\infty} \hat{d}(Bx_{2n-1}, w) = \lim_{n\to\infty} \hat{d}(Sx_{2n}, w) = 0. \tag{3.14}$$

Now without loss of generality, one can suppose that SX is a closed subset of the RbMS $(X, \hat{d})$. From(3.14), there exists $z \in X$ such that $w = Sz$.

Claim: $\hat{d}(Az, w) = 0$.

Suppose to the contrary, $\hat{d}(Az, w) > 0$. For this, there exists an $n_1 \in \mathbb{N}$ such that $\hat{d}(Az, y_{2n}) > 0$ for all $n \geq n_1$ (otherwise, there exists $n_2 \in \mathbb{N}$ such that $y_n = Az$ for all $n \geq n_2$, which implies that $y_n \to Az$. This is a contradiction, since $w \neq Az$).

Since $\hat{d}(Az, y_{2n}) > 0$, from contractive condition (3.3), one gets

$$\tau + F(\hat{d}(Az, y_{2n})) \leq F(\alpha(z, x_{2n})), \tag{3.15}$$

where,

$$\alpha(z, x_{2n}) = max\{\hat{d}(Sz, y_{2n-1}), \hat{d}(Az, Sz), \hat{d}(y_{2n}, y_{2n-1}), \hat{d}(Az, y_{2n-1})\}.$$

Taking $n \to \infty$ in (3.15), one concludes that

$$\tau + F(\lim_{n\to\infty} \hat{d}(Az, y_{2n})) \leq F(\lim_{n\to\infty} \hat{d}(Az, y_{2n-1})),$$

which contradict with $\tau > 0$. So,

$$Az = w = Sz. \tag{3.16}$$

Hence, A and S have coincidence point z. Since, $AX \subset TX$ and from equation (3.16), we have $w \in TX$. So, there exists $v \in X$ such that $w = Tv$. With the use of a similar procedure, one can deduce that $Bv = w = Tv$. Hence, B and T have coincidence point v.

Since, the pair $\{A, S\}$ is weakly compatible, from (3.16) one comes across

$$Aw = ASz = SAz = Sw.$$

Next, one claims that $\hat{d}(Aw, w) = 0$. Suppose, $\hat{d}(Aw, w) > 0$.

From contractive condition (3.3), one can derive

$$\tau + F(\hat{d}(Aw, y_{2n})) \leq F(\alpha(w, x_{2n})), \tag{3.17}$$

where,

$$\alpha(w, x_{2n}) = max\{\hat{d}(Sw, y_{2n-1}), \hat{d}(Aw, Sw), \hat{d}(y_{2n}, y_{2n-1}), \hat{d}(Aw, y_{2n-1})\}.$$

Taking $n \to \infty$, from (3.17), one has

$$\tau + F(\lim_{n\to\infty} \hat{d}(Aw, y_{2n}) \leq F(\lim_{n\to\infty} \hat{d}(Aw, y_{2n-1})).$$

Again it contradicts the fact that $\tau > 0$.

So, $Aw = w = Sw$. Therefore, w is the common fixed point of A and S. Similarly, $Bw = w = Tw$.

Hence, w is the common fixed point of A, B, S and T. It is easy to check that w is the unique common fixed point. □

If one puts $A = B$ and $S = T$, the contractive condition (3.3) leads to be following result:

Corollary 3.4.2. *Suppose that A and T are self-maps on a complete rectangular b-metric space X with $s > 1$ such that $AX \subset TX$ and if there exists $\tau > 0$ such that $\hat{d}(Ax, Ay) > 0$ implies*

$$\tau + F(\hat{d}(Ax, Ay)) \leq F(\alpha(x, y)) \quad ;\forall x, y \in X, \tag{3.18}$$

where,

$$\alpha(x, y) = max\{\hat{d}(Tx, Ty), \hat{d}(Ax, Tx), \hat{d}(Ay, Ty), \hat{d}(Ax, Ty)\}.$$

If range AX or TX is a closed subset of $(X, \hat{d})$, then A and T have a coincidence point. Moreover, if the pair $\{A, T\}$ is weakly compatible, then A and T have a unique common fixed point.

Example 3.4.3. *Let $X = U \cup V$, where $U = \{1, \frac{1}{2}, \frac{1}{3}\}$ and V={2,3}.*
Define $\hat{d} : X \times X \to [0, \infty)$ such that $\hat{d}(x, y) = \hat{d}(y, x)$;$\forall x, y \in X$ and

$$\begin{aligned}
&\hat{d}(x, y) = 0 \ \textit{if} \quad x = y,\\
&\hat{d}(1, \frac{1}{2}) = \hat{d}(1, \frac{1}{3}) = \hat{d}(\frac{1}{2}, \frac{1}{3}) = 1;\ \hat{d}(1, 2) = \hat{d}(\frac{1}{2}, 2) = \hat{d}(\frac{1}{3}, 2) = \frac{1}{8},\\
&\hat{d}(1, 3) = \hat{d}(\frac{1}{2}, 3) = \hat{d}(\frac{1}{3}, 3) = \frac{1}{12};\ \hat{d}(2, 3) = \frac{1}{2}.
\end{aligned}$$

Note that $(X, \hat{d})$ is a RbMS with coefficient $s = 2$, but not a RMS and metric space.
Define the mappings T, $A : X \to X$ by $T(x) = \frac{1}{x}$ and $A(x) = \begin{cases} 1, & ;x \in U \\ x - 1 & ;x \in V. \end{cases}$
We have $AX \subset TX = X$.

(i) *When $x \in U$ and $y \in V$ (vice versa).*

For $y = 2$, we have a trivial case. So take $y = 3$, one gets

$$\begin{aligned}
\hat{d}(Ax, Ay) &= \hat{d}(1, 2) = \frac{1}{8}, \\
\hat{d}(Tx, Ty) &= \hat{d}(\frac{1}{x}, \frac{1}{y}) = \hat{d}(\frac{1}{x}, \frac{1}{3}) = 1 \quad or \quad \frac{1}{8} \quad or \quad \frac{1}{12}, \\
\hat{d}(Ax, Tx) &= \hat{d}(1, \frac{1}{x}) = 0 \quad or \quad \frac{1}{8} \quad or \quad \frac{1}{12}, \\
\hat{d}(Ay, Ty) &= \hat{d}(y-1, \frac{1}{y}) = \frac{1}{8}, \\
\hat{d}(Ax, Ty) &= \hat{d}(1, \frac{1}{y}) = 1.
\end{aligned}$$

Hence,

$$\alpha(x, y) = 1.$$

From (3.18)

$$\tau + ln\frac{1}{8} \leq ln1.$$

Implies that

$$\tau \leq ln8.$$

(ii) *When $x \in V$ and $y \in V$.*

It is trivial if $x = y$. So, we should take $x = 2$ and $y = 3$ (vice versa).

One obtains

$$\begin{aligned}
\hat{d}(Ax, Ay) &= \hat{d}(1, 2) = \frac{1}{8}, \\
\hat{d}(Tx, Ty) &= \hat{d}(\frac{1}{x}, \frac{1}{y}) = \hat{d}(\frac{1}{2}, \frac{1}{3}) = 1, \\
\hat{d}(Ax, Tx) &= \hat{d}(1, \frac{1}{2}) = 1, \\
\hat{d}(Ay, Ty) &= \hat{d}(y-1, \frac{1}{y}) = \hat{d}(2, \frac{1}{3}) = \frac{1}{8}, \\
\hat{d}(Ax, Ty) &= \hat{d}(1, \frac{1}{y}) = \hat{d}(1, \frac{1}{3}) = 1.
\end{aligned}$$

From (3.18)

$$\tau + ln\frac{1}{8} \leq ln1.$$

That is

$$\tau \leq ln8.$$

(iii) *When $x \in U$ and $y \in U$, The case is trivial.*

let $\tau = ln8$ and $F(x) = lnx$. The equation (3.18) is satisfied. Hence, 1 is the unique common fixed point of A and T.

If we put $A = B$ and $S = T = I$ (the identity map on X) in (3.3), we obtain following:

Corollary 3.4.4. *Let A be a self-map on a complete rectangular b-metric space X with $s > 1$ and if there exists $\tau > 0$ such that $\hat{d}(Ax, Ay) > 0$ implies*

$$\tau + F(\hat{d}(Ax, Ay)) \leq F(\alpha(x, y)), \tag{3.19}$$

where,

$$\alpha(x, y) = max\{\hat{d}(x, y), \hat{d}(x, Ax), \hat{d}(y, Ay)\}.$$

Then A has a unique fixed point in X.

Example 3.4.5. *We have seen that the function A given in Example 3.4.3 with the same metric space satisfies corollary 3.4.4 for $\tau = ln2$ and $F(x) = lnx$.*

Next, in the sequel, the following is proven.

Theorem 3.4.6. *Suppose that T and S are self-maps on a complete rectangular b-metric space X with $s > 1$ and if there exists $\tau > 0$ such that $\hat{d}(Tx, Ty) > 0$ implies*

$$\tau + F(\hat{d}(Tx, Ty)) \leq F(\hat{d}(Sx, Sy)). \tag{3.20}$$

If range TX or SX is a closed subset of $(X, \hat{d})$, then T and S have a coincidence point. Moreover, if the pair $\{S, T\}$ is weakly compatible, then T and S have a unique common fixed point.

Proof. Consider the sequence $\{x_n\}$, where $y_n = Tx_n = Sx_{n+1}$. Adopting a similar process as in the previous theorem, it is easy to prove that S and T have a unique common fixed point. □

Now, If we take $S = I$(the identity map on X), we have Wardoski's F-contraction.

Corollary 3.4.7. *Let $(X, \hat{d})$ be a RbMS with $s > 1$ and $T : X \to X$ satisfying the following with $\tau > 0$,*

$$\tau + F(\hat{d}(Tx, Ty)) \leq F(\hat{d}(x, y)). \tag{3.21}$$

Then T has a unique fixed point in X.

Example 3.4.8. *Let $X = \{1, 2, 3, 4\}$. Define $\hat{d} : X \times X \to [0, \infty)$ such that $\hat{d}(x, y) = \hat{d}(y, x)$ for all $x, y \in X$ and*

$$\hat{d}(1, 2) = 10; \hat{d}(1, 3) = \hat{d}(2, 3) = 1; \hat{d}(1, 4) = \hat{d}(2, 4) = \hat{d}(3, 4) = 2.$$

Then, $(X, \hat{d})$ is a RbMS with coefficient $s = 2(> 1)$, but not a RMS.
Define $T : X \to X$ by $T(x) = \begin{cases} 1, & ; x = 1, 2, 3 \\ 3 & ; x = 4. \end{cases}$
T satisfies equation (3.21) for $\tau = ln2$ and $F(\alpha) = ln\alpha$. So, 1 is the unique fixed point of T.

3.5 Application

At last, we find the existence and uniqueness of the solution for the following system of the integral equation of Volterra type:

$$\begin{aligned} u(t) &= f(t) + \int_0^t K_1(t,s,u(s))\,ds \\ u(t) &= f(t) + \int_0^t K_2(t,s,u(s))\,ds \\ u(t) &= f(t) + \int_0^t K_3(t,s,u(s))\,ds \\ u(t) &= f(t) + \int_0^t K_4(t,s,u(s))\,ds \end{aligned} \tag{3.22}$$

where $t \in [0,a], a > 0$ and $K_i : [0,a] \times [0,a] \times R \to R$ $(i \in 1,2,3,4)$ and $f : R \to R$ are continuous functions. For $u \in C([0,a],R) = X(say)$, define supremum norm as:

$$\|u\|_\tau = \sup_{t\in[0,a]} |u(t)|^2 e^{-\tau t},$$

where $\tau > 0$ is taken arbitrary. Let $C([0,a],R)$ be endowed with the metric

$$\hat{d_\tau}(u,v) = \sup_{t\in[0,a]} \{|u(t)-v(t)|^2 e^{-\tau t}\} \qquad ;\forall u,v \in C([0,a],R).$$

Here $(X, \hat{d_\tau})$ is a complete RbMS with $s = 3/2$. Notice that it is not a metric space and RMS.

Let $I = [0,a]$ and defined $T_i : C(I,R) \to C(I,R)$ defined by

$$T_i u(t) = f(t) + \int_0^t K_i(t,s,u(s))ds, \tag{3.23}$$

$\forall\ u \in C(I,R),\ t \in I,\ i \in \{1,2,3,4\}$. Clearly, u^* is a solution of (3.22) if and only if it is a common fixed point of T_i for $i\ \in \{1,2,3,4\}$.

We are equipped with the following condition to prove our result.

Theorem 3.5.1. *Suppose that the following hypothesis hold:*

1. *For all* $t \in I, u \in C(I, R)$,

   $T_1T_4u(t) = T_4T_1u(t)$, *whenever* $T_1u(t) = T_4u(t)$,

   $T_2T_3u(t) = T_3T_2u(t)$, *whenever* $T_2u(t) = T_3u(t)$.

2. *Assume that there exist* $\tau > 1$, *such that*

$$|K_1(t, s, u) - K_2(t, s, v)|^2 \leq \tau e^{-\tau} |\alpha(u, v)|^2,$$

   $\forall t, s \in [0, a]$ *and* $u, v \in X$, *where*

$$\alpha(u, v) = max\{|Su - Tv|, |Au - Su|, |Bv - Tv|, |Au - Tv|\}.$$

Then (3.22) has a unique solution u^* *(say).*

Proof. By the above assumption, we have

$$\begin{aligned}
|T_1u(t) - T_2v(t)|^2 &\leq \int_0^t |K_1(t, s, u(s) - K_2(t, s, v(s)|^2 ds \\
&\leq \int_0^t \tau e^{-\tau}(|\alpha(u, v)|^2 e^{-\tau s})e^{\tau s} ds \\
&\leq \tau e^{-\tau} \|\alpha(u, v)\|_\tau \frac{1}{\tau} e^{\tau t} \\
\tau + ln\|T_1u(t) - T_2v(t)\|_\tau &\leq ln\|\alpha(u, v)\|_\tau.
\end{aligned}$$

This implies $\tau + F(\hat{d}(T_1u, T_2v)) \leq F(\alpha(u, v))$ where, $F(x) = lnx$.

Putting $A = T_1, B = T_2, T = T_3$ and $S = T_4$, Theorem 3.3 is satisfied. Therefore A, B, S and T have a unique common fixed point $u^* \in C(I, R)$; i.e, u^* is a unique solution of system (3.22). □

3.6 Conclusion

Throughout the chapter, we have generalized Wardoski's F-contraction fixed point theorem in Rectangular b-metric space. An example is also provided for the justification of our results. Finally, we successfully apply our result to examine the existence and uniqueness of the system of Volterra integral equations. System of Volterra integral equations appear in scientific applications in engineering, physics, chemistry, and populations growth models (one may refer Zakwv and Uniady (2016), Jerri (1999), Porter and Stirling (2004), Linz (1974), Linz (1985), Wazwaz (2011)).