Chapter 2

Fixed Point Theorems on Variant Spaces

In this chapter, we extend some fixed point theorems for Kannan type mappings with some additional conditions such as orbital continuity and asymptotic regularity on the mappings. We provide an example that gives strength to our result. In addition, we generalize Kannan (1968) theorem for generalized metric space (RMS). In section 2.1, we provide some results which will be needed for the subsequent sections. In section 2.2, we derive a common fixed point for two maps in which one map is orbitally continuous. In section 2.3, we find a unique fixed point for the asymptotically regular map. In the last section, we demonstrate a result for generalized metric space(RMS).

2.1 Introduction and Preliminaries

Edelstein (1962) proved the unique fixed point for mapping on compact metric space. Recently, Górnicki (2017) extended his result for Kannan contractive condition and derived a unique fixed point in compact metric space. We generalize

Górnicki (2017) result and establish a common fixed point for two maps in which one map is orbitally continuous in compact metric space of Ćirić (1974) type contractive condition, which was discussed in chapter 1, section 1.3 (1.1). In addition, Górnicki (2017) extended Kannan (1968) for three terms in the contractive condition and derived a unique fixed point for the asymptotically regular map in complete metric space. We generalize Górnicki (2017) result for four-term in contractive condition for the asymptotically regular map in complete metric space. Further, Azam and Arshad (2008) proved an important result for Kannan contractive condition for the RMS (rectangular metric space or generalized metric space). Inspired by Azam and Arshad (2008), we find a variant of his result in RMS. The definition of RMS was discussed in chapter 1 (1.4.2.1). Following that, we offer certain theorems that will be necessary for our research.

Theorem 2.1.1. *(Edelstein (1962)) Let $(X, \hat{d})$ be a compact metric space and the map $f : X \to X$ satisfies*

$$\hat{d}(fx, fy) < \hat{d}(x, y) \quad ; \forall x, y \in X, x \neq y.$$

Then, f has a unique fixed point.

Górnicki (2017) extended Theorem 2.1.1 as follows:

Theorem 2.1.2. *Let $(X, \hat{d})$ be a compact metric space and $f : X \to X$ be a continuous mapping and satisfies*

$$\hat{d}(fx, fy) < \frac{1}{2}[\hat{d}(x, fx) + \hat{d}(y, fy)].$$

Then, f has a unique fixed point $v \in X$ and for each $x \in X$ the sequence of iterates $\{f^n x\}$ converges to v.

The following theorem is also proved by Górnicki (2017) for the asymptotically regular map.

Theorem 2.1.3. *If $f : X \to X$ is an asymptotically regular map from a complete metric space X into itself such that there exists $K < 1$ and satisfies the following:*

$$\hat{d}(fx, fy) \leq K[\hat{d}(x, fx) + \hat{d}(y, fy) + \hat{d}(x, y)] \quad ; \forall x, y \in X.$$

Then, f has a unique fixed point $v \in X$.

Azam and Arshad (2008) proved a very strong result for Kannan contractive condition for the generalized metric space in the following way:

Theorem 2.1.4. *Let $(X, \hat{d})$ be a complete generalized metric space, and the mapping $f : X \to X$ satisfies*

$$\hat{d}(fx, fy) \leq \lambda[\hat{d}(x, fx) + \hat{d}(y, fy)] \qquad ; \forall x, y \in X,$$

where, $\lambda < \frac{1}{2}$. Then f has a unique fixed point in X.

We are now ready to provide our research findings in the next section.

2.2 Fixed Point Result with Orbital Continuity

In the following theorem, we extend Theorem 2.1.2 and obtain a common fixed point for two maps, one of which is orbitally continuous.

Theorem 2.2.1. *Let f and g be self mappings in compact metric space X, where g*

is an orbitally continuous map and

$$\hat{d}(fgx, fgy)^2 \leq \lambda\ max\{\hat{d}(fx, fgy)\hat{d}(x, y), \hat{d}(fy, fgx)\hat{d}(fy, fgy)\} + \mu[\hat{d}(gy, fgy)]^2, \quad (2.1)$$

where $0 < \lambda + \mu < \frac{1}{2}$. *Then* f *and* g *have a unique common fixed point.*

Proof. Firstly, we define a sequence $\{x_n\}$ of iterations of f and g as below:

$$fx_{2n} = x_{2n+1},\ gx_{2n-1} = x_{2n}\ ; n = 0, 1, 2, ...$$

By triangle inequality and Young's (Royden (1988)) inequality, we have

$$\begin{aligned}
&[\hat{d}(x_{2n+1}, x_{2n})]^2 = [\hat{d}(fgx_{2n-1}, fgx_{2n-2})]^2 \\
\leq\ &\lambda\ max\{\hat{d}(fx_{2n-1}, fgx_{2n-2})\hat{d}(x_{2n-1}, x_{2n-2}), \hat{d}(fx_{2n-2}, fgx_{2n-1})\hat{d}(fx_{2n-2}, fgx_{2n-2})\} \\
&+\mu[\hat{d}(gx_{2n-2}, fgx_{2n-2})]^2 \\
\leq\ &\lambda\ max\{\hat{d}(x_{2n}, x_{2n})\hat{d}(x_{2n-1}, x_{2n-2}), \hat{d}(x_{2n-1}, x_{2n+1})\hat{d}(x_{2n-1}, x_{2n})\} + \mu[\hat{d}(x_{2n-1}, x_{2n})]^2 \\
\leq\ &\lambda[\hat{d}(x_{2n-1}, x_{2n}) + \hat{d}(x_{2n}, x_{2n+1})]\hat{d}(x_{2n}, x_{2n-1}) + \mu[\hat{d}(x_{2n}, x_{2n-1})]^2 \\
\leq\ &\lambda[\hat{d}(x_{2n-1}, x_{2n})]^2 + \lambda\hat{d}(x_{2n}, x_{2n+1})\hat{d}(x_{2n}, x_{2n-1}) + \mu[\hat{d}(x_{2n}, x_{2n-1})]^2 \\
\leq\ &\lambda[\hat{d}(x_{2n-1}, x_{2n})]^2 + \lambda\{\frac{(\hat{d}(x_{2n}, x_{2n+1}))^2}{2} + \frac{(\hat{d}(x_{2n}, x_{2n-1}))^2}{2}\} + \mu[\hat{d}(x_{2n}, x_{2n-1})]^2,
\end{aligned}$$

which leads to

$$[\hat{d}(x_{2n+1}, x_{2n})]^2 \leq \{\frac{3\lambda + 2\mu}{2 - \lambda}\}[\hat{d}(x_{2n}, x_{2n-1})]^2.$$

That is

$$[\hat{d}(x_{2n+1}, x_{2n})]^2 \leq k[\hat{d}(x_{2n}, x_{2n-1})]^2 \quad \text{where, } k = \frac{3\lambda + 2\mu}{2 - \lambda} < 1,$$

which results in

$$\hat{d}(x_{2n}, x_{2n+1}) < \hat{d}(x_{2n-1}, x_{2n}). \tag{2.2}$$

Similarly, one obtains

$$\hat{d}(x_{2n+1}, x_{2n+2}) < \hat{d}(x_{2n}, x_{2n+1}). \tag{2.3}$$

From, equations (2.2) and (2.3), one gets

$$\hat{d}(x_{n+1}, x_n) < \hat{d}(x_n, x_{n-1}).$$

This proves that $\{\hat{d}(x_{n+1}, x_n)\}$ is a decreasing sequence of non negative real numbers and hence convergent to u.

$$\lim_{n\to\infty} x_n = u.$$

Since g is orbitally continuous, we have

$$\lim_{n\to\infty} gx_n = gu.$$

That is

$$\lim_{n\to\infty} x_{n+1} = u.$$

This is possible only if $gu = u$. Hence, u is a fixed point of g. Now, by a triangle and Young's inequality,

$$\begin{aligned}
[\hat{d}(u, fu)]^2 &\leq \{\hat{d}(u, x_{2n+1}) + \hat{d}(x_{2n+1}, fu)\}^2 \\
&\leq \{\hat{d}(u, x_{2n+1})\}^2 + \{\hat{d}(fgx_{2n-1}, fgu)\}^2 + 2\hat{d}(u, x_{2n+1})\hat{d}(fgx_{2n-1}, fgu) \\
&\leq \{\hat{d}(u, x_{2n+1})\}^2 + \{\hat{d}(fgx_{2n-1}, fgu)\}^2 \\
&\quad +2\{\frac{[\hat{d}(u, x_{2n+1})]^2}{2} + \frac{[\hat{d}(fgx_{2n-1}, fgu)]^2}{2}\}.
\end{aligned}$$

One arrives at,

$$[\hat{d}(u, fu)]^2 \leq [\hat{d}(u, x_{2n+1})]^2 + [\hat{d}(fgx_{2n-1}, fgu)]^2 + [\hat{d}(u, x_{2n+1})]^2 + [\hat{d}(fgx_{2n-1}, fgu)]^2. \tag{2.4}$$

By contractive condition, one gets

$$\begin{aligned}[\hat{d}(fgx_{2n-1}, fgu)]^2 &\leq \lambda\ max\{\hat{d}(fx_{2n-1}, fgu)\hat{d}(x_{2n-1}, u), \hat{d}(fu, fgx_{2n-1})\hat{d}(fu, fgu)\} \\ &\quad +\mu\{\hat{d}(gu, fu)\}^2 \\ &\leq \lambda\ \hat{d}(fx_{2n-1}, fgu)\hat{d}(x_{2n-1}, u) + \mu\{\hat{d}(gu, fu)\}^2,\end{aligned}$$

which together with equation (2.4) and applying limit on equation (2.4) both sides, we have

$$[\hat{d}(u, fu)]^2 \leq (\lambda + 2\ \mu)[\hat{d}(u, fu)]^2.$$

Since, $0 < \lambda + \mu < \frac{1}{2}$, one can write $u = fu$. Hence, u is a common fixed point of f and g. Uniqueness can be proved easily. □

Example 2.2.2. *Let $X = [0, 1]$ with the usual metric and $f,\ g : X \to X$ defined by $f(x) = x$ and*

$$gx = \begin{cases} \frac{1}{2} & ; x \in [0, 1) \\ \frac{1}{4} & ; x = 1. \end{cases}$$

It is easy to see, g is orbitally continuous. For $x \in [0, 1)$ and $y = 1$, we have

$$\hat{d}(fgx, fgy) = |fgx - fgy| = |f\frac{1}{2} - f\frac{1}{4}| = |\frac{1}{2} - \frac{1}{4}| = \frac{1}{4}.$$

So, $\hat{d}(fgx, fgy)^2 = \frac{1}{16}$ and

$$max\{\hat{d}(gx, fgy)\hat{d}(x, y), \hat{d}(fy, fgx)\hat{d}(fy, fgy)\} = \hat{d}(fx, f\frac{1}{4})\hat{d}(x, 1) = \frac{3}{4}.$$

One can see that all the conditions given in the previous theorem are satisfied. Hence, $\frac{1}{2}$ *is the unique common fixed point.*

2.3 Result with Asymptotically Regular Maps

Next, we extend the Theorem 2.1.3.

Theorem 2.3.1. *If* $f : X \to X$ *is an asymptotically regular map in complete metric space* X *with* $M < \frac{1}{2}$ *that satisfies following:*

$$\hat{d}(fx, fy) \leq M[\hat{d}(x, fx) + \hat{d}(y, fy) + \hat{d}(x, y) + \hat{d}(y, fx)]. \tag{2.5}$$

Then, f *has a unique fixed point* $v \in X$*.*

Proof. Let $x \in X$ and define a sequence $\{x_n = f^n x\}$. For $m > n$, one arrives at

$$\begin{aligned} \hat{d}(f^{n+1}x, f^{m+1}x) &\leq M[\hat{d}(f^n x, f^{n+1}x) + \hat{d}(f^m x + f^{m+1}x) + \hat{d}(f^n x, f^m x) \\ &\quad + \hat{d}(f^m x, f^{n+1}x)]. \end{aligned}$$

By the use of triangle inequality, we have

$$\begin{aligned} \hat{d}(f^{n+1}x, f^{m+1}x) &\leq M[\hat{d}(f^n x, f^{n+1}x) + \hat{d}(f^m x + f^{m+1}x) + \hat{d}(f^n x, f^{n+1}x) \\ &\quad + \hat{d}(f^{n+1}x, f^{m+1}x) + \hat{d}(f^{m+1}x, f^m x) + \hat{d}(f^m x, f^n x) \\ &\quad + \hat{d}(f^n x, f^{n+1}x)]. \end{aligned}$$

Again using triangle inequality, we have

$$\hat{d}(f^{n+1}x, f^{m+1}x) \leq \frac{M}{1-2M}[4\hat{d}(f^n x, f^{n+1}x) + 3\hat{d}(f^m x, f^{m+1}x)].$$

Since f is asymptotically regular map, we have

$$\hat{d}(f^{n+1}x, f^{m+1}x) \to 0 \; ; n \to \infty.$$

This means that $\{f^n x\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that

$$\lim_{n\to\infty} f^n x = v.$$

Then,

$$\begin{aligned} \hat{d}(v, fv) &\leq \hat{d}(v, f^{n+1}x) + \hat{d}(f^{n+1}x, fv) \\ &\leq \hat{d}(v, f^{n+1}x) + M[\hat{d}(f^n x, f^{n+1}x) + \hat{d}(v, fv) + \hat{d}(f^n x, v) + \hat{d}(v, f^{n+1}x)], \end{aligned}$$

which results in

$$\hat{d}(v, fv) \leq (\frac{1+M}{1-M})\hat{d}(v, f^{n+1}x) + \frac{M}{1-M}[\hat{d}(f^n x, f^{n+1}x) + \hat{d}(f^n x, v)].$$

Hence,

$$\hat{d}(v, fv) \to 0 \; ; n \to \infty.$$

So, v is a fixed point of f. For the uniqueness, let us assume that u is another fixed point of f and $u \neq v$. Then

$$\hat{d}(u, v) = \hat{d}(fu, fv) \leq M\left\{\hat{d}(u, fu) + \hat{d}(v, fv) + \hat{d}(u, v) + \hat{d}(v, fu)\right\}.$$

This implies, $M \geq \frac{1}{2}$, which is a contradiction with our assumption.
Hence, we proved. □

The following example provides the strength of the above theorem.

Example 2.3.2. *let* $X = [0,1] \cup [\frac{3}{2}, \frac{5}{3}]$ *with* $\hat{d}(x,y) = |x-y|$ *and* $f : X \to X$ *be given by*

$$fx = \begin{cases} 0 & ; x \in [0,1] \\ 1 & ; x \in [\frac{3}{2}, \frac{5}{3}] \end{cases}$$

Here, f *is asymptotically regular map.*

(a) f *does not satisfy Banach contraction principle at* $x = 1, y = \frac{3}{2}$.

(b) f *does not satisfy Kannan fixed point theorem at* $x = 0, y = \frac{3}{2}$.

(c) If $x \in [0,1]$ *and* $y \in [\frac{3}{2}, \frac{5}{3}]$, *then*
$\hat{d}(fx, fy) = \hat{d}(0,1) = 1$, $\hat{d}(y, fy) = y - 1$, $\hat{d}(x, fx) = x$ *and* $\hat{d}(x,y) \geq \frac{1}{2}$.
Which gives $[\hat{d}(x, fx) + \hat{d}(y, fy) + \hat{d}(x,y)] \geq x + y - \frac{1}{2} \geq 1$.
If $[\hat{d}(x, fx) + \hat{d}(y, fy) + \hat{d}(x,y)] = 1$, *then there is no such* $K < 1$, *which satisfies Theorem 2.1.3.*

(d) If $x, y \in [0,1]$ *and* $x, y \in [\frac{3}{2}, \frac{5}{3}]$, *then it is obvious.*
If $x \in [0,1]$ *and* $y \in [\frac{3}{2}, \frac{5}{3}]$, *then*
$\hat{d}(fx, fy) = \hat{d}(0,1) = 1$, $\hat{d}(x, fx) = x$, $\hat{d}(y, fy) = y - 1$, $\hat{d}(y, fx) = y$ *and* $\hat{d}(x,y) \geq \frac{1}{2}$.
Now, $[\hat{d}(x, fx) + \hat{d}(y, fy) + \hat{d}(x,y) + \hat{d}(y, fx)] \geq x + 2y - \frac{1}{2} \geq \frac{5}{2}$.
So, for $M = \frac{2}{5}$, *all the conditions of the previous theorem are satisfied. Hence, 0 is the unique fixed point of* f.

2.4 Result in Rectangular Metric Space

We find a variant of Theorem 2.1.4 as follows:

Theorem 2.4.1. *Let* $(X, \hat{d})$ *be a complete generalized metric space and the mapping*

$f : X \to X$ satisfying the following:

$$\hat{d}(fx, fy) \leq \lambda[\hat{d}(x, fx) + \hat{d}(y, fy) + \hat{d}(x, y) + \hat{d}(y, fx)], \tag{2.4}$$

where $0 < \lambda < \frac{1}{3}$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\}$ of iterations of f as follows:

$$fx_n = f^{n+1}x_0 = x_{n+1},\ x_n \neq x_{n+1} \quad ; n = 0, 1, 2, ...$$

Using the inequality (2.4), we have

$$\begin{aligned} \hat{d}(x_n, x_{n+1}) &= \hat{d}(fx_{n-1}, fx_n) \\ &\leq \lambda[\hat{d}(x_{n-1}, fx_{n-1}) + \hat{d}(x_n, fx_n) + \hat{d}(x_{n-1}, x_n) + \hat{d}(x_n, fx_{n-1}] \\ &\leq \frac{2\lambda}{1-\lambda}\hat{d}(x_{n-1}, x_n). \end{aligned}$$

Now assume that x_0 is not a periodic point. Actually if $x_n = x_0$, then

$$\begin{aligned} \hat{d}(x_0, fx_0) = \hat{d}(x_n, fx_n) &= \hat{d}(f^n x_0, f^{n+1}x_0) \\ &\leq \frac{2\lambda}{1-\lambda}\hat{d}(f^{n-1}x_0, f^n x_0) \\ &\vdots \\ &\leq \left\{\frac{2\lambda}{1-\lambda}\right\}^n \hat{d}(x_0, fx_0). \end{aligned}$$

Put $k = \frac{2\lambda}{1-\lambda}$, clearly $k < 1$ as $\lambda < \frac{1}{3}$. This implies

$$\hat{d}(f^n x_0, f^{n+1}x_0) \leq k^n \hat{d}(x_0, fx_0), \tag{2.5}$$

and

$$[1-k^n]\hat{d}(x_0, fx_0) \leq 0.$$

It follows that x_0 is a fixed point of f. Thus we can suppose $f^n x_0 \neq x_0$ for $n = 1, 2, 3, ...$

From inequality (2.4), one arrives at

$$\begin{aligned}\hat{d}(f^n x_0, f^{n+m} x_0) &\leq \lambda[\hat{d}(f^{n-1}x_0, f^n x_0) + \hat{d}(f^{n+m-1}x_0, f^{n+m}x_0)\\ &\quad +\hat{d}(f^{n-1}x_0, f^{n+m-1}x_0) + \hat{d}(f^{n+m-1}x_0, f^n x_0)].\end{aligned}$$

By rectangular property and (2.5), one gets

$$(1-\lambda)\hat{d}(f^n x_0, f^{n+m}x_0) \leq 2\lambda[k^{n-1}\hat{d}(x_0, fx_0) + k^{n+m-1}\hat{d}(x_0, fx_0)].$$

Thus, $\hat{d}(x_n, x_{n+m}) \to 0$ as $n \to \infty$. It means that $\{x_n\}$ is a Cauchy sequence in X. Using completeness of X, there exists a $u \in X$ such that $x_n \to u$. From rectangular property, we have

$$\begin{aligned}\hat{d}(fu, u) &\leq \hat{d}(fu, f^n x_0) + \hat{d}(f^n x_0, f^{n+1}x_0) + \hat{d}(f^{n+1}x_0, u)\\ &\leq \lambda[\hat{d}(u, fu) + \hat{d}(f^{n-1}x_0, f^n x_0) + \hat{d}(u, f^{n-1}x_0)]\\ &\quad +k^n\hat{d}(x_0, fx_0) + \hat{d}(f^{n+1}x_0, u)\\ &\leq \frac{k^n}{2}\hat{d}(x_0, fx_0) + \frac{\lambda}{1-\lambda}\hat{d}(u, x_{n-1}) + \frac{k^n}{1-\lambda}\hat{d}(x_0, fx_0)\\ &\quad +\frac{1}{1-\lambda}\hat{d}(u, x_{n+1}).\end{aligned}$$

Now, taking limit $n \to \infty$, we have $u = fu$. For the uniqueness, suppose $v \in X$ is another fixed point with $v = fv$ and $v \neq u$.

$$\hat{d}(v, u) = \hat{d}(fv, fu) \leq \lambda[\hat{d}(v, fv) + \hat{d}(u, fu) + \hat{d}(v, u)].$$

This results in

$$\hat{d}(v, u) = 0 \quad ; \lambda \neq 1.$$

Hence, u is the unique fixed point of f. □

The following example illustrates the above result.

Example 2.4.2. *let* $X = \{1, 2, 3, 4\}$. *Define* $\hat{d} : X \times X \to R$ *as follows:*

$\hat{d}(2, 3) = \hat{d}(3, 2) = \hat{d}(1, 3) = \hat{d}(3, 1) = 1,$

$\hat{d}(1, 4) = \hat{d}(4, 1) = \hat{d}(2, 4) = \hat{d}(4, 2) = \hat{d}(3, 4) = \hat{d}(4, 3) = 4,$

$\hat{d}(1, 2) = \hat{d}(2, 1) = 3.$

Then $(X, \hat{d})$ *is a complete rectangular metric space but not a metric space as it does not satisfy the triangular property:*

$$3 = \hat{d}(1, 2) \nleq \hat{d}(1, 3) + \hat{d}(3, 2) = 2.$$

Let $f : X \to X$, *defined as*

$$fx = \begin{cases} 3 & ; x \neq 1 \\ 2 & ; x = 1. \end{cases}$$

Now, $\hat{d}(f2, f3) = \hat{d}(f2, f4) = \hat{d}(f3, f4) = 0$ *and* $\hat{d}(f1, f2) = \hat{d}(f1, f3) = \hat{d}(f1, f4) = 1$.

Hence, in all cases $\hat{d}(fx, fy) = 0$ *or* 1.

Also, $\hat{d}(1, f1) = 3$, $\hat{d}(2, f2) = 1$, $\hat{d}(3, f3) = 0$, $\hat{d}(4, f4) = 4$.

If $\hat{d}(fx, fy) = 0$, *then it is obvious. For* $\hat{d}(fx, fy) = 1$ *if we take* $\frac{1}{4} \leq \lambda < \frac{1}{3}$, *all conditions of previous theorem are satisfied. Therefore, 3 is a unique fixed point of* f.