Preface

When we mathematically model real-world problems, imprecision may occur in estimating parameters. Before 1965, people were used to think probability theory is the only tool to handle such kind of uncertainty. But not all the uncertainties may be of possibilistic nature. In 1965, Zadeh introduced the fuzzy theory to handle such possibilistic uncertainty. The proposed theory allowed the handling of possibilistic errors as in measuring the length of a certain object, counting fishes in the ponds or quantifying the coldness in the temperature etc.

After, the concept of the fuzzy set was first discussed in a seminal paper by Zadeh in 1965. The concept of fuzzy derivative was first introduced by Chang and Zadeh in 1972. Dubois and Prade defined derivative based on the extension principle in 1982. Puri and Ralescue introduced H-derivative of fuzzy valued function based on Hukuhara difference. Kaleva and Seikkala, first simultaneously solved the fuzzy initial value problem with fuzzy initial condition. Kandel and Byatt used this theory for the applications involving fuzzy dynamical system. The basic and most popular approach to solve fuzzy differential equation (FDE) is Hukuhara differentiability which is based on H-difference. Fuzzy differential equations are also studied under other derivatives like Seikkala derivative, strongly generalized Hukuhara derivative and generalized Hukuhara derivatives.

Most of the authors solved scalar differential equations with fuzzy initial condition, the one is given as follows,

$$\dot{\tilde{x}} = f(t, x);\ \tilde{x}(0) = \tilde{x}_0$$

using different techniques like analytical, numerical, transformation and semi-analytical.

We started our work, with a system of fuzzy differential equations involving fuzzy initial condition then extended it to fully fuzzy system i.e., a system in which parameters as well as initial condition both are taken as fuzzy. Such a system in general form is given as follows,

$$\dot{\tilde{X}} = \tilde{f}\left(t, \tilde{X}\right);\ \tilde{X}(0) = \tilde{X}_0$$

For this thesis, we have focused on the techniques for the solution of the system of linear and nonlinear fuzzy differential equations. The summary of work is given as follows.

In our initial work, Prey-Predator Model with fuzzy initial conditions is considered and solved by analytical technique. Prey- Predator model contains a system of nonlinear differential equations. So, first we have linearized the system about its equilibrium point then given solution with the help of the eigenvalue and eigenvector method.

Next, we have used two numerical techniques, first by simply discretizing the Hukuhara derivative and in second we proposed and proved the equivalent Improved Euler method to solve fully fuzzy dynamical system. We have extended the convergence result of the numerical scheme for scalar fuzzy differential equation to the system of fuzzy differential equations which is based on complete error analysis. We have also solved a disease model by these numerical techniques.

We then proceeded, for the transform technique and used the fuzzy Laplace transform method for solving fuzzy linear dynamical system. In this, we considered a linear dynamical system with both cases homogeneous and nonhomogeneous. The condition for existence of fuzzy solution using Laplace transform is also proposed and proved for the fuzzy linear dynamical system. We have solved the arms race model in a fuzzy environment using this technique.

In the beginning, we worked on fuzzy systems by taking existing fuzzy derivatives like Hukuhara derivative, strongly generalized derivative and generalized Hukuhara derivative. These fuzzy derivatives have limitations. In Hukuhara derivative, as time increases solution becomes non-fuzzy. In other mentioned derivatives, we always obtain a possible set of solutions and one needs to select the solution that best fits to the problem, to overcome these limitations we then proposed a new fuzzy derivative i.e., the Modified Hukuhara derivative. The advantage of this new fuzzy derivative is that the solution always remains fuzzy and unique.

We redefined fuzzy Laplace transform under this Modified Hukuhara derivative. All the results related to fuzzy Laplace transform like the existence of fuzzy Laplace transform, its derivative and convolution theorem are proposed and proved under newly defined Modified Hukuhara derivative. These results are compared with the crisp counter parts and verified at the core.

Later, we renamed this new derivative as Modified generalized Hukuhara (mgH) derivative. Under this derivative, we proposed the semi-analytical technique Adomian Decomposition Method, in the parametric form, to solve the fuzzy dynamical system.

In all the above-mentioned work, we have applied the techniques on fuzzy dynamical system by taking parametric form. That is, the technique requires, the fuzzy differential equations to be converted into a system of the ordinary differential equation for solving them. This motivated us to develop a theory that solves the fuzzy dynamical system in its fuzzy form, instead of converting them into crisp.

We next proposed and proved results pertaining to fuzzy calculus and used it to solve nonlinear fuzzy differential equations by fuzzy Adomian Decomposition Method with convergence result under our proposed mgH derivative. For development of results, we do take help of parametric form, but results once established are directly applicable in fuzzy form.

For the development of this theory, we consider a fuzzy valued scalar function with fuzzy argument $\tilde{f}: E \to E$, E is the collection of a fuzzy number. Its parametric form can be defined as follows,

$$^{\alpha}\tilde{f}(\tilde{x}) = [\underline{f}(\tilde{x}), \overline{f}(\tilde{x})], \text{ where, } \underline{f}(\tilde{x}) = \min \tilde{f}(\tilde{x}) \text{ and } \overline{f}(\tilde{x}) = \max \tilde{f}(\tilde{x}).$$

$$= \left[\underline{\underline{f}}(\underline{x}, \overline{x}), \overline{\overline{f}}(\underline{x}, \overline{x})\right]$$

where,

$$\underline{\underline{f}}(\underline{x}, \overline{x}) = \min\left(\underline{f}(\underline{x}, \overline{x}), \overline{f}(\underline{x}, \overline{x})\right) \text{ and } \overline{\overline{f}}(\underline{x}, \overline{x}) = \max\left(\underline{f}(\underline{x}, \overline{x}), \overline{f}(\underline{x}, \overline{x})\right).$$

Based on this definition, our proposed modified generalized Hukuhara derivative is defined as follows.

A function $\tilde{f}: E \to E$ is said to be modified generalized Hukuhara differentiable for an element $\dot{\tilde{f}}(\tilde{x}_0) \in E$, such that for small $h > 0$, $\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)$, $\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)$ should exist and

$$\lim_{h \to 0^+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)}{h} = \lim_{h \to 0^-} \frac{\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)}{h} = \dot{\tilde{f}}(\tilde{x}_0)$$

The equivalent parametric form for the first limit is given as,

$$\lim_{h \to 0^+} \frac{^{\alpha}\tilde{f}(\tilde{x}_0 + h) \ominus {}^{\alpha}\tilde{f}(\tilde{x}_0)}{h}$$

$$= \lim_{h \to 0^+} \left[\min\left(\frac{\underline{\underline{f}}(\tilde{x}_0 + h) - \underline{f}(\tilde{x}_0)}{h}, \frac{\overline{f}(\tilde{x}_0 + h) - \overline{f}(\tilde{x}_0)}{h}\right), \max\left(\frac{\underline{\underline{f}}(\tilde{x}_0 + h) - \underline{f}(\tilde{x}_0)}{h}, \frac{\overline{f}(\tilde{x}_0 + h) - \overline{f}(\tilde{x}_0)}{h}\right)\right]$$

Further,

$$= \lim_{h \to 0^+} \left[\min\left(\frac{\underline{\underline{f}}(\underline{x}_0 + h, \overline{x}_0 + h) - \underline{\underline{f}}(\underline{x}_0, \overline{x}_0)}{h}, \frac{\overline{\overline{f}}(\underline{x}_0 + h, \overline{x}_0 + h) - \overline{\overline{f}}(\underline{x}_0, \overline{x}_0)}{h}\right), \max\left(\frac{\underline{\underline{f}}(\underline{x}_0 + h, \overline{x}_0 + h) - \underline{\underline{f}}(\underline{x}_0, \overline{x}_0)}{h}, \frac{\overline{\overline{f}}(\underline{x}_0 + h, \overline{x}_0 + h) - \overline{\overline{f}}(\underline{x}_0, \overline{x}_0)}{h}\right)\right]$$

Similarly, for the second limit,

$$\lim_{h \to 0^-} \frac{^{\alpha}\tilde{f}(\tilde{x}_0) \ominus {}^{\alpha}\tilde{f}(\tilde{x}_0 - \mathrm{h})}{h}$$

$$= \lim_{h \to 0^-} \left[\min\left(\frac{\underline{\underline{f}}(\tilde{x}_0) - \underline{f}(\tilde{x}_0 - h)}{h}, \frac{\overline{f}(\tilde{x}_0) - \overline{f}(\tilde{x}_0 - h)}{h}\right), \max\left(\frac{\underline{\underline{f}}(\tilde{x}_0) - \underline{f}(\tilde{x}_0 - h)}{h}, \frac{\overline{f}(\tilde{x}_0) - \overline{f}(\tilde{x}_0 - h)}{h}\right)\right]$$

Further,

$$= \lim_{h \to 0^-} \left[\min\left(\frac{\underline{\underline{f}}(\underline{x}_0, \overline{x}_0) - \underline{\underline{f}}(\underline{x}_0 - h, \overline{x}_0 - h)}{h}, \frac{\overline{\overline{f}}(\underline{x}_0, \overline{x}_0) - \overline{\overline{f}}(\underline{x}_0 - h, \overline{x}_0 - h)}{h}\right), \max\left(\frac{\underline{\underline{f}}(\underline{x}_0, \overline{x}_0) - \underline{\underline{f}}(\underline{x}_0 - h, \overline{x}_0 - h)}{h}, \frac{\overline{\overline{f}}(\underline{x}_0, \overline{x}_0) - \overline{\overline{f}}(\underline{x}_0 - h, \overline{x}_0 - h)}{h}\right)\right]$$

Other results required for the fuzzy Adomian Decomposition Method, like fuzzy power series along with its convergence and fuzzy Taylor's series under mgH derivative, are proposed and proved further in the thesis.

Using these developed results, we have solved a real-life application based on mathematical modelling of solar air collector with fuzzy parameters.

In the solar air collector, the ambient temperature depends on weather and also varies from morning to evening, so taking a range of values instead of one specific is more realistic. Similarly, the air mass flow rate also varies. Due to this reason, we have proposed the mathematical model of solar air collector involving fuzzy initial temperature and fuzzy rate of mass of airflow, as given below.

$$\left(\left(\frac{\dot{\tilde{m}}}{W}\right)\otimes a\otimes\left(\frac{d\tilde{T}}{dx}\right)\right)\oplus\left(\left(\frac{\dot{\tilde{m}}}{W}\right)\otimes b\otimes\left(\frac{d\tilde{T}}{dx}\right)\otimes\tilde{T}(x)\right)\oplus\left(\tilde{F}\otimes U_l\otimes\tilde{T}(x)\right)=\tilde{F}\otimes(S+T_aU_l)\ ;\ \ \tilde{T}(0).$$