Chapter 5

Fixed Point Results in A_b-metric Space

Ughade et al. (2016) introduced the A_b-metric space and derived some fixed point results in A_b-metric space by considering a continuous map. Motivated by his work, in this chapter, we present an important result in A_b-metric space and then we obtain the Banach type contraction principle and Kannan type fixed point theorem as corollaries in which map need not be continuous. At last, we derive the fixed point theorem having rational terms, which is the answer to the open problem given in Saluja (2021). Moreover, we find common fixed point theorems for four maps that involve rational terms. Our results extend and generalize several results from the existing literature, especially the results of Ughade et al. (2016). In addition, we provide examples for the justification of our results.

5.1 Introduction

Metric spaces are extremely useful in mathematics and applied sciences. As a result, some authors have attempted to provide generalizations of metric spaces in a variety of methods. For example, Gahler (1963) and Dhage (1992) introduced the concepts of 2-metric spaces and D-metric spaces respectively, but some authors pointed out that these attempts are not valid (see Mustafa (2005), Naidu et al. (2005)). Mustafa and Sims (2006) proposed a new structure of generalized metric spaces known as G-metric spaces as a generalization of metric spaces (X, d) in order to create and offer a new fixed point theory for various mappings in this new structure. Sedghi et al. (2007) established D^*-metric spaces, which are likely modifications of the definition of D-metric spaces introduced by Dhage (1992), and demonstrated some basic properties in D^*-metric spaces.

Sedghi et al. (2012) created a three-dimensional metric space called S-metric space, which is defined by modifying D-metric and G-metric spaces. After that, Kim et al. (2016) derived a common fixed point theorem for two single-valued mappings in S-metric spaces. In the same year, Souayah and Mlaiki (2016) introduced the concept of S_b- metric space which is a combination of b-metric space and S-metric space. In the definition of S_b-metric space, condition(2) (see 1.4.8.1) is not true in general. In order to make a general one, Rohen et al. (2017) modified. Also, many writers generalized the Banach contraction principle by employing various contractive conditions in extended S_b- metric space, S-metric space, and cone S-metric space (see Mlaiki (2018), Özgür and Tas (2017), Saluja (2020), Kim et al. (2015), Sedghi and Dung (2014), Sedghi et al. (2014), Sedghi et al. (2015), Mustafa et al. (2019b)). Such generalizations are established via contractive conditions formulated by rational terms (see, Latif et al. (2015), Shahkoohi and Razani (2014), Saluja (2019)).

Abbas et al. (2015) introduced n-dimensional metric space, namely A-metric space which is a generalization of S-metric space. Ughade et al. (2016) introduced the A_b-metric space, which is a mixture of A-metric space and S_b-metric space and defined as below:

5.2 A_b-metric Space and its Properties

Definition 5.2.1. *(Ughade et al. (2016)) Let X be a nonempty set and let $s \geq 1$, the function $A_b : X^n \to [0, \infty)$ that satisfies, for all $x_1, x_2, ..., x_n, a \in X$,*

(1) $A_b(x_1, x_2, ..., x_n) \geq 0$,

(2) $A_b(x_1, x_2, ..., x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_n$,

(3) $A_b(x_1, x_2, ..., x_n) \leq$
$s[A_b(x_1, x_1, .., x_{1_{(n-1)}}, a) + A_b(x_2, x_2, .., x_{2_{(n-1)}}, a) + \ldots + A_b(x_n, ..., x_{n_{(n-1)}}, a)]$.

Then (X, A_b) is called an A_b-metric space.

Remark 5.2.1.1. *(i) S_b-metric space is the particular case of A_b-metric space with $n = 3$.*

(ii) Every A-metric space will be A_b-metric space with $s = 1$. However, the converse need not be true.

Example 5.2.2. *(Ughade et al. (2016)) Let $X = [1, \infty)$ and $A_b : X^n \to [0, \infty)$ defined by*

$$A_b(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2 \quad ; \forall x_i \in X, \quad i = 1, 2, ..., n.$$

Then (X, A_b) is an A_b-metric space with $s = 2$.

Example 5.2.3. *(Ughade et al. (2016)) Let $X = R$ and $A_b : X^n \to [0, \infty)$ defined by*

$$\begin{aligned} A_b(x_1, x_2, ..., x_{n-1}, x_n) &= |\sum_{i=n}^{2} x_i - (n-1)x_1|^2 + |\sum_{i=n}^{3} x_i - (n-2)x_2|^2 + \ldots \\ &\quad + |\sum_{i=n}^{n-3} x_i - 3x_{n-3}|^2 + |\sum_{i=n}^{n-2} x_i - 2x_{n-2}|^2 + |x_n - x_{n-1}|^2, \end{aligned}$$

for all $x_i \in X$, $i = 1, 2, ..., n$. Then (X, A_b) is an A_b-metric space with $s = 2$.

Example 5.2.4. *Let $X = R$ and $A_b(x_1, x_2, ..., x_n) = |x_1 - x_n|^2 + |x_2 - x_n|^2 + \ldots + |x_{n-1} - x_n|^2$. Then (X, A_b) is an A_b-metric space with $s = 2$.*

Proof.

$$\begin{aligned} A_b(x_1, x_2, ..., x_n) &= |x_1 - x_n|^2 + |x_2 - x_n|^2 + \ldots + |x_{n-1} - x_n|^2 \\ &\leq 2\{|x_1 - a|^2 + |x_n - a|^2\} + \ldots + 2\{|x_{n-1} - a|^2 + |x_n - a|^2\} \\ &\leq 2(n-1)\{|x_1 - a|^2 + |x_2 - a|^2 + \ldots + |x_n - a|^2\} \\ &= 2[A_b(x_1, x_1, ..., a) + A_b(x_2, x_2, ..., a) + \ldots + A_b(x_n, x_n, ..., a)]. \end{aligned}$$

Hence, (X, A_b) is an A_b-metric space with $s = 2$. □

Lemma 5.2.5. *(Ughade et al. (2016)) Let (X, A_b) be an A_b-metric space with $s \geq 1$. Then for all $x, y \in X$,*

$$A_b(x, x, ..., x, y) \leq s A_b(y, y, ..., y, x).$$

The concepts of convergence, Cauchy sequence, and completeness in an A_b-metric space are defined in a similar manner.

Definition 5.2.6. *(Ughade et al. (2016)) Let (X, A_b) be an A_b-metric space and $\{x_n\}$ be a sequence in X. Then*

(i) A sequence $\{x_n\}$ is called convergent to $u \in X$ if $\lim_{n\to\infty} A_b(x_n, x_n, ..., x_n, u) = 0$. That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $A_b(x_n, x_n, ..., x_n, u) \leq \epsilon$, and we write $\lim_{n\to\infty} x_n = u$.

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{n\to\infty} A_b(x_n, x_n, ..., x_n, x_m) = 0$.

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $A_b(x_n, x_n, ..., x_n, x_m) \leq \epsilon$.

(iii) *(X, A_b) is said to be a complete A_b- metric space if every Cauchy sequence $\{x_n\}$ is convergent to a point $u \in X$.*

In our last result, the following lemma will be helpful to manage the discontinuity of the A_b-metric space.

Lemma 5.2.7. *If (X, A_b) is a A_b-metric space with $s \geq 1$, then we have the following assertions:*

(i) *Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x, y_n \to y$ and the elements of $\{x, y, x_n, y_n : n \in \mathbb{N}\}$are totally distinct. Then, we have*

$$\begin{aligned} s^{-2}A_b(x, x, ..., x, y) \leq \liminf_{n\to\infty} A_b(x_n, x_n, ..., x_n, y_n) &\leq \limsup_{n\to\infty} A_b(x_n, x_n, ..., x_n, y_n) \\ &\leq s^2 A_b(x, x, ..., x, y). \end{aligned}$$

(ii) *If $\{x_n\}$ is a Cauchy sequence in X converging to x and it has infinitely many distinct terms, then*

$$\begin{aligned} s^{-2}A_b(x, x, ..., x, y) \leq \liminf_{n\to\infty} A_b(x_n, x_n, ..., x_n, y) &\leq \limsup_{n\to\infty} A_b(x_n, x_n..., x_n, y) \\ &\leq s^2 A_b(x, x, ..., x, y), \end{aligned}$$

for each $y \in X$ with $x \neq y$.

Proof. (i) Using the second condition from the definition of A_b-metric space, we have

$$\begin{aligned} A_b(\underbrace{x,x,...,x}_{\text{(N-1) terms}},y) &\leq s[(N-1)A_b(x,x,...,x,x_n)+A_b(y,y,...,y,x_n)] \\ &\leq s(N-1)A_b(x,x,...,x,x_n) \\ &\quad +s^2[(N-1)A_b(y,y,...,y,y_n)+A_b(x_n,x_n,...,x_n,y_n)], \end{aligned}$$

and

$$\begin{aligned} A_b(x_n,x_n,...,x_n,y_n) &\leq s[(N-1)A_b(x_n,x_n,...,x_n,x)+A_b(y_n,y_n,...,y_n,x)] \\ &\leq s(N-1)A_b(x_n,x_n,...,x_n,x) \\ &\quad +s^2[(N-1)A_b(y_n,...,y_n,y)+A_b(x,...,x,y)]. \end{aligned}$$

Applying the lower and upper limit as $n \to \infty$ in the first inequality and second inequality respectively, we reach to the required result.

(ii) Using the second condition from the definition of A_b-metric space, we get

$$\begin{aligned} A_b(x,x,...,x,y) &\leq s[(N-1)A_b(x,x,...,x,x_n)+A_b(y,y,...,y,x_n)] \\ &\leq s(N-1)A_b(x,x,...,x,x_n) \\ &\quad +s^2[(N-1)A_b(y,y,...,y,y)+A_b(x_n,x_n,...,x_n,y)]. \end{aligned}$$

That is

$$A_b(x,x,...,x,y) \leq s[(N-1)A_b(x,x,...,x,x_n)+sA_b(x_n,x_n,...,x_n,y)], \quad (5.1)$$

and

$$A_b(x_n,x_n,...,x_n,y) \leq s[(N-1)A_b(x_n,x_n,...,x_n,x)+A_b(y,y,...,y,x)]. \quad (5.2)$$

From Lemma 5.2.5, one obtains

$$A_b(x_n, x_n, ..., x_n, y) \leq s[(N-1)A_b(x_n, x_n, ..., x_n, x) + sA_b(x, x, ..., x, y)].$$

Taking the lower limit as $n \to \infty$ in the equation (5.1) and the upper limit in the equation (5.2), we obtain desired result.

□

Ughade et al. (2016) proved the following theorem for continuous maps.

Theorem 5.2.8. *If (X, A_b) is a complete A_b-metric space and let f be a continuous self map on X that satisfies:*

$$A_b(fx^1, fx^2, ..., fx^n) \leq \psi(A_b(x^1, x^2, ..., x^n)),$$

for all $x^1, x^2, ..., x^n \in X$, where $\psi : [0, +\infty) \to [0, +\infty)$ is an increasing function such that
$\lim_{k\to\infty} \psi^k(t) = 0$, *for each fixed $t > 0$. Then f has a unique fixed point in X.*

5.3 Fixed Point Theorems on A_b-metric Space

Now, we start with the main result in which the map need not be continuous. Also, note that throughout, $N \geq 2$.

Theorem 5.3.1. *If (X, A_b) is a complete A_b-metric space with $s \geq 1$ and T be a* *continuous self map on X that satisfies:*

$$
\begin{aligned}
&A_b(Tu_1, Tu_2, ..., Tu_N)\\
&\leq \lambda_1[A_b(u_1, u_1, ..., Tu_1) + A_b(u_2, u_2, ..., Tu_2) + ... + A_b(u_N, u_N, ..., Tu_N)]\\
&+\lambda_2 A_b(u_1, u_2, ..., u_N)\\
&+\lambda_3[A_b(u_1, u_1, ..., Tu_2) + A_b(u_2, u_2, ..., Tu_3) + ... + A_b(u_N, u_N, ..., Tu_1)],
\end{aligned}
\tag{5.3}
$$

$\forall\ u_1, u_2, ..., u_N \in X$, *where* λ_1, λ_2 *and* λ_3 *are non negative real numbers such that* $0 < \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$*; and,*
$\alpha = s(N-1)^2 + 1, \beta = s(N-1), \gamma = s(N-1)(N-2) + s^2(N-1)^2 + s$. *Then* T *has a unique fixed point in* X*.*

Note: Here $\alpha \geq 2, \beta \geq 1, \gamma \geq 2$, for any value of s and N. So, according to the value of α, β, γ, one can choose $\lambda_1,\ \lambda_2$ and λ_3 such that $0 < \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$.

Proof. Let us define sequence $\{y_n\}$ as $Ty_n = y_{n+1}$. From definition of A_b-metric space and for $n > m$, we have

$$
\begin{aligned}
A_b(\underbrace{y_n, y_n, ..., y_n}_{\text{(N-1) terms}}, y_m) \leq\ & s^2 A_b(\underbrace{y_m, y_m, ..., y_m}_{\text{(N-1) terms}}, y_{m+1})\\
&+s^3(N-1)A_b(\underbrace{y_{m+1}, y_{m+1}, ..., y_{m+1}}_{\text{(N-1) terms}}, y_{m+2})\\
&+s^4(N-1)^2 A_b(\underbrace{y_{m+2}, y_{m+2}, ..., y_{m+2}}_{\text{(N-1) terms}}, y_{m+3})\\
&+s^5(N-1)^3 A_b(\underbrace{y_{m+3}, y_{m+3}, ..., y_{m+3}}_{\text{(N-1) terms}}, y_{m+4})\\
&+s^5(N-1)^4 A_b(\underbrace{y_n, y_n, ..., y_n}_{\text{(N-1) terms}}, y_{m+4}).
\end{aligned}
$$

Continuing in a similar way, we obtain

$$\begin{aligned} A_b(y_n, y_n, ..., y_m) \;\leq\; & s^2 A_b(y_m, y_m, ..., y_{m+1}) \\ & + s^3(N-1)A_b(y_{m+1}, y_{m+1}, ..., y_{m+2}) \\ & + s^4(N-1)^2 A_b(y_{m+2}, y_{m+2}, ..., y_{m+3}) \\ & + s^5(N-1)^3 A_b(y_{m+3}, y_{m+3}, ..., y_{m+4}) \\ & + s^6(N-1)^4[A_b(y_{m+4}, ..., y_{m+5}) + (N-1)A_b(y_n, y_n, ..., y_{m+4})]. \end{aligned}$$

We arrive at

$$A_b(\underbrace{y_n, y_n, ..., y_n}_{\text{(N-1) terms}}, y_m) \leq \sum_{i=1}^{n-m} S^{i+1}(N-1)^{i-1} A_b(\underbrace{y_{m+i-1}, y_{m+i-1}, ..., y_{m+i-1}}_{\text{(N-1) terms}}, y_{m+i}). \quad (5.4)$$

Again using definition of A_b-metric space and the contractive condition, one gets

$$\begin{aligned} & A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty_{n-1}, Ty_n) \\ & \leq \lambda_1[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_n, y_n, ..., y_n, y_{n+1})] \\ & + \lambda_2 A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) \\ & + \lambda_3[\{(N-2) + s(N-1)\}A_b(y_{n-1}, ..., y_{n-1}, y_n) + sA_b(y_n, ..., y_n, y_{n+1})]. \end{aligned}$$

That implies

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \frac{\lambda_1(N-1) + \lambda_2 + \lambda_3[N-2+s(N-1)]}{1-\lambda_1-\lambda_3 s} A_b(y_{n-1}, ..., y_{n-1}, y_n).$$

That is

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \mu \; A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n), \quad (5.5)$$

where, $\mu = \frac{\lambda_1(N-1)+\lambda_2+\lambda_3[N-2+s(N-1)]}{1-\lambda_1-\lambda_3 s}$.

Then, the equation (5.5) results in

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \mu^n A_b(y_0, y_0, ..., y_0, y_1).$$

This yields the following by the use of equation(5.4),

$$A_b(y_n, y_n, ..., y_m) \leq \sum_{i=1}^{n-m} s^{i+1}(N-1)^{i-1}\mu^{m+i-1} A_b(y_0, y_0, ..., y_1). \tag{5.6}$$

Let, $a_i = s^{i+1}(N-1)^{i-1}\mu^{m+i-1}$. Then,

$$\lim_{i\to\infty} \frac{a_i}{a_{i+1}} = \frac{1}{s(N-1)\mu}.$$

Since, $\alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$, where, $\alpha = s(N-1)^2 + 1$, $\beta = s(N-1)$, and $\gamma = s(N-1)(N-2) + s^2(N-1)^2 + s$.

We have

$$\begin{aligned}
& \{s(N-1)^2 + 1\}\lambda_1 + s(N-1)\lambda_2 + \{s(N-1)(N-2) + s^2(N-1)^2 + s\}\lambda_3 < 1 \\
\implies & s(N-1)^2\lambda_1 + s(N-1)\lambda_2 + \{s(N-1)(N-2) + s^2(N-1)^2\}\lambda_3 < 1 - \lambda_1 - \lambda_3 s \\
\implies & (N-1)\lambda_1 + \lambda_2 + \{(N-2) + s(N-1)\}\lambda_3 < \frac{1 - \lambda_1 - \lambda_3 s}{s(N-1)} \\
\implies & \frac{(N-1)\lambda_1 + \lambda_2 + \{(N-2) + s(N-1)\}\lambda_3}{1 - \lambda_1 - \lambda_3 s} < \frac{1}{s(N-1)} \\
\implies & \mu < \frac{1}{s(N-1)}
\end{aligned}$$

This yields, $\lim_{i\to\infty} \frac{a_i}{a_{i+1}} > 1$. Therefore utilizing the ratio test, $\sum a_i$ is convergent. So, from the equation (5.6), we conclude that

$$\lim_{n,m\to\infty} A_b(y_n, y_n, ..., y_m) \to 0.$$

Hence, $\{y_n\}$ is a Cauchy sequence. With the use of completeness, one gets

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

Again by contractive condition, one finds that

$$\begin{aligned} A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty) \;\leq\; & \lambda_1[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y, y, ..., Ty)] \\ & +\lambda_2[A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y)] \\ & +\lambda_3[(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_n) + A_b(y_{n-1}, y_{n-1}, ..., Ty)] \\ & +\lambda_3 A_b(y, y, ..., y_n). \end{aligned}$$

Taking limit on both sides, we arrive at $(1 - \lambda_1 - \lambda_3)A_b(y, y, ..., Ty) \leq 0$.

Here, $1 - \lambda_1 - \lambda_3 > 0$ as $0 < \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$. Hence, $y = Ty$.

Suppose $y*$ is another fixed point of T, then one can come across

$$A_b(Ty*, Ty*, ..., Ty) \leq (\lambda_2 + \lambda_3)A_b(y*, y*, ..., y*, y) + \lambda_3 A_b(y, y, ..., y*).$$

With the use of Lemma 5.2.5, we have

$$(1 - \lambda_2 - (1 + s)\lambda_3)A_b(y*, y*, ..., y) \leq 0. \tag{5.7}$$

If $(1 - \lambda_2 - (1 + s)\lambda_3) < 0$, then $1 < \lambda_2 + (1 + s)\lambda_3$. That implies

$$\alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < \lambda_2 + (1 + s)\lambda_3,$$

which is not possible for any value of s and N. So, $(1 - \lambda_2 - (1 + s)\lambda_3) > 0$.

Therefore from (5.7), we have

$$A_b(y*, y*, ..., y) = 0.$$

Hence, y is the unique fixed point of T in X. □

If we put $\lambda_2 = 0$ and $\lambda_3 = 0$ in the previous Theorem 5.3.1, then we have the Kannan theorem as a corollary.

Corollary 5.3.2. *Let (X, A_b) be a complete A_b-metric space and T be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N) \leq \lambda[A_b(u_1, u_1, ..., Tu_1)+A_b(u_2, ..., Tu_2)+...+A_b(u_N, u_N, ..., Tu_N)], \tag{5.8}$$

$\forall u_1, u_2, ..., u_n \in X$ and $0 < \lambda < \frac{1}{1+s(N-1)^2}$. Then T has a unique fixed point in X.

Proof. Here $max\ \{\frac{1}{1+s(N-1)^2}\} = \frac{1}{2}$.
Let us define sequence $\{y_n\}$ as $Ty_n = y_{n+1}$. With the same process adopted in the previous theorem, one observes that $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say). So, finally one can write

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

Now our claim is to prove that y is the fixed point of T. So, by contractive condition, one gets

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty) \leq \lambda(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_n) + \lambda A_b(y, y, ..., y, Ty).$$

Letting limit $n \to \infty$, we have

$$(1-\lambda)A_b(y, y, ..., y, Ty) \leq 0.$$

Since, $\lambda < \frac{1}{2}$. Therefore, $A_b(y, y, ..., y, Ty) = 0$.

That is $y = Ty$. It is easy to prove that, T has a unique fixed point in X. □

Example 5.3.3. *Let $X = R - (-1, 1) \cup \{0\} \cup \{\frac{1}{6}\}$ and $A_b(x_1, x_2, x_3, x_4) = |x_1 - x_4|^2 + |x_2 - x_4|^2 + |x_3 - x_4|^2$. Then, it is clear from Example* 5.2.4 *that (X, A_b) is an A_b-metric space with $s = 2$. Define $T : X \to X$ by*

$$T(x) = \begin{cases} \frac{1}{6} & ; if \quad x = 0, \frac{1}{6} \\ 0 & ; otherwise. \end{cases}$$

Here, T is discontinuous at $\{\frac{1}{6}\}$ and 0. Now, we have the following possibilities:

Case 1: $x_i = \frac{1}{6}$ *or* 0 *;* $i = 1, 2, 3, 4$.

We have $A_b(Tx_1, Tx_2, Tx_3, Tx_4) = 0$. It is trivially true.

Case 2: $x_i \neq \frac{1}{6}$ *and* 0 *;* $i = 1, 2, 3, 4$. *It is trivially true.*

Case 3: $x_i \neq \frac{1}{6}, 0$ *;* $i = 1, 2, 3$ *and* $x_4 = 0$.

$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(0, 0, 0, \frac{1}{6}) = \frac{3}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 3|x_1|^2$,

$A_b(x_2, x_2, x_2, Tx_2) = 3|x_2|^2, A_b(x_3, x_3, x_3, Tx_3) = 3|x_3|^2$,

$A_b(x_4, x_4, x_4, Tx_4) = \frac{3}{36}$.

From contractive condition, one observe that

$$\frac{3}{36} \leq 3\lambda\{|x_1|^2 + |x_2|^2 + |x_3|^2 + \frac{1}{36}\}.$$

Case 4: $x_i \neq \frac{1}{6}, 0$ *;* $i = 1, 2$ *and* $x_3 = 0, x_4 = 0$.

$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(0, 0, \frac{1}{6}, \frac{1}{6}) = \frac{2}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 3|x_1|^2$,

$A_b(x_2, x_2, x_2, Tx_2) = 3|x_2|^2$, $A_b(x_3, x_3, x_3, Tx_3) = \frac{3}{36}$, $A_b(x_4, x_4, x_4, Tx_4) = \frac{3}{36}$.

From contractive condition, one gets

$$\frac{3}{36} \leq 3\lambda\{|x_1|^2 + |x_2|^2 + \frac{1}{36} + \frac{1}{36}\}.$$

Case 5: $x_1 \neq \frac{1}{6}, 0$ *and* $x_i = 0 \quad ; i = 2, 3, 4.$

$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) = \frac{1}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 3|x_1|^2$,

$A_b(x_2, x_2, x_2, Tx_2) = \frac{3}{36}$, $A_b(x_3, x_3, x_3, Tx_3) = \frac{3}{36}$, $A_b(x_4, x_4, x_4, Tx_4) = \frac{3}{36}$.

This implies

$$\frac{3}{36} \leq 3\lambda\{|x_1|^2 + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}\}.$$

Case 6: $x_1 = \frac{1}{6}$ *and* $x_i \neq 0, \frac{1}{6} \quad ; i = 2, 3, 4.$

$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(\frac{1}{6}, 0, 0, 0) = \frac{1}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 0$,

$A_b(x_2, x_2, x_2, Tx_2) = 3|x_2|^2$, $A_b(x_3, x_3, x_3, Tx_3) = 3|x_3|^2$, $A_b(x_4, x_4, x_4, Tx_4) = 3|x_4|^2$.

We arrive at

$$\frac{1}{36} \leq 3\lambda\{0 + |x_2|^2 + |x_3|^2 + |x_4|^2\}.$$

Case 7: $x_1 = \frac{1}{6}, x_2 = \frac{1}{6}$ *and* $x_i \neq 0, \frac{1}{6} \quad ; i = 3, 4.$

$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(\frac{1}{6}, \frac{1}{6}, 0, 0) = \frac{2}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 0$,

$A_b(x_2, x_2, x_2, Tx_2) = 0$, $A_b(x_3, x_3, x_3, Tx_3) = 3|x_3|^2$, $A_b(x_4, x_4, x_4, Tx_4) = 3|x_4|^2$.

We have

$$\frac{2}{36} \leq 3\lambda\{|x_3|^2 + |x_4|^2\}.$$

Case 8: $x_1, x_2, x_3 = \frac{1}{6}$ *and* $x_4 \neq 0, \frac{1}{6}$.

$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0) = \frac{3}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 0$,

$A_b(x_2, x_2, x_2, Tx_2) = A_b(x_3, x_3, x_3, Tx_3) = 0$, $A_b(x_4, x_4, x_4, Tx_4) = 3|x_4|^2$.

So,

$$\frac{2}{36} \leq 3\lambda\{|x_4|^2\}.$$

So, for $0 < \lambda < \frac{1}{19}$*, all the above cases are satisfied. Hence, all the conditions required in Theorem* 5.3.2 *are satisfied. Thus,* $\frac{1}{6}$ *is the unique fixed point of* T *in* X *at which the map is discontinuous.*

Remark 5.3.3.1. *In the next results, continuity of the* A_b-*metric space is not necessary.*

If we put $\lambda_1 = 0$ and $\lambda_2 = 0$ in the Theorem 5.3.1, we have following result.

Corollary 5.3.4. *Let* (X, A_b) *be a complete* A_b*-metric space with* $s \geq 1$ *and* T *be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N) \leq \lambda[A_b(u_1, u_1, ..., Tu_2) + A_b(u_2, u_2, ..., Tu_3) + ... + A_b(u_N, u_N, ..., Tu_1)], \tag{5.9}$$

for all $u_1, u_2, ..., u_N \in X$*, where,* $0 < \lambda < \frac{1}{s[1+(N-1)\{(N-2)+(N-1)s\}]}$*. Then* T *has a unique fixed point in* X*.*

Proof. With the same process adopted in the previous theorem, one observes that $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say). So, one can write

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

From the contractive condition, one finds that

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty) \leq \lambda(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_n) + \lambda A_b(y, y, ..., y, y_n),$$

which together with Lemma 5.2.5, we have

$$A_b(y_n, y_n, ..., Ty) \leq \lambda(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_n) + \lambda s A_b(y_n, y_n, ..., y_n, y).$$

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., Ty) = 0.$$

Therefore, $\{y_n\}$ converges to both y and Ty. It must be the case $y = Ty$. Suppose $y*$ is another fixed point of T, then

$$A_b(Ty*, Ty*, ..., Ty) \leq \lambda[A_b(y*, y*, ..., y*, y) + sA_b(y*, y*, ..., y*, y)].$$

$$[1-(1+s)\lambda]A_b(y*, y*, ..., y*, y) \leq 0. \tag{5.10}$$

If $[1-(1+s)\lambda] < 0$ then

$$\frac{1}{1+s} < \lambda < \frac{1}{s[1+(N-1)\{(N-2)+(N-1)s\}]},$$

which is not possible for any $N \geq 2$. That means $[1-(1+s)\lambda] > 0$. Hence, from equation (5.10), it is easy to say that y is the unique fixed point of T in X. □

If we put $\lambda_1 = 0$ and $\lambda_3 = 0$ in the Theorem 5.3.1, then the theorem turns into Banach type contractive condition. In a similar way, one can easily prove the following corollary.

Corollary 5.3.5. *Let (X, A_b) be a complete A_b-metric space and T be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N) \leq \lambda A_b(u_1, u_2, ..., u_N) \quad ; \forall u_1, u_2, ..., u_N \in X. \tag{5.11}$$

where, $0 < \lambda < \frac{1}{s(N-1)}$. Then T has a unique fixed point.

Proof. Here *max* $\{\frac{1}{s(N-1)}\} = 1$. □

Example 5.3.6. *Let* $X = [0, \infty)$ *and* $A_b(x_1, x_2, x_3, x_4) = (max\{x_1, x_2, x_3\} - x_4)^2$. *It is clear that* (X, A_b) *is an* A_b*-metric space with* $s = 2$. *Also define* $T : X \to X$ *by* $T(x) = \frac{x}{4}$.

We have

$$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = (max\{\frac{x_1}{4}, \frac{x_2}{4}, \frac{x_3}{4}\} - \frac{x_4}{4})^2,$$

and

$$A_b(x_1, x_2, x_3, x_4) = (max\{x_1, x_2, x_3\} - x_4)^2.$$

So, for $\frac{1}{16} \leq \lambda < \frac{1}{6}$, *the equation(5.11) is satisfied. Hence,* 0 *is the unique fixed point of* T *in* X.

Saluja (2021) presented the following open problem:

Open Question: "Can we extend the results for rational contraction/ rational type contraction/ contraction involving rational expression?"

We try to give the answer to the open problem in the next theorem.

Theorem 5.3.7. *Let* (X, A_b) *be a complete* A_b*-metric space and* T *be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N) \leq \lambda \cdot \frac{M^*}{N^*} \quad ; \forall u_1, u_2, ..., u_N \in X,$$

where,

$$M^* = [A_b(u_1, u_1, ..., Tu_1) + A_b(u_2, u_2, ..., Tu_2) + ... + A_b(u_N, u_N, ..., Tu_N)]A_b(u_1, u_2, ..., u_N),$$

$$\begin{aligned} N^* = \quad & [A_b(u_1, u_1, ..., Tu_2) + ... + A_b(u_{N-2}, u_{N-2}, ..., Tu_{N-1})] + A_b(u_1, u_2, ..., u_N) \\ & + A_b(Tu_1, Tu_2, ..., Tu_N), \end{aligned}$$

$0 < \lambda < \frac{1}{s(N-1)}$ *and* $N^* \neq 0$. *Then* T *has a unique fixed point in* X.

Proof. Let us define sequence $\{y_n\}$ as $Ty_n = y_{n+1}$. Using the definition of A_b-metric space and the contractive condition, one can see that

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty_{n-1}, Ty_n) \leq \lambda \cdot$$
$$\{\frac{[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_n, y_n, ..., y_n, y_{n+1})]A_b(y_{n-1}, ..., y_{n-1}, y_n)}{(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_n, ..., y_n, y_{n+1})}\},$$

yielding thereby

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \lambda^n A_b(y_0, y_0, ..., y_0, y_1).$$

With the same process adopted in previous theorems, one observes that $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say).

That is $\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0$.

Again using contractive condition, we have

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty_{n-1}, Ty) \leq$$
$$\lambda\{\frac{[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y, y, ..., y, Ty)]A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y)}{(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y) + A_b(y_n, y_n, ..., y_n, Ty)}\}.$$

That is

$$A_b(y_n, y_n, ..., y_n, Ty) \leq$$
$$\lambda\{\frac{[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y, y, ..., y, Ty)]A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y)}{(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y) + A_b(y_n, y_n, ..., y_n, Ty)}\}.$$

Taking limit both sides, we conclude that $A_b(y, y, ..., Ty) = 0$. Hence $y = Ty$. It is easy to check that, T has a unique fixed point in X. □

At last, we find a common fixed point theorem for the use of four maps.

Theorem 5.3.8. *Let (X, A_b) be a complete A_b-metric space with coefficient $s \geq 1$ and S, T, A and B be self mappings of X such that $TX \subseteq AX, SX \subseteq BX$ and*

$$A_b(Su_1, Su_2, .., Su_{N-1}, Tu_N) \leq \frac{\lambda A_b(Au_1, Au_2, .., Au_{N-1}, Bu_N) + \mu A_b(Au_1, Au_2, ..., Au_{N-1}, Su_1) A_b(Bu_N, ..., Bu_N, Tu_N)}{1 + A_b(Au_1, Au_2, ..., Au_{N-1}, Bu_N)},$$

for all $u_1, u_2, ..., u_N \in X$ where, $0 < s(N-1)\lambda + \mu < 1$. If either range AX or BX is a closed subset of (X, A_b), then

(i) A and S have a coincidence point.

(ii) B and T have a coincidence point.

Furthermore, if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible then $A, B, S,$ and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $TX \subseteq AX$, there exists $x_1 \in X$ such that $Ax_1 = Tx_0$, and $SX \subseteq BX$, there exists $x_2 \in X$ such that $Bx_2 = Sx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Ax_{2n+1} = Tx_{2n}, \qquad y_{2n+1} = Bx_{2n+2} = Sx_{2n+1} \qquad ; \forall n \in \mathbb{N}.$$

Using contractive condition, one can see that

$$A_b(Sx_{2n+1}, Sx_{2n+1}, ..., Sx_{2n+1}, Tx_{2n+2}) \leq \frac{\lambda A_b(Ax_{2n+1}, ..., Ax_{2n+1}, Bx_{2n+2}) + \mu \cdot l^*}{1 + A_b(Ax_{2n+1}, Ax_{2n+1}, ..., Ax_{2n+1}, Bx_{2n+2})},$$

where,

$$l^* = A_b(Ax_{2n+1}, ..., Ax_{2n+1}, Sx_{2n+1}) A_b(Bx_{2n+2}, Bx_{2n+2}, ..., Tx_{2n+2}).$$

That implies

$$A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}) \leq \frac{\lambda A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1}) + \mu \cdot m^*}{1 + A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1})},$$

where,

$$m^* = A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1}) A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}),$$

which results in

$$A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}) \leq \lambda A_b(y_{2n}, ..., y_{2n}, y_{2n+1}) + \mu A_b(y_{2n+1}, ..., y_{2n+1}, y_{2n+2}).$$

We arrive at

$$A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}) \leq \frac{\lambda}{1-\mu} A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1}). \tag{5.12}$$

Similarly, one gets

$$A_b(y_{2n+2}, y_{2n+2}, ..., y_{2n+2}, y_{2n+3}) \leq \frac{\lambda}{1-\mu} A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}). \tag{5.13}$$

Therefore, from (5.12) and (5.13),

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \frac{\lambda}{1-\mu} A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n).$$

Likewise,

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq [\frac{\lambda}{1-\mu}]^2 A_b(y_{n-2}, y_{n-2}, ..., y_{n-2}, y_{n-1}).$$

Continuing this process, one arrives at

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq [\frac{\lambda}{1-\mu}]^n A_b(y_0, y_0, ..., y_0, y_1).$$

Say, $h = \frac{\lambda}{1-\mu}$ and with the use of the equation (5.4),

$$A_b(y_n, y_n, ..., y_m) \leq \sum_{i=1}^{n-m} s^{i+1}(N-1)^{i-1} h^{m+i-1} A_b(y_0, y_0, ..., y_1). \tag{5.14}$$

Let, $a_i = s^{i+1}(N-1)^{i-1}h^{m+i-1}$. Then

$$\lim_{i\to\infty} \frac{a_i}{a_{i+1}} = \frac{1}{s(N-1)h}.$$

Since, $s(N-1)\lambda + \mu < 1$. We have $\lim_{i\to\infty} \frac{a_i}{a_{i+1}} > 1$. By Ratio test, $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say). That means

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

Thus, one finds that

$$\lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Bx_{2n+2} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Ax_{2n+1} = y. \tag{5.15}$$

Now without loss of generality, one can suppose that AX is a closed subset of (X, A_b). From the equation (5.15), there exists $z \in X$ such that $y = Az$. Employing the definition of A_b-metric space, we have

$$A_b(Sz, Sz, ..., Sz, y) \leq s[(N-1)A_b(Sz, Sz, ..., Sz, Tx_{2n}) + A_b(y, y, ..., y, Tx_{2n})].$$

From the contractive condition,

$$\begin{aligned}
&A_b(Sz, Sz, ..., Sz, y)\\
&\leq s(N-1)\{\frac{\lambda A_b(Az, ..., Az, Bx_{2n}) + \mu A_b(Az, ..., Az, Sz)A_b(Bx_{2n}, ..., Bx_{2n}, Tx_{2n})}{1 + A_b(Az, ..., Az, Bx_{2n})}\}\\
&\quad + sA_b(y, y, ..., y, Tx_{2n}).
\end{aligned}$$

From Lemma 5.2.7 (ii) and letting limit supremum on both sides, one obtains

$$A_b(Sz, Sz, ..., Sz, y) = 0.$$

That is, $y = Sz = Az$. Since, $SX \subseteq BX$, there exists $w \in X$ such that $Bw = y$. Again using contractive condition, one arrives at

$$A_b(y, y, ..., y, Tw) = A_b(Sz, Sz, ...Sz, Tw) = 0.$$

Thus, $y = Tw = Bw = Sz = Az$. That is, A and S have coincidence point z and B and T have coincidence point w.

Let, A and S are weakly compatible, so we have

$$Ay = ASz = SAz = Sy.$$

Now our claim is to prove $Sy = y$.

$$\begin{aligned}
A_b(Sy, Sy, ..., Sy, y) &= A_b(Sy, Sy, ..., Sy, Tw)\\
&\leq \frac{\lambda A_b(Ay, ..., Ay, Bw) + \mu A_b(Ay, ..., Ay, Sy)A_b(Bw, ..., Bw, Tw)}{1 + A_b(Ay, ..., Ay, Bw)},
\end{aligned}$$

which results in

$$A_b(Sy, Sy, ..., Sy, y) = 0.$$

Therefore, $Sy = y = Ay$. Similarly, B and T are weakly compatible, one concludes that $Ty = y = By$. Finally, we have $Sy = Ay = Ty = By = y$. So, y is the common fixed point of S, T, A and B. It is easy to check that y is the unique common fixed point. □

5.4 Future Work

It is observed that our results can be derived for Wardowski F-contraction for the discontinuous map in A_b-metric space and even without taking continuity of the A_b-metric space.