Chapter 6

Fixed Point Theorems on S_p- and A_p- metric Spaces

In this chapter, we will discuss S_p- and A_p- metric space which are the generalization of S-metric space, S_b- metric space, A-metric space, and A_b-metric space. In section 6.1, we prove the fixed point theorem derived by Mustafa et al. (2019b) in S_p-metric space without taking the continuity of the class of functions defined by Jleli et al. (2014).

In section 6.2 of this chapter, we derive a common fixed point theorem for two maps in which one map is orbitally continuous in A_p-metric space (not necessarily continuous) using altering distance function ϕ. In addition, we prove a fixed point result for Banach and Kannan type contraction condition without considering orbitally continuous map. An example is also given. In our results, we are dealing with the discontinuity of metric space.

6.1 Fixed Point Theorem on S_p - metric Space

6.1.1 Introduction and Preliminaries

Recently, Mustafa et al. (2019b) introduced S_p-metric space, at which he replaced constant s in the definition of $S-$metric space by one variable continuous increasing function and derived some results in that space. Jleli et al. (2014) introduced new contractive maps and proved a new fixed point result in generalized metric spaces (RMS). Later on, it is known as JS-type contraction mappings. Motivated by Jleli et al. (2014), Mustafa et al. (2019b) dealt with a different type of JS-contraction and proved a fixed point result.

Definition 6.1.1.1. *(Jleli et al. (2014)) Let the class Θ_0 that contains all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following:*

- (θ_1) *θ is non decreasing.*
- (θ_2) *For each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$.*
- (θ_3) *There exists $r \in (0, 1)$ and $l \in (0, \infty)$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.*
- (θ_4) *θ is continuous.*

Mustafa et al. (2019b) redefined the class Θ_0 as following and denote it by Θ.

Definition 6.1.1.2. *(Mustafa et al. (2019b)) Let the class Θ consisting of all functions $\theta : [0, \infty) \to [1, \infty)$ satisfying only the following conditions:*

- (θ_1^*) *θ is a continuous strictly increasing function.*
- (θ_2^*) *For each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$.*

The definition of modified S-metric space is given below, a proper generalization of the S-metric space and S_b-metric space.

Definition 6.1.1.3. **S_p-metric space***(Mustafa et al. (2019b))*
Let X be a nonempty set and $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $t \leq \Omega(t)$ for all $t > 0$ and $0 = \Omega(0)$. Suppose that a mapping $\tilde{S} : E \times E \times E \to R^+$ satisfies:

(1) $\tilde{S}(x, y, z) = 0$ *if and only if* $x = y = z$,

(2) $\tilde{S}(x, y, z) \leq \Omega[\tilde{S}(x, x, a) + \tilde{S}(y, y, a) + \tilde{S}(z, z, a)]$; *for all* $x, y, z, a \in E$.

Then, $(X, \tilde{S})$ *is called an* S_p*-metric space.*

Note:(Mustafa et al. (2019b)) If we put (x, x, y, x) in place of (x, y, z, a) in condition (2) of definition of S_p-metric space, then $\tilde{S}(x, x, y) \leq \Omega[\tilde{S}(y, y, x)]$.

Definition 6.1.1.4. *(Mustafa et al. (2019b)) Let* (X, S_p) *be an* S_p*-metric space and* $\{x_n\}$ *be a sequence in* X*. Then*

(i) A sequence $\{x_n\}$ *is called convergent to* $u \in X$ *if, for every* $\epsilon > 0$ *there exists some positive integer* N_0 *such that* $\tilde{s}(x_n, x_n, u) < \epsilon$, $\forall n \geq N_0$.

(ii) A sequence $\{x_n\}$ *is called a Cauchy sequence if, for every* $\epsilon > 0$ *there exists some positive integer* N_0 *such that* $\tilde{s}(x_m, x_n, x_n) < \epsilon$, $\forall m, n \geq N_0$.

(iii) (X, S_p) *is said to be a complete* S_p*- metric space if every Cauchy sequence* $\{x_n\}$ *is convergent to a point* $u \in X$.

Lemma 6.1.1.1. *(Mustafa et al. (2019b)) Let* (X, S_p) *be an* S_p*-metric space, then we have the following:*

(i) Suppose that $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $x_n \to x$, $y_n \to y$. *Then, we have*

$$\begin{aligned} \frac{\Omega^{-1}[\frac{1}{2}\Omega^{-1}[\tilde{S}(x,y,y)]]}{2} &\le \liminf_{n\to\infty} \tilde{S}(x_n,y_n,y_n) \le \limsup_{n\to\infty} \tilde{S}(x_n,y_n,y_n) \\ &\le \Omega[2\ \Omega[\tilde{S}(x,y,y)]]. \end{aligned}$$

(ii) Let $\{x_n\}$ *be a Cauchy sequence in* X *converging to* x *and* $z \in X$ *arbitrary. Then*

$$\begin{aligned} \frac{\Omega^{-1}[\tilde{S}(x,z,z)]}{2} &\le \liminf_{n\to\infty} \tilde{S}(x_n,z,z) \le \limsup_{n\to\infty} \tilde{S}(x_n,z,z) \\ &\le \Omega[2\tilde{S}(x,z,z)]. \end{aligned}$$

Definition 6.1.1.5. *(Mustafa et al. (2019b)) Let* $(X, \preceq, \tilde{S})$ *be an ordered* S_p*-metric space. A map* $f : X \to X$ *is called an* S_p *rational-JS contraction if*

$$\theta(\Omega[2\tilde{S}(fx,fy,fz)]) \le \theta(M(x,y,z))^k, \tag{6.1}$$

for all mutually comparable elements $x, y, z \in X$, *where* $\theta \in \Theta$, $k \in [0,1)$ *and*

$$M(x,y,z) = max\{\tilde{S}(x,y,z), \frac{\tilde{S}(x,x,fx)\tilde{S}(y,y,fy)}{1+\tilde{S}(x,y,y)+\tilde{S}(x,z,z)}, \frac{\tilde{S}(y,y,fy)\tilde{S}(z,z,fz)}{1+\tilde{S}(y,fz,fz)+\tilde{S}(y,x,x)}\}.$$

Definition 6.1.1.6. *(Mustafa et al. (2019b)) An ordered* S_p*- metric space* $(X, \preceq, \tilde{S})$ *is said to have the sequential limit comparison property (s.l.c. property) if* $\{x_n\}$ *is an increasing sequence in* X *such that* $x_n \to u \in X$, *then* $x_n \preceq u$ *for all* $n \in \mathbb{N}$.

Theorem 6.1.1.1. *(Mustafa et al. (2019b)) Let* $(X, \preceq, \tilde{S})$ *be an ordered* S_p*-metric space and* $f : X \to X$ *be an increasing map with respect to* $\preceq$ *such that there exists an element* $x_0 \in X$ *with* $x_0 \preceq fx_0$. *Assume that* f *is an* S_p- *rational* JS*-contractive* *mapping. If*

(I) f is continuous, or

(II) $(X, \preceq, \tilde{S})$ enjoys the s.l.c. property,

then f has a fixed point. In addition, f has one and only one fixed point if and only if the set of fixed points of f is well ordered.

6.1.2 A Remark on the fixed point theorem of Mustafa

In this section, we show that the assumption of continuity of the function θ is not necessary. So, hereby we prove the Theorem 6.1.1.1 of Mustafa et al. (2019b) for the class Θ_0 containing of all functions $\theta : [0, \infty) \to [1, \infty)$ satisfying the conditions (θ_1) and (θ_2) defined by Jleli et al. (2014).

Now, the proof of the Theorem 6.1.1.1 is as follows:

Proof. Let us define $x_n = f^n x_0$. We can assume, without losing generality, that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then by (6.1), we conclude that

$$\begin{aligned} \theta(\tilde{S}(x_n, x_n, x_{n+1})) \leq \theta(\Omega[2\tilde{S}(x_n, x_n, x_{n+1})]) &\leq \theta(M(x_{n-1}, x_{n-1}, x_n))^k \\ &\leq \theta(\tilde{S}(x_{n-1}, x_{n-1}, x_n))^k, \quad (6.2) \end{aligned}$$

because

$$M(x_{n-1}, x_{n-1}, x_n) \leq max\{\tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_n, x_n, x_{n+1})\}.$$

From (6.2), one deduce that

$$1 \leq \theta(\tilde{S}(x_n, x_n, x_{n+1})) \leq \theta(\tilde{S}(x_{n-1}, x_{n-1}, x_n))^k \leq \ldots \theta(\tilde{S}(x_0, x_0, x_1))^{k^n}. \quad (6.3)$$

Taking the limit $n \to \infty$,

$$\theta(\tilde{S}(x_n, x_n, x_{n+1})) = 1.$$

Since $\theta \in \Theta$, we obtain

$$\lim_{n\to\infty} \tilde{S}(x_n, x_n, x_{n+1}) = 0. \tag{6.4}$$

Now, our claim is to prove that $\{x_n\}$ is an S_p-Cauchy sequence. Assume, on the other hand, i.e., there exists $\epsilon > 0$ for which one can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that m_i is the smallest index for which $m_i > n_i > i$ and $\tilde{S}(x_{m_i}, x_{m_i}, x_{n_i}) \geq \epsilon$.

This means that $\tilde{S}(x_{m_{i-1}}, x_{m_i-1}, x_{n_i}) < \epsilon$.

Using the rectangular inequality, we have

$$\epsilon \leq \tilde{S}(x_{m_i}, x_{m_i}, x_{n_i}) \leq \Omega[2\tilde{S}(x_{m_i}, x_{m_i}, x_{n_i+1}) + \tilde{S}(x_{n_i}, x_{n_i}, x_{n_i+1})].$$

Taking the upper limit, one obtains

$$\frac{1}{2}\Omega^{-1}(\epsilon) \leq \limsup_{i\to\infty} \tilde{S}(x_{m_i}, x_{m_i}, x_{n_i+1}), \tag{6.5}$$

and

$$\limsup_{i\to\infty} M(x_{m_i-1}, x_{m_i-1}, x_{n_i}) \leq \epsilon. \tag{6.6}$$

Since θ is non decreasing, one can write from the equation (6.1),

$$\Omega[2\tilde{S}(fx, fy, fz)] \leq (M(x, y, z))^k.$$

Put, $x = x_{m_i-1}, y = x_{m_i-1}, z = x_{n_i}$ in above equation, we have,

$$\Omega[2\tilde{S}(x_{m_i}, x_{m_i}, x_{n_i+1})] \leq (M(x_{m_i-1}, x_{m_i-1}, x_{n_i}))^k.$$

Taking limit supremum both sides,

$$\limsup_{i\to\infty}(\Omega[2\tilde{S}(x_{m_i}, x_{m_i}, x_{n_i+1})]) \leq \limsup_{i\to\infty}(M(x_{m_i-1}, x_{m_i-1}, x_{n_i}))^k.$$

Again using the condition (θ_1). i.e θ is non decreasing, we obtain

$$\theta\{\limsup_{i\to\infty}\Omega[2\tilde{S}(x_{m_i}, x_{m_i}, x_{n_i+1})]\} \leq \theta\{\limsup_{i\to\infty}(M(x_{m_i-1}, x_{m_i-1}, x_{n_i}))^k\}.$$

Hence, from the equations (6.5),(6.6) and above inequality, one arrives at

$$\begin{aligned}
\theta(\Omega[2.\frac{1}{2}\Omega^{-1}(\epsilon)]) \quad &\leq \quad \theta\{\Omega[2\limsup_{i\to\infty}\tilde{S}(x_{m_i}, x_{m_i}, x_{n_i+1})]\} \\
&\leq \theta\{\limsup_{i\to\infty}\Omega[2\tilde{S}(x_{m_i}, x_{m_i}, x_{n_i+1})]\} \\
&\leq \theta\{\limsup_{i\to\infty}(M(x_{m_i-1}, x_{m_i-1}, x_{n_i}))^k\} \\
&\leq \theta(\epsilon)^k.
\end{aligned}$$

Hence,

$$\theta(\epsilon) \leq \theta(\epsilon)^k,$$

which is possible only if $\epsilon = 0$, a contradiction. So, $\{x_n\}$ is an S_p-Cauchy sequence. Using completeness of S_p-metric space, $\{x_n\}$ converges to a point $u \in X$.

Now our claim is to prove that, u is a fixed point of f. When f is continuous, it is easy to prove. Now, from condition (II) in Theorem 6.1.1.1 and Lemma 6.1.1.1,

$$\begin{aligned}
\theta(\Omega[2\cdot\frac{\Omega^{-1}[\tilde{S}(u,u,fu)]}{2}]) \quad &\leq \quad \limsup_{i\to\infty}\theta(\Omega[2\tilde{S}(x_{n+1}, x_{n+1}, fu)]) \\
&\leq \quad \limsup_{i\to\infty}\theta(M(x_n, x_n, u))^k,
\end{aligned}$$

where,

$$\lim_{n\to\infty} M(x_n, x_n, u)) =$$
$$\lim_{n\to\infty} max\{\tilde{S}(x_n, x_n, u), \frac{\tilde{S}(x_n, x_n, fx_n)\tilde{S}(x_n, x_n, fx_n)}{1 + \tilde{S}(x_n, x_n, x_n) + \tilde{S}(x_n, u, u)}, \frac{\tilde{S}(x_n, x_n, fx_n)\tilde{S}(u, u, fu)}{1 + \tilde{S}(x_n, fu, fu) + \tilde{S}(x_n, x_n, x_n)}\}$$
$$= 0.$$

Therefore, we deduce that, $\tilde{S}(u, u, fu) = 0, i.e.\ \ u = fu$. From the definition of S_p rational-JS contraction (6.1), uniqueness can be proven easily. □

6.2 Fixed Point Theorems on A_p - metric Space

6.2.1 Altering distance function

Altering distance function introduced by Khan et al. (1984), which is a control function that alters the distance between two points in a metric space.

Definition 6.2.1.1. *(Faraji and Nourouzi (2017)) Let $\phi : [0, \infty) \to [0, \infty)$ is said to be altering distance function if*

(i) ϕ is increasing and continuous,

(ii) $\phi(t) = 0$ if and only if $t = 0$.

We denote the class of all functions satisfying (i) and (ii) by φ.

Next, we present an A_p- metric space which is the combination of A-metric space, A_b-metric space, and S_p- metric space.

6.2.2 A_p-metric space

Definition 6.2.2.1. *(Adewale et al. (2020)) Let X be a nonempty set, $\omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $\omega^{-1}(t) \leq t \leq \omega(t)$ for all $t > 0$ and $0 = \omega(0)$ and let $A_p : X^N \to [0, \infty)$ be a mapping such that for all $x_1, x_2, ..., x_N, a \in X$ satisfies the following conditions:*

(1) $A_p(x_1, x_2, ..., x_N) \geq 0$,

(2) $A_p(x_1, x_2, ..., x_N) = 0$ if and only if $x_1 = x_2 = ... = x_N$,

(3) $A_p(x_1, x_2, ..., x_N) \leq \omega[A_p(x_1, x_1, .., x_{1_{(N-1)}}, a) + A_p(x_2, x_2, .., x_{2_{(N-1)}}, a) + \ldots + A_p(x_N, x_N, ..., x_{N_{(N-1)}}, a)]$.

Then (X, A_p) is called an A_p-metric space.

Lemma 6.2.2.1. *(Adewale et al. (2020)) Let (X, A_p) be an A_p-metric space. Then for all $x, y \in X$,*

$$A_p(x, x, ..., x, y) \leq \omega[A_p(y, y, ..., y, x)].$$

The concepts of convergence, Cauchy sequence, and completeness in an A_p-metric space are defined in a similar manner.

Definition 6.2.2.2. *(Adewale et al. (2020)) Let (X, A_p) be an A_p-metric space and $\{x_n\}$ be a sequence in X. Then*

(i) A sequence $\{x_n\}$ is called convergent to $u \in X$ if $\lim_{n\to\infty} A_p(x_n, x_n, ..., x_n, u) = 0$. That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $A_p(x_n, x_n, ..., x_n, u) \leq \epsilon$, and we write $\lim_{n\to\infty} x_n = u$.

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{n\to\infty} A_p(x_n, x_n, ..., x_n, x_m) = 0$. That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $A_p(x_n, x_n, ..., x_n, x_m) \leq \epsilon$.

(iii) (X, A_p) is said to be a complete A_p- metric space if every Cauchy sequence $\{x_n\}$ is convergent to a point $u \in X$.

In our result, the problems that arise in proving fixed point results due to the possible discontinuity of the A_p-metric space can be fortunately managed with the following lemma.

Lemma 6.2.2.2. *Let (X, A_p) be an A_p-metric space with the function ω, then we have the following:*

(i) Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x, y_n \to y$ and

the elements of $\{x, y, x_n, y_n : n \in \mathbb{N}\}$ *are totally distinct. Then, we have*

$$\begin{aligned}\omega^{-1}[\omega^{-1}(A_p(x, x, ..., x, y))] &\leq \liminf_{n\to\infty} A_p(x_n, ..., x_n, y_n) \\ &\leq \limsup_{n\to\infty} A_p(x_n, ..., x_n, y_n) \\ &\leq \omega[\omega(A_p(x, x, ..., x, y))].\end{aligned}$$

(ii) Let $\{x_n\}$ *be a Cauchy sequence in* X *converging to* x*. If* x_n *has infinitely many distinct terms, then*

$$\begin{aligned}\omega^{-1}[\omega^{-1}(A_p(x, x, ..., x, y))] &\leq \liminf_{n\to\infty} A_p(x_n, x_n, ..., x_n, y) \\ &\leq \limsup_{n\to\infty} A_p(x_n, x_n, ..., x_n, y) \\ &\leq \omega[\omega(A_p(x, x, ..., x, y))],\end{aligned}$$

for all $y \in X$ *with* $x \neq y$*.*

Proof. (i) Using the second condition from the definition of A_p-metric space, we have

$$A_p(x, x, ..., x, y) \leq \omega[(N-1)A_p(x, x, ..., x, x_n) + A_p(y, y, ..., y, x_n)]$$
$$\leq \omega[(N-1)A_p(x, ..., x, x_n) + \omega\{(N-1)A_p(y, ..., y, y_n) + A_p(x_n, ..., x_n, y_n)\}].$$

Using Lemma 6.2.2.1,

$$\begin{aligned}[A_p(x, x, ..., x, y)] \leq\ & \omega[(N-1)\ \omega\{A_p(x_n, x_n, ..., x_n, x)\} \\ & +\omega\{(N-1)\ \omega(A_p(y_n, ..., y_n, y)) + A_p(x_n, x_n, ..., x_n, y_n)\}],\end{aligned}$$

and

$$A_p(x_n, x_n, ..., x_n, y_n) \leq \omega[(N-1)A_p(x_n, x_n, ..., x_n, x) + A_p(y_n, y_n, ..., y_n, x)]$$
$$\leq \omega[(N-1)A_p(x_n, ..., x_n, x) + \omega\{(N-1)A_p(y_n, ..., y_n, y) + A_p(x, x, ..., x, y)\}].$$

Taking the lower and upper limit as $n \to \infty$ in the first and second inequality respectively, we obtain the necessary result.

(ii) Using the second condition from the definition of A_p-metric space, we get

$$\begin{aligned} A_p(x, x, ..., x, y) &\leq \omega[(N-1)A_p(x, x, ..., x, x_n) + A_p(y, y, ..., y, x_n)] \\ &\leq \omega[(N-1)A_p(x, x, ..., x, x_n) \\ &\quad +\omega\{(N-1)A_p(y, y, ..., y, y) + A_p(x_n, x_n, ..., x_n, y)\}]. \end{aligned}$$

That is

$$A_p(x, x, ..., x, y) \leq \omega[(N-1)A_p(x, x, ..., x, x_n) + \omega\{A_p(x_n, x_n, ..., x_n, y)\}], \quad (6.7)$$

and

$$A_p(x_n, x_n, ..., x_n, y) \leq \omega[(N-1)A_p(x_n, x_n, ..., x_n, x) + A_p(y, y, ..., y, x)]. \quad (6.8)$$

From Lemma 6.2.2.1, one obtains

$$A_p(x_n, x_n, ..., x_n, y) \leq \omega[(N-1)A_p(x_n, x_n, ..., x_n, x) + \omega\{A_p(x, x, ..., x, y)\}].$$

Taking the lower limit as $n \to \infty$ in the equation (6.7) and the upper limit in the equation (6.8), we obtain desired result.

□

6.2.3 Fixed point theorems with altering distance function

Theorem 6.2.3.1. *Let (X, A_p) be a complete A_p-metric space with non-trivial function $\omega(i.e., \omega(t) \neq t)$. Let f and g be commuting mappings into itself which satisfies the following:*

(i)

$$\varphi[\omega(\omega(A_p(fu_1, fu_2, ..., fu_N)))] \leq \lambda\, \varphi(A_p(gu_1, gu_2, ..., gu_N)) \quad ; \forall u_1, u_2, ..., u_N \in X, \tag{6.9}$$

where, $0 < \lambda < 1$.

(ii) Either f or g is orbitally continuous and the range of g contains the range of f.

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Then fx_0 and gx_0 are well defined. Since $fx_0 \in g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. Continuing this process, if x_n is chosen, then we choose a point x_{n+1} in X such that $gx_{n+1} = fx_n$.
Step I : We will prove that $\lim_{n\to\infty} A_p(gx_{n+1}, gx_{n+1}, ..., gx_n) = 0$.
From contractive condition (6.9) and $t \leq \omega(t)$, one can have

$$\begin{aligned}
\varphi\{A_p(gx_{n+1}, gx_{n+1}, ..., gx_n)\} &= \varphi\{A_p(fx_n, fx_n, ..., fx_{n-1})]\} \\
&\leq \varphi\{\omega(\omega(A_p(fx_n, fx_n, ..., fx_{n-1})))\} \\
&\leq \lambda\varphi\{A_p(gx_n, gx_n, ..., gx_{n-1})\} \\
&\leq \lambda\varphi\{\omega(\omega(A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-2})))\} \\
&\leq \lambda^2\varphi\{A_p(gx_{n-1}, gx_{n-1}, ..., gx_{n-2})\} \\
&\vdots \\
&\leq \lambda^n\varphi\{A_p(gx_1, gx_1, .., gx_1, gx_0)\}.
\end{aligned}$$

Then

$$\begin{aligned}\limsup_{n\to\infty}\varphi(A_p(gx_{n+1},gx_{n+1},...,gx_{n+1},gx_n)) &= \liminf_{n\to\infty}\varphi(A_p(gx_{n+1},...,gx_{n+1},gx_n))\\ &= \lim_{n\to\infty}\varphi(A_p(gx_{n+1},...,gx_{n+1},gx_n))\\ &= 0.\end{aligned}$$

This implies

$$\lim_{n\to\infty} A_p(gx_{n+1},gx_{n+1},...,gx_{n+1},gx_n) = 0.$$

Step 2 : We will show that $gx_n \neq gx_m$ for $n \neq m$.

Suppose, $gx_n = gx_m$ for some $n > m$, then $fx_{n-1} = fx_{m-1} = gx_n = gx_m$.

This yields

$$\begin{aligned}\varphi\{\ A_p(gx_m,gx_m,...,gx_m,gx_{m+1})\} &= \varphi\{\ A_p(gx_n,gx_n,...,gx_n,gx_{n+1})\}\\ &= \varphi\{\ A_p(fx_{n-1},fx_{n-1},...,fx_{n-1},fx_n)\}\\ &\leq \varphi\{\omega(\omega(A_p(fx_{n-1},fx_{n-1},...,fx_{n-1},fx_n)))\}\\ &\leq \lambda\varphi\{A_p(gx_{n-1},gx_{n-1},...,gx_n)\}\\ &\leq \lambda^2\varphi\{A_p(gx_{n-2},gx_{n-2},...,gx_{n-1})\}.\end{aligned}$$

Since $0 < \lambda < 1$, one obtains

$$\varphi\{\ A_p(gx_m,gx_m,...,gx_m,gx_{m+1})\} < \varphi\{\ A_p(gx_{n-2},gx_{n-2},...,gx_{n-1})\}.$$

Here $n \neq m$, so after a few steps, we arrive at

$$\varphi\{\ A_p(gx_m,gx_m,...,gx_m,gx_{m+1})\} < \varphi\{\ A_p(gx_m,gx_m,...,gx_{m+1})\},$$

which is not possible. Therefore, we can assume that $gx_n \neq gx_m$ for $n \neq m$.

Step 3 : Now it is shown that $\{gx_n\}$ is an A_p-Cauchy sequence. Assume, on the other hand, i.e. that there exists $\epsilon > 0$, we can find two subsequences $\{gx_{m_i}\}$ and $\{gx_{n_i}\}$ of $\{gx_n\}$ such that n_i is the smallest index, where

$$n_i > m_i > i \quad and \quad A_p(gx_{m_i}, gx_{m_i}, ..., gx_{m_i}, gx_{n_i}) \geq \epsilon. \tag{6.10}$$

It indicates that

$$A_p(gx_{m_i}, gx_{m_i}, ..., gx_{m_i}, gx_{n_{i-1}}) < \epsilon. \tag{6.11}$$

Using definition of A_p- metric space, one obtains

$$\begin{aligned} \epsilon &\leq A_p(gx_{m_i}, gx_{m_i}, ..., gx_{m_i}, gx_{n_i}) \\ &\leq \omega[(N-1)A_p(gx_{m_i}, gx_{m_i}, ..., gx_{m_i}, gx_{m_i+1}) + A_p(gx_{n_i}, gx_{n_i}, ..., gx_{n_i}, gx_{m_i+1})]. \end{aligned}$$

Taking the upper limit as $i \to \infty$, we have

$$\omega^{-1}(\epsilon) \leq \limsup_{i\to\infty} A_p(gx_{n_i}, gx_{n_i}, ..., gx_{n_i}, gx_{m_i+1})].$$

That is

$$\omega^{-1}(\epsilon) \leq \limsup_{i\to\infty} A_p(fx_{n_i-1}, fx_{n_i-1}, ..., fx_{n_i-1}, fx_{m_i})]. \tag{6.12}$$

From Lemma 6.2.2.1 and the equations (6.9), (6.11) and (6.12), we have

$$\begin{aligned}
\varphi[\omega(\epsilon)] &= \varphi[\omega(\omega(\omega^{-1}(\epsilon)))] \\
&\leq \varphi[\omega(\omega(\limsup_{n\to\infty} A_p(fx_{n_{i-1}}, fx_{n_{i-1}}, ..., fx_{n_{i-1}}, fx_{m_i}))] \\
&\leq \lambda\varphi(\limsup_{n\to\infty} A_p(gx_{n_{i-1}}, gx_{n_{i-1}}, ..., gx_{m_i})) \\
&\leq \lambda\varphi[\omega(\limsup_{n\to\infty} A_p(gx_{m_i}, gx_{m_i}, ..., gx_{n_{i-1}}))] \\
&< \varphi[\omega(\epsilon)],
\end{aligned}$$

a contradiction. Thus, $\{gx_n\}$ is a A_p-Cauchy sequence in X. Since (X, A_p) is a complete A_p-metric space. So, there exists $z \in X$ such that

$$\lim_{n\to\infty} gx_n = u, \tag{6.13}$$

which yields

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n-1} = u.$$

Step 4 : We will prove that u is the coincidence point of f and g. i.e. $fu = gu$. With the use of $\omega-$ inequality, one finds that

$$A_p(fu, fu, ..., fu, gu) \leq \omega[(N-1)A_p(fu, fu, ..., fu, fgx_{n-1})+A_p(gu, gu, ..., gu, fgx_{n-1})].$$

Letting limit supremum, one has

$$\begin{aligned}
\limsup_{n\to\infty} A_p(fu, ..., fu, gu) &\leq \omega[(N-1)\limsup_{n\to\infty} A_p(fu, ..., fu, fgx_{n-1})] \\
&\quad +\omega[\limsup_{n\to\infty} A_p(gu, ..., gu, fgx_{n-1})].
\end{aligned}$$

Without loss of generality, one can assume that f is orbitally continuous. Also, we have f and g are commutative. Then one gets

$$\lim_{n\to\infty} A_p(fu, fu, ..., fu, gu) \leq \omega(0),$$

one conclude that $fu = gu$.

Step 5 : At last we will prove that, u is the unique common fixed point of f and g. At first, we will prove that $gu = u$.

$$\begin{aligned}
\varphi[(A_p(gx_n, gx_n, ..., gx_n, gu))] &= \varphi[A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-1}, fu)] \\
&< \varphi[\omega(\omega(A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-1}, fu)))] \\
&< \lambda\varphi[A_p(gx_{n-1}, gx_{n-1}, ..., gx_{n-1}, gu))] \\
&< \lambda^2\varphi[A_p(gx_{n-2}, gx_{n-2}, ..., gx_{n-2}, gu))] \\
&\vdots \\
&< \lambda^n\varphi[A_p(gx_0, gx_0, ..., gx_0, gu)].
\end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \varphi(A_p(gx_n, gx_n, ..., gx_n, gu)) = 0.$$

This implies

$$\lim_{n\to\infty} A_p(gx_n, gx_n, ..., gx_n, gu) = 0.$$

This means Cauchy sequence $\{gx_n\}$ converges to both u and gu, it is clear that $gu = u$. Thus $gu = u = fu$. It is easy to check that u is the unique common fixed point.

□

If we put $gx = I_x$(identity map) then Theorem 6.2.3.1 turns into Banach type contractive condition. To prove the below theorem, orbital continuity is not required.

Theorem 6.2.3.2. *Let (X, A_p) be a complete A_p-metric space with non-trivial function $\omega(i.e., \omega(t) \neq t)$. Let f be a self map that satisfies the following:*

$$\varphi[\omega(\omega(A_p(fu_1, fu_2, ..., fu_N)))] \leq \lambda\varphi(A_p(u_1, u_2, ..., u_N)) \qquad ; \forall u_1, u_2, ..., u_N \in X, \tag{6.14}$$

where, $0 < \lambda < 1$. Then f has a unique fixed point.

In the next theorem, we have taken the Kannan type contractive condition.

Theorem 6.2.3.3. *Let (X, A_p) be a complete A_p-metric space with non-trivial function $\omega(i.e., \omega(t) \neq t)$. Let f be a self map that satisfies the following:*

$$\begin{aligned} \varphi[\omega(\omega(A_p(fu_1, fu_2, ..., fu_N)))] \leq\ & \lambda\{\varphi(A_p(u_1, u_1, ..., fu_1)) \\ & +\varphi(A_p(u_2, u_2, ..., fu_2)) \\ & \vdots \\ & +\varphi(A_p(u_N, u_N, ..., fu_N))\}, \end{aligned} \tag{6.15}$$

for all $u_1, u_2, ..., u_N \in X$ and $0 < \lambda < 1$. Then f has a unique fixed point.

Proof. Let the sequence $\{x_n\}$ defined by $fx_n = x_{n+1}$. First, we will prove

$$\lim_{n\to\infty} A_p(x_n, x_n, ..., x_n, x_{n+1}) = 0. \tag{6.16}$$

From contractive condition (6.15),

$$\begin{aligned}\varphi(A_p(x_n, x_n, ..., x_n, x_{n+1})) &= \varphi(A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-1}, fx_n)) \\ &\leq \lambda(N-1)\varphi\{A_p(x_{n-1}, x_{n-1}, ..., x_n)\} \\ &\quad +\lambda\varphi\{A_p(x_n, x_n, ..., x_{n+1})\}.\end{aligned}$$

We arrive at,

$$\begin{aligned}\varphi(A_p(x_n, x_n, ..., x_n, x_{n+1})) &\leq (N-1)\{\frac{\lambda}{1-\lambda}\}\varphi\{A_p(x_{n-1}, x_{n-1}, ..., x_n)\} \\ &\leq (N-1)^2\{\frac{\lambda}{1-\lambda}\}^2\varphi\{A_p(x_{n-2}, x_{n-2}, ..., x_{n-1})\} \\ &\vdots \\ &\leq (N-1)^n\{\frac{\lambda}{1-\lambda}\}^n\varphi\{A_p(x_0, x_0, ..., x_1)\}. \qquad (6.17)\end{aligned}$$

Here, $\frac{\lambda}{1-\lambda} < 1$ as $\lambda < \frac{1}{2}$. Then,

$$\lim_{n\to\infty} \varphi(A_p(x_n, x_n, ..., x_{n+1})) = 0.$$

Now, our claim is to prove $\{x_n\}$ is a Cauchy sequence. For that we will show

$$\lim_{n\to\infty} A_p(x_n, x_n, ..., x_{n+p}) = 0 \quad ; p > 0.$$

Case 1 : Let p is odd, i.e. $p = 2m+1, \quad m \geq 1$.

From the contractive condition and equation (6.17),

$$\begin{aligned}\varphi[(A_p(x_n, x_n, ..., x_n, x_{n+2m+1}))] &\leq \varphi[\omega(\omega A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-1}, fx_{n+2m}))] \\ &\leq \lambda(N-1)\varphi[A_p(x_{n-1}, x_{n-1}, ..., x_{n-1}, x_n))] \\ &\quad +\lambda\varphi[A_p(x_{n+2m}, ..., x_{n+2m}, x_{n+2m+1}))] \\ &\vdots \\ &\leq \lambda(N-1)^n(\frac{\lambda}{1-\lambda})^{n-1}\varphi[A_p(x_0, x_0, ..., x_0, x_1))] \\ &\quad +\lambda(\frac{\lambda}{1-\lambda})^{n+2m}(N-1)^{n+2m}\varphi[A_p(x_0, ..., x_0, x_1))].\end{aligned}$$

Thus, we obtain

$$\lim_{n\to\infty} \varphi[A_p(x_n, x_n, ..., x_{n+2m+1})] = 0 \quad ; m \geq 1.$$

Case 2 : let p is even, i.e. $p = 2m, \quad m \geq 1$.

$$\begin{aligned}\varphi[(A_p(x_n, x_n, ..., x_n, x_{n+2m}))] &\leq \varphi[\omega(\omega A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-1}, fx_{n+2m-1}))] \\ &\leq \lambda(N-1)\varphi[A_p(x_{n-1}, x_{n-1}, ..., x_{n-1}, x_n))] \\ &\quad +\lambda\varphi[A_p(x_{n+2m-1}, ..., x_{n+2m-1}, x_{n+2m}))] \\ &\vdots \\ &\leq \lambda(N-1)^n(\frac{\lambda}{1-\lambda})^{n-1}\varphi[A_p(x_0, x_0, ..., x_0, x_1))] \\ &+ \lambda(\frac{\lambda}{1-\lambda})^{n+2m-1}(N-1)^{n+2m-1}\varphi[A_p(x_0, ..., x_0, x_1))].\end{aligned}$$

So,

$$\lim_{n\to\infty} \varphi[A_p(x_n, x_n, ..., x_{n+2m})] = 0 \quad ; m \geq 1,$$

which means $\{x_n\}$ is a Cauchy sequence and hence convergent to z.

$$\lim_{n\to\infty} x_n = z.$$

Case 3 : Finally, our claim is to prove $fz = z$. From the definition of A_p-metric space, one gets

$$A_p(z, z, ..., z, fz) \leq \omega[(N-1)A_p(z, z, ..., z, x_n) + A_p(fz, fz, ..., fz, x_n)].$$

Letting $n \to \infty$,

$$A_p(z, z, ..., z, fz) \leq \lim_{n\to\infty} \omega[A_p(fz, fz, ..., fz, fx_{n-1})].$$

From Lemma 6.2.2.1

$$A_p(z, z, ..., z, fz) \leq \lim_{n\to\infty} \omega[\omega(A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-1}, fz))].$$

We can write

$$\varphi(A_p(z, z, ..., z, fz) \leq \lim_{n\to\infty} \varphi[\omega[\omega(A_p(fx_{n-1}, fx_{n-1}, ..., fx_{n-1}, fz))]].$$

From the contractive condition (6.15),

$$\varphi(A_p(z, z, ..., z, fz) \leq \lim_{n\to\infty} \lambda[(N-1)\varphi(A_p(x_{n-1}, x_{n-1}, ..., x_{n-1}, x_n)+\varphi(z, z, ..., z, fz)],$$

which results in

$$(1-\lambda)\varphi(z, z, ..., z, fz) \leq 0.$$

Since, $\lambda < 1$ and φ is non negative,

$$\varphi(z, z, ..., z, fz) = 0.$$

Thus, z is the fixed point of f. It is clear z is the unique fixed point of f. $\square$