Chapter 4

F-contraction Fixed Point Results in Ω-extended Rectangular b-metric Space

In the present chapter, we prove a common fixed point theorem for F-contraction by considering two maps in which one map is orbitally continuous in $\Omega-$Extended rectangular b-metric space. In addition, we find a fixed point for Banach and Kannan type contraction inequality without consideration of orbitally continuous map. Also, our contractive condition is strong enough to generate a fixed point but does not force the mapping to be continuous at the fixed point. Examples are also provided to demonstrate the utility of our findings. This chapter is inspired by Mustafa et al. (2019a) and Lukács and Kajántó (2018).

4.1 Introduction

Kannan's fixed point theorems are regarded as the origin of the question of the continuity of contractive mappings at the fixed point (Kannan (1968), Kannan (1969)).

Following that, Rhoades (1988) raised the question as an open problem "Does there exists a contractive definition that is powerful enough to yield a fixed point but does not force the mapping to be continuous at the fixed point?". Pant (1999) established two fixed point theorems in which the mappings were discontinuous at the fixed points, yielding positive solutions to the Rhoades problem. Some new solutions to this problem with neural network applications have been published in [Bisht and Pant (2017b), Bisht and Pant (2017a), Bisht and Özgür (2020), Bisht and Rakočević (2018b), Bisht and Rakočević (2018a), Rashid et al. (2018), Özgür and Tas (2019), Garai et al. (2020), Pant et al. (2019), Zheng and Wang (2017)]. Fixed point theorems for discontinuous mappings have a wide range of applications, such as neural networks, which are commonly employed in character recognition, stock market prediction, image compression, and solving non-negative sparse approximation issues (Ding et al. (2017), Forti and Nistri (2003), Nie and Zheng (2014), Nie and Zheng (2015a), Nie and Zheng (2015b)).

The Banach contraction principle has been generalized in many ways over the years. In some generalizations, the contractive nature of the map is weakened. In other generalizations, the topology is weakened. In the previous chapter, we discussed that RbMS is not necessarily Hausdorff topology, and we used Wardowski (2012) F-contraction to derive some results.

After Wardowski (2012) contribution to fixed point theory, several papers have dealt with the F-contraction mappings and their extensions (see Erhan et al. (2012), Khan et al. (2016)). In F-contraction, Cosentino et al. (2015) introduced the additional condition (F_4) and obtain some results in b-metric spaces. Then Lukács and Kajántó (2018) defined F-Contraction as follows:

Definition 4.1.1. *(Lukács and Kajántó (2018)) Let* $(X, \hat{d})$ *be a b-metric space with constant* $s \geq 1$*and* $T : X \to X$ *is said to be a F-contraction if there exists* $\tau > 0$

such that $\hat{d}(Tx, Ty) > 0$ *implies*

$$\tau + F(s \cdot \hat{d}(Tx, Ty)) \leq F(\hat{d}(x, y)) \quad ;\forall x, y \in X, \tag{4.1}$$

where, $F : (0, \infty) \to R$ *belongs to* $\mathcal{F}_{s,\tau}$ *satisfying* (F_1) *to* (F_3) *[chapter-3: Definition (3.3.1)] and the following additional condition* (F_4).

(F_4) *Let* $s \geq 1$ *be a real number. For each sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *of positive numbers such that* $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ *for all* $n \in \mathbb{N}$ *and some* $\tau > 0$*, then* $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$ *for all* $n \in \mathbb{N}$.

Goswami et al. (2019) introduced F-contractive type mapping which is a combination of F-contraction by Wardowski as well as Kannan contraction mappings and obtained a fixed point in b-metric space with all conditions (F_1) to (F_4). It is easy to see that, F-contraction mapping is continuous as F is increasing but F-contractive type mapping need not be continuous. So, Goswami et al. (2019) weakened the contractive nature of the map. We use a class of all functions that satisfy (F_1) and (F_2), which further weakens the nature of contractive mapping.

In chapter-3, we discussed the concept of rectangular b-metric space. Recently Mustafa et al. (2019a) introduced the $\Omega-$extended rectangular b-metric space by multiplying with the function Ω in place of constant s in RbMS. $\Omega-$extended rectangular b-metric space is a generalization of both rectangular metric space and rectangular b-metric space. In the next section, we discuss Ω-extended RbMS and its properties.

4.2 Ω-extended RbMS and its Properties

Definition 4.2.1. *(Mustafa et al. (2019a)) Let X be a nonempty set, $\Omega : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $t \leq \Omega(t)$ for all $t > 0$ and $0 = \Omega(0)$ and let $\tilde{r} : X \times X \to [0,\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y satisfies the following conditions:*

(1) $\tilde{r}(x,y) = 0$ *iff* $x = y$,

(2) $\tilde{r}(x,y) = \tilde{r}(y,x)$,

(3) $\tilde{r}(x,y) \leq \Omega[\tilde{r}(x,u) + \tilde{r}(u,v) + \tilde{r}(v,y)]$.

Then $(X,\tilde{r})$ is called an Ω-extended rectangular b- metric space(in short Ω-ERbMS).

Remark 4.2.1.1. *(i)* $\Omega^{-1}(t) \leq t$ *for all* $t > 0$ *and* $0 = \Omega(0)$.

(ii) Each rectangular b-metric space is an ERbMS with $\Omega(t) = st$, $s \geq 1$.

In $\Omega-$ERbMS, the concepts of convergence, Cauchy sequence, and completeness are specified in a standard manner.

It is clear from the Example 3.2.2 that a sequence in $\Omega-$ERbMS may have more than one limit. One may see where this is not possible by using the following theorem.

Theorem 4.2.2. *(Mustafa et al. (2019a)) Let $(X,\tilde{r})$ be an Ω-ERbMS and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

The following lemma can be used to control the discontinuity of the Ω-ERbMS while proving our results.

Lemma 4.2.3. *(Mustafa et al. (2019a)) Let $(X,\tilde{r})$ be an ERbMS with the function Ω, then we have the following:*

(i) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $x_n \to x, y_n \to y$ *and the elements of* $\{x, y, x_n, y_n : n \in \mathbb{N}\}$*are totally distinct. Then, we have*

$$\Omega^{-1}(\tilde{r}(x,y)) \leq \liminf_{n\to\infty} \tilde{r}(x_n, y_n) \leq \limsup_{n\to\infty} \tilde{r}(x_n, y_n) \leq \Omega(\tilde{r}(x,y)).$$

(ii) *Let* $\{x_n\}$ *be a Cauchy sequence in* X *converging to* x. *If* x_n *has infinitely many distinct terms, then*

$$\Omega^{-1}(\tilde{r}(x,y)) \leq \liminf_{n\to\infty} \tilde{r}(x_n, y) \leq \limsup_{n\to\infty} \tilde{r}(x_n, y) \leq \Omega(\tilde{r}(x,y)),$$

for all $y \in X$ *with* $x \neq y$.

Pant et al. (2020) and Bisht and Rakočević (2020) derived some fixed point results in metric space by considering orbitally continuous map. Also, they demonstrated that the map does not force it to be continuous at a fixed point. In this chapter, we try to weaken the continuity of one map by taking orbitally continuous and deriving a common fixed point result for two maps. Next, we discuss the examples related to orbitally continuous maps.

4.3 Orbitally Continuous

In chapter-1 (1.2.5), we already defined orbitally continuous map. As we know, a continuous function is always orbitally continuous but the converse may not be true. The following example illustrates this fact.

Example 4.3.1. *(Bisht and Pant (2017b)) Let* $X = [0, 2]$, *the map* $f : X \to X$ *defined by*

$$f(x) = 1 \quad if \quad x \in [0, 1], f(x) = 0 \quad if \quad x \in (1, 2].$$

It is clear that f is orbitally continuous but not continuous at $x = 1$.

Next, the following examples which we are going to present are not orbitally continuous.

Example 4.3.2. *Let $X = [0, 1)$ and the map $f : X \to X$ defined by*

$$f(x) = \begin{cases} 0 & ; x = 0, \\ \frac{1}{2} - x, & ; 0 < x \leq \frac{1}{2}, \\ \frac{3}{2} - x, & ; \frac{1}{2} < x < 1. \end{cases}$$

Sequence $\{\frac{1}{2^n}\} \to 0$, but $f(\frac{1}{2^n}) \nrightarrow f(0)$, So f is not orbitally continuous.

Example 4.3.3. *Let $X = A \cup B$, where $A = \{\frac{1}{n}\ ; n = 2, 3, 4, ...\}$ and $B = \{0, 1, 2, 3, ...\}$. The map $f : X \to X$ defined by*

$$f(x) = \begin{cases} 1 & ; x = 0, \\ \frac{x}{2}, & ; x \in A, \\ \frac{1}{x}, & ; x \in B - \{0\}. \end{cases}$$

Here $\{\frac{1}{n}\} \to 0$, but $f(\frac{1}{n}) \nrightarrow f(0)$, hence f is not orbitally continuous.

Now, we are ready to present the main result of this chapter.

4.4 Fixed Point Results on Ω-extended RbMS

Theorem 4.4.1. *Let $(X, \tilde{r})$ be a complete ERbMS with non-trivial function Ω (i.e., $\Omega(t) \neq t$). Let f and g be commuting mappings into itself which satisfy the following:*

(i) if their exists $\tau > 0$ such that

$$\tau + F(\Omega[\tilde{r}(fx, fy)]) \leq F(\tilde{r}(gx, gy)) \qquad ; \forall x, y \in X. \tag{4.2}$$

(ii) Either f or g is orbitally continuous and the range of g contains the range of f.

Then f and g have a unique common fixed point, say z. Moreover, f and g are continuous at common fixed point z iff $\lim_{x\to z} max\{\tilde{r}(x, fx)+\tilde{r}(x, gx), \tilde{r}(z, fz)\} = 0$.

Proof. Let $x_0 \in X$ be arbitrary. Then fx_0 and gx_0 are well defined. Since $fx_0 \in g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. Continuing this process, if x_n is chosen, then we choose a point x_{n+1} in X such that $gx_{n+1} = fx_n$.

Step I : We will prove that $\lim_{n\to\infty} \tilde{r}(gx_{n+1}, gx_n) = 0$.

From the contractive condition (4.2), one has

$$\begin{aligned} \tau + F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) &= \tau + F(\Omega[\tilde{r}(fx_n, fx_{n-1})]) \\ &\leq F(\tilde{r}(gx_n, gx_{n-1})). \end{aligned}$$

This implies

$$F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) \leq F(\tilde{r}(gx_n, gx_{n-1})) - \tau. \tag{4.3}$$

Since, $t \leq \Omega(t)$, F is strictly increasing, one can write with the use of (4.3)

$$\begin{aligned} F(\tilde{r}(gx_{n+1}, gx_n)) &< F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) \\ &\leq F(\tilde{r}(gx_n, gx_{n-1})) - \tau \\ &\vdots \\ &< F(\tilde{r}(gx_1, gx_0)) - n\tau. \end{aligned}$$

Then

$$\limsup_{n\to\infty} F(\tilde{r}(gx_{n+1}, gx_n)) = \liminf_{n\to\infty} F(\tilde{r}(gx_{n+1}, gx_n)) = \lim_{n\to\infty} F(\tilde{r}(gx_{n+1}, gx_n)) = -\infty,$$

which together with (F2) of definition 3.3.1 gives

$$\lim_{n\to\infty} \tilde{r}(gx_{n+1}, gx_n) = 0. \tag{4.4}$$

Step 2 : We will show that $gx_n \neq gx_m$ for $n \neq m$.

Case (i) If $gx_n = gx_{n+1}$ for some n, then $fx_n = gx_n = u$, for some n.

This yields

$$fu = fgx_n = gfx_n = gu. \tag{4.5}$$

Now our claim is to prove $\tilde{r}(u, fu) = 0$. On the other hand, let $\tilde{r}(u, fu) > 0$.

Using contractive condition (4.2), one comes across

$$F(\ \tilde{r}(fx_n, fu)) < F(\Omega\ \tilde{r}(fx_n, fu)) \leq F(\tilde{r}(gx_n, gu)) - \tau.$$

By (4.5)

$$F(\ \tilde{r}(fx_n, fu)) < F(\tilde{r}(fx_n, fu)),$$

which is absurd. Hence our assumption is wrong.

$$fu = u = gu.$$

Hence, u is the common fixed point of f and g.

Case (ii): If $gx_n \neq gx_{n+1}$ for all $n \geq 0$, then $gx_n \neq gx_{n+k}$ for all $n \geq 0, k \geq 1$.

If $gx_n = gx_{n+k}$ for some $n \geq 0, k \geq 1$, then

$$\begin{aligned} F(\tilde{r}(gx_{n+1}, gx_n)) < F(\Omega[\tilde{r}(gx_{n+1}, gx_n)]) &= F(\Omega[\tilde{r}(gx_{n+k+1}, gx_{n+k})]) \\ &\leq F(\tilde{r}(gx_{n+k}, gx_{n+k-1})) - \tau \\ &< F(\Omega[\tilde{r}(gx_{n+k}, gx_{n+k-1})]) - \tau \\ &\leq F(\tilde{r}(gx_{n+k-1}, gx_{n+k-2})) - 2\tau \\ &\vdots \\ &< F(\tilde{r}(gx_{n+1}, gx_n)) - k\tau \end{aligned}$$

$F(\tilde{r}(gx_{n+1}, gx_n)) < F(\tilde{r}(gx_{n+1}, gx_n))$, which is a contradiction.

Hence $gx_n \neq gx_{n+k}$ for all $n \geq 0, k \geq 1$. Therefore, we can assume that $gx_n \neq gx_m$ for $n \neq m$.

Step 3 : Now it is shown that $\{gx_n\}$ is an $\tilde{r}$-Cauchy sequence. Suppose to the contrary that there exists $\epsilon > 0$ for which we can find two subsequences $\{gx_{m_i}\}$ and $\{gx_{n_i}\}$ of $\{gx_n\}$ such that m_i is the smallest index, where

$$m_i > n_i > i \quad and \quad \tilde{r}(gx_{n_i}, gx_{m_i}) \geq \epsilon. \tag{4.6}$$

It means that

$$\tilde{r}(gx_{n_i}, gx_{m_i-2}), \tilde{r}(gx_{n_i}, gx_{m_i-1}) < \epsilon. \tag{4.7}$$

Using the $\Omega-$rectangular inequality and (4.6), one obtains

$$\epsilon \leq \tilde{r}(gx_{n_i}, gx_{m_i}) \leq \Omega[\tilde{r}(gx_{n_i}, gx_{n_i+1}) + \tilde{r}(gx_{n_i+1}, gx_{m_i-1}) + \tilde{r}(gx_{m_i-1}, gx_{m_i})],$$

which together with (4.4) and taking the upper limit as $i \to \infty$, we have

$$\Omega^{-1}(\epsilon) \leq \limsup_{i\to\infty} \tilde{r}(gx_{n_i+1}, gx_{m_i-1}). \tag{4.8}$$

Again from the $\Omega-$rectangular inequality, one finds that

$$\tilde{r}(gx_{n_i+1}, gx_{m_i-2}) \leq \Omega[\tilde{r}(gx_{n_i+1}, gx_{n_i}) + \tilde{r}(gx_{n_i}, gx_{m_i-1}) + \tilde{r}(gx_{m_i-1}, gx_{m_i-2})].$$

Taking the upper limit as $i \to \infty$, From (4.4) and (4.7), one arrives at

$$\limsup_{i\to\infty} \tilde{r}(gx_{n_i+1}, gx_{m_i-2}) \leq \Omega(\epsilon). \tag{4.9}$$

Since F is strictly increasing and with the use of inequalities (4.7) and (4.8), one gets

$$\begin{aligned} F(\epsilon) &= F(\Omega[\Omega^{-1}(\epsilon)]) \\ &\leq F(\Omega[\limsup_{n\to\infty} \tilde{r}(gx_{n_i+1}, gx_{m_i-1})]) \\ &\leq F(\limsup_{n\to\infty} \tilde{r}(gx_{n_i}, gx_{m_i-2})) \\ &< F(\epsilon), \end{aligned}$$

a contradiction. Thus, $\{gx_n\}$ is a $\tilde{r}-$Cauchy sequence in X. Since $(X, \tilde{r})$ is a complete $\Omega-$ERbMS. So, there exists $z \in X$ such that

$$\lim_{n\to\infty} gx_n = z. \tag{4.10}$$

Which yields $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n-1} = z$.

Step 4 : In this we will prove that z is the coincidence point of f and g. i.e. $fz = gz$. With the use of $\Omega-$rectangular inequality, one finds that

$$\tilde{r}(fz, gz) \leq \Omega[\tilde{r}(fz, fgx_n) + \tilde{r}(fgx_n, fgx_{n-1}) + \tilde{r}(fgx_{n-1}, gz)].$$

Letting limit supremum, one has

$$\limsup_{n\to\infty}\tilde{r}(fz,gz)\leq\Omega[\limsup_{n\to\infty}\tilde{r}(fz,fgx_n)+\limsup_{n\to\infty}\tilde{r}(fgx_n,fgx_{n-1})+\limsup_{n\to\infty}\tilde{r}(fgx_{n-1},gz)].$$

Without loss of generality, we can assume that f is orbitally continuous. Also, we have, f and g are commutative. Applying Lemma 4.2.3 and equation (4.10), we get

$$\lim_{n\to\infty}\tilde{r}(fz,gz)=\limsup_{n\to\infty}\tilde{r}(fz,gz)\leq\Omega(0).$$

One conclude that $fz=gz$.

Step 5 : At last we will prove that, z is the unique common fixed point of f and g. At first, we will prove that $gz=z$.

$$\begin{aligned}
F[(\tilde{r}(gx_n,gz))]=F[(\tilde{r}(fx_{n-1},fz))] &< F[\Omega(\tilde{r}(fx_{n-1},fz))]\\
&\leq F(\tilde{r}(gx_{n-1},gz))-\tau\\
&< F[\Omega(\tilde{r}(gx_{n-2},gz))]-2\tau\\
&\vdots\\
&< F[\tilde{r}(gx_0,gz)]-n\tau.
\end{aligned}$$

Taking limit as $n\to\infty$ and from (F_2) of definition 3.3.1, we have

$$\lim_{n\to\infty}F[\tilde{r}(gx_n,gz))]=-\infty.$$

This implies

$$\lim_{n\to\infty}\tilde{r}(gx_n,gz)=0.$$

Since, Cauchy sequence $\{gx_n\}$ converges to both z and gz, it is clear that $gz=z$. Thus $gz=z=fz$. It is easy to check that z is the unique common fixed point.

For the second part, let f and g are continuous at fixed point z, then for the sequence $\{gx_n\}$ of (4.10), we have $\lim_{n\to\infty} fgx_n = fz = z$ and $\lim_{n\to\infty} ggx_n = gz = z$. That is

$$\tilde{r}(gx_n, fgx_n) = 0 \quad \textit{and} \quad \tilde{r}(gx_n, ggx_n) = 0.$$

So,

$$\lim_{x\to z} max\{\tilde{r}(gx_n, fgx_n) + \tilde{r}(gx_n, ggx_n), \tilde{r}(z, fz)\} = 0.$$

Conversely, let $\lim_{x\to z} max\{\tilde{r}(x, fx) + \tilde{r}(x, gx), \tilde{r}(z, fz)\} = 0$. Then,

$$\lim_{n\to\infty} \{\tilde{r}(gx_n, fgx_n) + \tilde{r}(gx_n, ggx_n)\} = 0.$$

Then it is obvious that

$$\lim_{n\to\infty} \tilde{r}(gx_n, fgx_n) = 0, \lim_{n\to\infty} \tilde{r}(gx_n, ggx_n) = 0.$$

It is clear that f and g are continuous at fixed point z. □

The following example shows the novelty of our result.

Example 4.4.2. *Let $X = A \cup B$, where $A = [0, \frac{1}{3}]$ and $B = (\frac{1}{3}, 1)$. Define $\tilde{r} : X \times X \to [0, \infty)$ such that $\tilde{r}(x, y) = \tilde{r}(y, x)$ for all $x, y \in X$ and*

$$\tilde{r}(x, y) = \begin{cases} 0, & ; x = y \\ \frac{1}{16} & ; x, y \in A \\ 1 & ; x, y \in B \\ \frac{1}{4} & ; otherwise. \end{cases} \tag{4.11}$$

Then $(X, \tilde{r})$ is a $\Omega-ERbMS$ with $\Omega(t) = 2t$, which is not a rectangular metric space.

The mappings $f, g : X \to X$ *defined by* $f(x) = \frac{1}{3}$ $; x \in A \cup B$ *and*

$$g(x) = \begin{cases} \frac{1}{3} & ; x \in A - \{0\} \\ \frac{4}{3} - x & ; x \in B \\ 0 & ; x = 0. \end{cases}$$

Here we have $R(f) \subset R(g)$*,* f *and* g *are commutative and* f *is orbitally continuous. A sequence* $\{\frac{1}{3^n}\} \to 0$*, but* $g(\frac{1}{3^n}) \nrightarrow g(0)$*, so* g *is not orbitally continuous map. For all* $x, y \in X$*, we have* $\tilde{r}(fx, fy) = \tilde{r}(\frac{1}{3}, \frac{1}{3}) = 0$*, which is trivially hold. We conclude that the equation* (4.2) *is satisfied. Thus* f *and* g *have unique common fixed point* $\frac{1}{3}$*.*

If we put $gx = I_x$(identity map) then (4.2) turns into Banach type contractive condition. To prove the below theorem, the map need not be orbitally continuous.

Theorem 4.4.3. *Let* f *be a self map on a complete extended rectangular b-metric space* X *with the non-trivial function* Ω*, and if there exists* $\tau > 0$ *such that*

$$\tau + F(\Omega[\tilde{r}(fx, fy)]) \leq F(\tilde{r}(x, y)) \qquad ; \forall x, y \in X. \tag{4.12}$$

Then f *has a unique fixed point.*
Moreover, f *is continuous at fixed point* z *iff* $\lim_{x \to z} max\{\tilde{r}(x, fx), \tilde{r}(z, fz)\} = 0$.

Proof. Let the sequence $\{x_n\}$ defined by $fx_n = x_{n+1}$. It is easy to prove that

$$\lim_{n \to \infty} \tilde{r}(x_{n+1}, x_n) = 0. \tag{4.13}$$

Next, we will prove $x_n \neq x_m$ for $n \neq m$. Suppose to the contrary, $x_n = x_m$ for some $n > m$ then $x_{n+1} = fx_n = fx_m = x_{m+1}$. By continuing this process, one has

$x_{n+k} = x_{m+k}$ for all $k \in N$. Then from inequality (4.12),

$$\begin{aligned} F(\tilde{r}(x_n, x_{n+1})) < F(\Omega[\tilde{r}(x_m, x_{m+1})]) &\leq F(\tilde{r}(x_n, x_{n+1})) - \tau \\ &< F(\tilde{r}(x_n, x_{n+1})), \end{aligned}$$

a contradiction. Hence $x_n \neq x_m$ for $n \neq m$. In a similar way, as the previous theorem, one can easily prove that $\{x_n\}$ is a Cauchy sequence and hence convergent to z.

$$\lim_{n\to\infty} x_n = z. \tag{4.14}$$

Next, one arrives at

$$\begin{aligned} F(\Omega[\tilde{r}(fx_{n-1}, fz)]) &\leq F(\tilde{r}(x_{n-1}, fz)) - \tau \\ &< F(\Omega[\tilde{r}(fx_{n-2}, fz)]) - \tau \\ &\leq F(\tilde{r}(x_{n-2}, fz)) - 2\tau \\ &\vdots \\ &< F[\tilde{r}(x_0, fz)] - n\tau. \end{aligned}$$

Taking limit as $n \to \infty$ and from the definition of F-contraction (F_2) (3.3.1), we have

$$\lim_{n\to\infty} F[\tilde{r}(fx_{n-1}, fz))] = -\infty.$$

This implies

$$\lim_{n\to\infty} \tilde{r}(x_n, fz) = 0.$$

Since the Cauchy sequence $\{x_n\}$ converges to both z and fz, it must be the case $fz = z$. It is easy to check that z is the unique fixed point. The second part can be similarly proved as the previous Theorem 4.4.1. □

Next, we present an example, which supports the above result.

Example 4.4.4. *Let* $X = A \cup B$*, where* $A = [0, \frac{1}{3}]$ *and* $B = (\frac{1}{3}, 1)$*. Define* $\tilde{r} : X \times X \to [0, \infty)$ *such that* $\tilde{r}(x, y) = \tilde{r}(y, x)$ *for all* $x, y \in X$ *and*

$$\tilde{r}(x,y) = \begin{cases} 0, & ; x = y \\ \frac{1}{16} & ; x, y \in A \ ; x \ or \ y \neq 0. \\ 1 & ; x, y \in B \\ \frac{1}{4} & ; otherwise. \end{cases} \tag{4.15}$$

Then $(X, \tilde{r})$ *is a* $\Omega-ERbMS$ *with* $\Omega(t) = 2t$*, which is not a rectangular metric space. The mappings* $f : X \to X$ *defined by*

$$f(x) = \begin{cases} \frac{1}{3} & ; x \in A - \{0\} \\ \frac{x}{3} & ; x \in B \\ \frac{1}{4} & ; x = 0. \end{cases}$$

Here f *is not orbitally continuous. For* $F(x) = lnx(x > 0)$ *and* $\tau = ln2$*, all the conditions required in Theorem* 4.4.3 *are satisfied. Hence* $\{\frac{1}{3}\}$ *is the unique fixed point at which the map is discontinuous.*

In the sequel, we have taken Kannan type contractive condition.

Theorem 4.4.5. *Let* f *be a self map on a complete extended rectangular b-metric space* X *with non-trivial function* $\Omega(i.e., \Omega(t) \neq t)$ *and if there exists* $\tau > 0$ *such that*

$$\tau + F(\Omega[\tilde{r}(fx, fy)]) \leq \frac{1}{2}\{F(\tilde{r}(x, fx)) + F(\tilde{r}(y, fy))\} \qquad ; \forall x, y \in X. \tag{4.16}$$

Then f *has a unique fixed point. Moreover,* f *is continuous at fixed point* z *iff*

$\lim_{x\to z} max\{\tilde{r}(x, fx), \tilde{r}(z, fz)\} = 0$.

Proof. Let the sequence $\{x_n\}$ defined by $fx_n = x_{n+1}$. It is clear that

$$\lim_{n\to\infty} \tilde{r}(x_{n+1}, x_n) = 0. \tag{4.17}$$

Now we will show that $x_n \neq x_m$ for $n \neq m$. Suppose to the contrary, $x_n = x_m$ for some $n > m$ then $x_{n+1} = fx_n = fx_m = x_{m+1}$. By continuing this process, one has $x_{n+k} = x_{m+k}$, for all $k \in \mathbb{N}$. Let $\mu_n = \tilde{r}(x_n, x_{n+1})$, then from inequality (4.16),

$$\begin{aligned} F(\mu_m) = F(\mu_n) < F(\Omega[\mu_n]) &\leq F(\mu_{n-1}) - 2\tau \\ &< F(\mu_{n-2}) - 4\tau \\ &\vdots \\ &< F(\mu_m), \end{aligned} \tag{4.18}$$

a contradiction. Hence $x_n \neq x_m$ for $n \neq m$. From equation (4.16),

$$\tau + F(\Omega[\tilde{r}(x_{n+1}, x_{m+1})]) \leq \frac{1}{2} F((\mu_n) + (\mu_m)).$$

Take limit $n \to \infty$, we get

$$F(\tilde{r}(x_{n+1}, x_{m+1})) = -\infty,$$

or

$$\tilde{r}(x_{n+1}, x_{m+1}) = 0.$$

This means $\{x_n\}$ is a Cauchy sequence and hence convergent to z.

$$\lim_{n\to\infty} x_n = z.$$

Next, one arrives at

$$F(\tilde{r}(fx_{n-1}, fz)) < F(\Omega[\tilde{r}(fx_{n-1}, fz)]) \leq \frac{1}{2}\{F(\tilde{r}(x_{n-1}, x_n)) + F(\tilde{r}(z, fz))\} - \tau.$$

Taking limit as $n \to \infty$ and from the definition of F-contraction (F_2) (3.3.1), we have

$$\lim_{n\to\infty} F[\tilde{r}(fx_{n-1}, fz))] = -\infty.$$

This implies

$$\lim_{n\to\infty} \tilde{r}(x_n, fz) = 0.$$

Since the Cauchy sequence $\{x_n\}$ converges to both z and fz, it must be the case $fz = z$. It is easy to check that z is the unique common fixed point. It is similar to proving the second part as a Theorem 4.4.1. □

Example 4.4.6. *Let* $X = A \cup B$, *where* $A = [0, \frac{1}{3}]$ *and* $B = (\frac{1}{3}, 1)$. *Define* $\tilde{r} : X \times X \to [0, \infty)$ *such that* $\tilde{r}(x, y) = \tilde{r}(y, x)$ *for all* $x, y \in X$ *and*

$$\tilde{r}(x, y) = \begin{cases} 0 & ; x = y \\ \frac{1}{12} & ; x, y \in A \; ; x \text{ or } y \neq 0 \\ 1 & ; x, y \in B \\ \frac{1}{2} & ; \text{otherwise.} \end{cases} \tag{4.19}$$

Then $(X, \tilde{r})$ *is a* $\Omega-ERbMS$ *with* $\Omega(t) = 2t$, *which is not a rectangular metric space. Let the mapping* $f : X \to X$ *defined in previous Example 4.4.4, then for* $F(x) = lnx (x > 0)$ *and* $\tau = 0.20$, *all the conditions required in* (4.4.5) *are satisfied. Hence* $\{\frac{1}{3}\}$ *is the unique fixed point at which the map is discontinuous.*