# 2. Prey-Predator Model with Fuzzy initial condition

## 2.1. Introduction

In this chapter, we consider a system of differential equations representing a prey-predator model as described by Lotka-Volterra equation. Such a system mathematically is represented in general semi-linear form. For such a prey-predator model a more realistic depiction of the phenomena will be by considering initial conditions fuzzy, because error may occur in estimating the initial populations. All other parameters involved are considered crisp given in the description below.

In the population model, predator eats prey and prey depends on other food (grass, fruits, and herbs), the prey is assumed to have unlimited food supply and to reproduce exponentially until they interact with a predator. This exponential growth rate is represented by the parameter $a$. The rate of predation upon the prey is assumed to be proportional to the rate at which prey and predator interact, represented as $b$. On the other hand, the growth term of a predator when it encounters prey has a proportionality constant $d$ and $c$ represents the loss rate of the predator due to natural death or absence of prey.

Thus, the two species population model can be represented as a system, with two first-order nonlinear differential equations, which is also known as the Lotka-Volterra equation.

$$\dot{x} = ax - bxy$$

$$\dot{y} = -cy + dxy$$

with initial conditions, $x(0) = x_0$ and $y(0) = y_0$ (2.1)

where, $a, b, c$ and $d$ are positive constants as described above, $x$ denotes the population of prey species and $y$ denotes the population of predator species, $x_0$ and $y_0$ is the initial estimates of the species.

For the system, as given by equation (2.1), it may not be possible to have the exact estimates of the initial population, then such a scenario fits into a fuzzy setup where the initial estimates are represented by fuzzy numbers.

Equation (2.1) with the fuzzy initial condition is given by,

$$\dot{x} = ax - bxy$$
$$\dot{y} = -cy + dxy \qquad (2.2)$$

with initial conditions, $\tilde{x}(0) = \tilde{x}_0$ and $\tilde{y}(0) = \tilde{y}_0$.

Equation (2.2) can be written in compact form as,

$$\dot{X} = AX + f(X)$$

where, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ and $f = \begin{pmatrix} -bxy \\ dxy \end{pmatrix}$ with $\tilde{X}(0) = \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix}$.

In this chapter, we adopt an analytical approach to solve the proposed model, which gives the estimate of the numbers of prey and number of predators at time $t$, starting with an initial approximate population $\tilde{X}(0)$. To get the approximate solution first we obtain the solution of crisp prey-predator model and use it to obtain the solution of proposed fuzzy prey-predator model.

## 2.2. Linearized Crisp Prey- Predator Model

The equilibrium state is obtained by considering a change in state variable concerning $t$ as 0.

$$AX + f(X) = 0$$

i.e. $$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -bxy \\ dxy \end{pmatrix} = 0$$

Giving, the nontrivial equilibrium state of the system as,

$$\begin{pmatrix} x_e \\ y_e \end{pmatrix} = \begin{pmatrix} c/d \\ a/b \end{pmatrix}$$

To obtain the approximate solution of such a system we linearize the equation (2.1) about the equilibrium state $(x_e, y_e)$ with the help of Taylor's expansion considering the first-order term and neglecting the higher-order terms, as

$$\dot{X} = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix} \qquad (2.3)$$

where, $A_L$ is $2 \times 2$ matrix and given as $\begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$ and $B_L$ is $2 \times 1$ vector and given as $\begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix}$.

The eigenvalues of $A_L$ are $\lambda_1 = i\sqrt{ac}$ and $\lambda_2 = -\,i\sqrt{ac}$. We construct fundamental matrix $\Psi(t)$ to obtain transition matrix $\emptyset(t,0)$. The solution of system (2.3) is then given by,

$$X(t) = \emptyset(t,0)\ X(0) + \int_0^t \emptyset(t,\tau)\, B_L\, dt.$$

Here, $\emptyset(t,\tau)$ is the state transition matrix given as $\emptyset(t,\tau) = \psi(t)\psi^{-1}(\tau)$, which exists at $\psi(t)$ is made up of a linearly independent solution of a homogeneous part of the equation (2.3).

In the next section, we have applied the above analytical technique for the prey-predator model with fuzzy initial conditions.

### 2.3. Linearized Fuzzy Prey-Predator Model

The Prey-Predator model with a fuzzy initial condition in the linearized form with the fuzzy initial condition is, represented by equation as in (2.3) with $\tilde{X}(0)$.

$$\dot{X} = A_L X + B_L \tag{2.4}$$

with $\tilde{X}(0) = [\tilde{x}_0, \tilde{y}_0]^T$.

The solution of equation (2.4) with fuzzy initial condition is given by,

$$\tilde{X}(t) = \emptyset(t,0)\tilde{X}(0) + \int_0^t \emptyset(t,\tau)\, B_l\, dt.$$

Here, $\emptyset(t,0) = \psi(t)\psi^{-1}(0)$, where $\psi(t)$ is a fundamental matrix of linearly independent columns of $A_L$.

Writing parametric form of the above equation,

$${}^{\alpha}\tilde{X}(t) = [\overline{X}, \underline{X}] = \emptyset(t,0)\,[\overline{X}_0, \underline{X}_0] + \int_0^t \emptyset(t,\tau)\, B_l\, dt.$$

Comparing the elements of an interval, we get,

$$\underline{X} = \emptyset(t,0)\,\underline{X_0} + \int_0^t \emptyset(t,\tau)\, B_l\, dt \tag{2.5}$$

$$\overline{X} = \emptyset(t,0)\ \overline{X_0} + (t)\int_0^t \emptyset(t,\tau)\, B_l\, dt \tag{2.6}$$

The state vector $\tilde{X}(t)$ can be constructed from (2.5) and (2.6) using, the First Decomposition theorem, Klir [70].

The solution $\tilde{X} = [\tilde{x}, \tilde{y}]^T$ obtained using (2.5) and (2.6) will be a fuzzy solution if, for $t > 0$, the following conditions are satisfied.

- $\forall\, \alpha \in (0\ \ 1], {}^{\alpha}\underline{x} \leq {}^{\alpha}\overline{x}$ and ${}^{\alpha}\underline{y} \leq {}^{\alpha}\overline{y}$
- $\forall \alpha, \beta \in (0\ \ 1], \alpha \leq \beta.\ {}^{\alpha}\underline{x} \leq {}^{\beta}\underline{x} \leq {}^{\beta}\overline{x} \leq {}^{\alpha}\overline{x}$ and ${}^{\alpha}\underline{y} \leq {}^{\beta}\underline{y} \leq {}^{\beta}\overline{y} \leq {}^{\alpha}\overline{y}$

Hence, $\forall\, \alpha \in (0\ \ 1]$, ${}^{\alpha}\tilde{x} = \left[\underline{x}, \overline{x}\right]$ and ${}^{\alpha}\tilde{y} = \left[\underline{y}, \overline{y}\right]$.

In the next section, we discuss the stability analysis of equation (2.2).

## 2.4. Stability Analysis

Consider, $f = ax - bxy$ and $g = -cy + dxy$ from the prey-predator model as in equation (2.2). Its equilibrium state is $\left(\frac{c}{d}, \frac{a}{b}\right)$.

Let's take perturbations $p(t)$ and $q(t)$ are in an equilibrium state, then equation (2.2) becomes,

$$\frac{d\big(x_e + p(t)\big)}{dt} = a\big(x_e + p(t)\big) - b\big(x_e + p(t)\big)\big(y_e + q(t)\big);$$

$$\frac{d\big(y_e + q(t)\big)}{dt} = -c\big(y_e + q(t)\big) + d\big(x_e + p(t)\big)\big(y_e + q(t)\big);$$

Now, putting $x_e = \frac{c}{d}, y_e = \frac{a}{b}$ in the above equations, we have,

$$\frac{dp(t)}{dt} = a\left(\frac{c}{d} + p(t)\right) - b\left(\frac{c}{d} + p(t)\right)\left(\frac{a}{b} + q(t)\right);$$

$$\frac{dq(t)}{dt} = -c\left(\frac{a}{b} + q(t)\right) + d\left(\frac{c}{d} + p(t)\right)\left(\frac{a}{b} + q(t)\right); \tag{2.7}$$

Simplifying equation (2.7) and taking appropriate assumptions, we have,

$$\frac{dp(t)}{dt} = -\frac{bc}{d}q$$

$$\frac{dq(t)}{dt} = \frac{da}{b}p \tag{2.8}$$

Now, Jacobian of equation (2.8) is,

$$\begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{bmatrix}$$

Eigenvalues of the above matrix are, $\lambda_1 = \sqrt{ac}\, i, \lambda_2 = -\sqrt{ac}\, i.$

The sum of eigenvalues is 0 i.e., $\lambda_1 + \lambda_2 = 0$ and product of eigenvalues $\lambda_1.\lambda_2 > 0$ i.e., positive.

Thus, equation (2.2) represents a stable system.

### 2.5. Illustrative examples

We solve two numerical examples on Prey-Predator Model by the proposed technique.

**1) Classical Prey-Predator Model**

Consider the following example of the Prey-Predator Model [73] in a crisp setup.

$$\dot{x}(t) = 0.1x - 0.005xy$$
$$\dot{y}(t) = -0.4y + 0.008xy \quad (2.9)$$

with initial condition $x(0) = 130; \; y(0) = 40.$

The system has two critical points, the trivial one is the origin and the other is $(50, 20)$. First, we linearize this problem at $(50, 20)$ by Taylor's expansion and get,

$$\dot{x}(t) = -0.25y + 5 \quad (2.10)$$
$$\dot{y}(t) = 0.16x - 8$$

The linearized system has a $2 \times 2$ coefficient matrix $C$ which is given as $\begin{bmatrix} 0 & -0.25 \\ 0.16 & 0 \end{bmatrix}$ and $B$ is $2 \times 1$ vector which is given as $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$.

Eigenvalues of this matrix are $-0.2i$ and $0.2i$ and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -0.8i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0.8i \end{bmatrix}$ respectively and the fundamental matrix is given by,

$$\Psi(t) = \begin{bmatrix} \cos(0.2t) & \sin(0.2t) \\ 0.8\sin(0.2t) & -0.8\cos(0.2t) \end{bmatrix}.$$

Thus, the solution of equation (2.6) is given by,

$$X(t) = \Psi(t)\Psi^{-1}(0)X(0) + \Psi(t)\int_0^t \Psi^{-1}(s)B\,ds.$$

i.e.

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} 130 \\ 40 \end{bmatrix}$$

$$+ \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \int_0^t \left( \begin{bmatrix} cos(0.2s) & -\frac{sin(0.2s)}{8i} \\ sin(0.2s) & \frac{cos(0.2s)}{8i} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) ds.$$

And we get,

$$x(t) = 50 + 80\,cos(0.2t) - 25\,sin(0.2t),$$

$$y(t) = 20 + 20\,cos(0.2t) + 64\,sin(0.2t).$$

The evolution of the system in a small-time interval is as shown in Fig. (2.1).

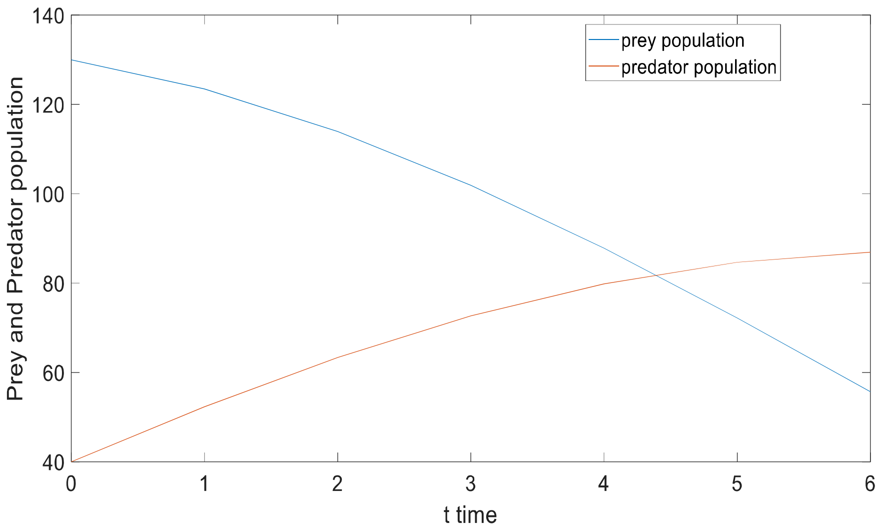


**Figure 2.1: Evolution of Crisp Prey Predator Population**

In equation (2.9), when we consider the fuzzy initial condition as below,

$$\tilde{x}_0 = \begin{cases} \dfrac{x-120}{10} & 120 < x \le 130 \\ \dfrac{150-x}{20} & 130 < x \le 150 \end{cases}$$

$$\tilde{y}_0 = \begin{cases} \frac{y-20}{10} & 20 < y \le 40 \\ \frac{50-y}{10} & 40 < y \le 50 \end{cases} \qquad (2.11)$$

Then, $\alpha - cut$ of above equation (2.11), initial conditions are obtained as follows,

$${}^{\alpha}\tilde{x}_0 = [10\alpha + 120\ ,150 - 20\alpha],$$

$${}^{\alpha}\tilde{y}_0 = [20\alpha + 20, 50 - 10\alpha],$$

where, $\alpha \in (0,1]$.

The solution of equation (2.9) with fuzzy initial conditions as in (2.11) by the proposed technique, is obtained as follows,

$${}^{\alpha}\tilde{X}(t) = \begin{bmatrix} {}^{\alpha}\tilde{x}(t) \\ {}^{\alpha}\tilde{y}(t) \end{bmatrix} = \begin{bmatrix} [\underline{x}, \overline{x}] \\ [\underline{y}, \overline{y}] \end{bmatrix}$$

$$= \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} 10\alpha + 120\ ,150 - 20\alpha \\ 20\alpha + 20, 50 - 10\alpha \end{bmatrix}$$

$$+ \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix}$$

$$\int_0^t \left( \begin{bmatrix} cos(0.2s) & -\frac{sin(0.2s)}{8i} \\ sin(0.2s) & \frac{cos(0.2s)}{8i} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) \text{ds} \qquad (2.12)$$

For, $\alpha = 1$, the solution of above equation is,

$$x(t) = 50 + 80\,cos(0.2t) - 25\,sin(0.2t),$$

$$y(t) = 20 + 20\,cos(0.2t) + 64\,sin(0.2t)$$

which is same as crisp solution of equation (2.9).

Put, $\alpha = 0$, in equation (2.12),

$${}^{0}\tilde{X}(t) = \begin{bmatrix} {}^{0}\tilde{x}(t) \\ {}^{0}\tilde{y}(t) \end{bmatrix} = \begin{bmatrix} [\underline{x}, \overline{x}] \\ [\underline{y}, \overline{y}] \end{bmatrix} = \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} [120\ ,150] \\ [20, 50] \end{bmatrix}$$

$$+ \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \int_0^t \left( \begin{bmatrix} cos(0.2s) & -\frac{sin(0.2s)}{8i} \\ sin(0.2s) & \frac{cos(0.2s)}{8i} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) ds$$

Now, comparing the components, we get

$$\underline{x} = 70\,cos(0.2t) + 50,$$

$$\underline{y} = 20 + 56\sin(0.2t),$$

$$\overline{x} = 50 + 100\,cos(0.2t) - 37.5\,sin(0.2t),$$

$$\overline{y} = 20 + 80\,sin(0.2t) + 30\cos(0.2t).$$

The evolution of the Prey and Predator population in the system (2.9) with fuzzy initial conditions, in a small-time interval, are as shown in Fig. (2.2) and Fig. (2.3) respectively.

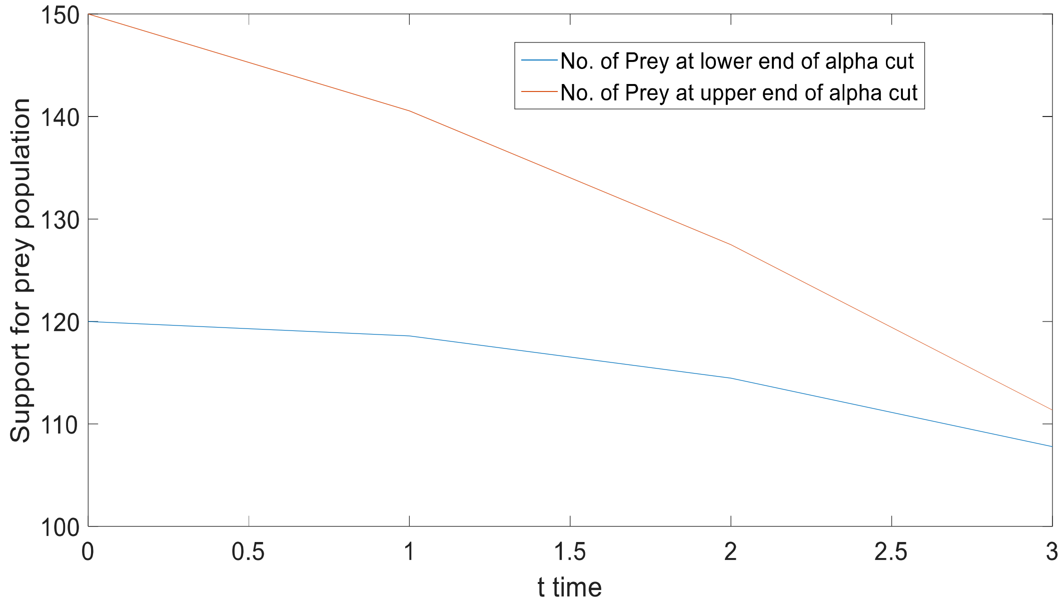


**Figure 2.2: Evolution of Fuzzy Prey Population**

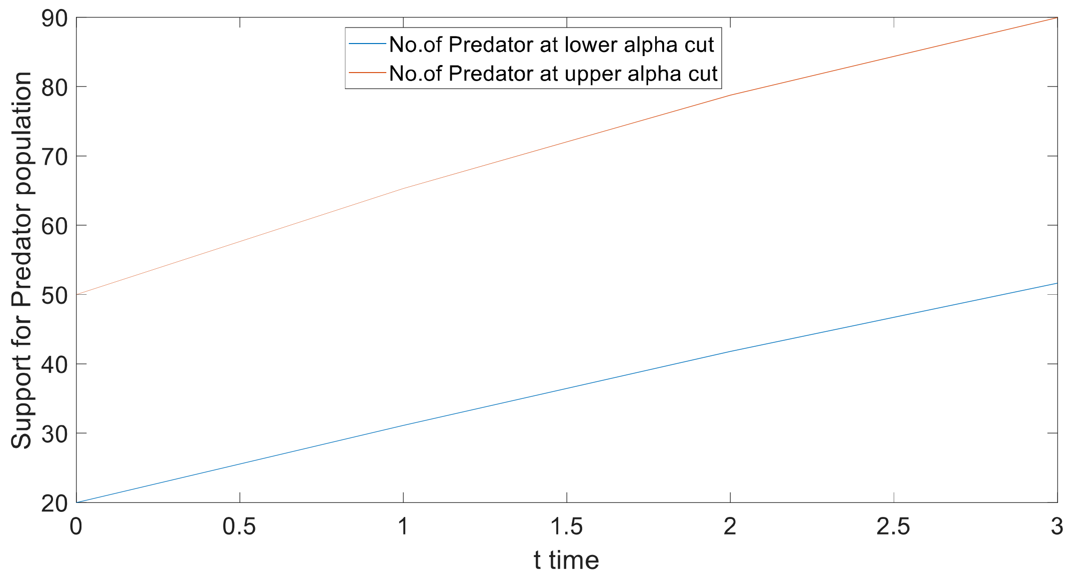


**Figure 2.3: Evolution of Fuzzy Predator Population**

Next, we solve a diabetes model in fuzzy setup by proposed technique, as follows.

For stability of equation (2.9), we have imaginary eigenvalues. So, the result from section 2.4, the system is stable.

**2) Mathematical Model of Diabetes Mellitus**

Diabetes mellitus is commonly known as diabetes. This chronic disease is a major concern among all the countries either poor or rich across the globe. In India, this disease has been spreading very rapidly. According to the International diabetes federation (IDF) projects that the number of Indians with diabetes will rise to 123 million by 2040 and currently 5% population of India is suffering from this disease. The major reason for being diabetic in India is food habits. Indian diet consists of fat and carbohydrates. Also, lacking in exercise or any other physical activity, people gain weight which is the key factor of being prey for this disease. The mathematical model for diabetes mellitus [75] is given below,

$$\dot{C}(t) = -0.68\, C(t) + 0.66N(t),$$
$$\dot{N}(t) = 6 \times 10^7 - 0.1\, C(t) - 0.02N(t). \tag{2.13}$$

For fuzzy initial populations,

$${}^{\alpha}\tilde{C}_0 = [2 \times 10^6\alpha + 45 \times 10^6, 45 \times 10^6 - 2 \times 10^6\alpha],$$

$${}^{\alpha}\widetilde{N}_0 = \left[605 \times 10^5 + 6 \times 10^5\alpha, 614 \times 10^5 - 3 \times 10^5\alpha\right],$$

where, $N(t)$ denotes the total number of diabetic patients with and without complications. $C(t)$ denotes the number of diabetic patients with complications.

For fuzzy initial conditions, on applying our proposed techniques, the following solutions are obtained.

$$\underline{N} = 31.762 \times 10^7 \times e^{-(0.145t)} + 18.876 \times 10^7 \times e^{-(0.555t)} - 50.7 \times 10^7 - 0.76 \times 605 \times 10^5 \times e^{-(0.145t)} + 1.76 \times 605 \times 10^5 \times e^{-(0.555t)},$$

$$\overline{N} = 31.762 \times 10^7 \times e^{-(0.145t)} + 18.876 \times 10^7 \times e^{-(0.555t)} - 50.7 \times 10^7 - 0.76 \times 614 \times 10^5 \times e^{-(0.145t)} + 1.76 \times 614 \times 10^5 \times e^{-(0.555t)},$$

$$\underline{C} = -49.62 + 39.10 \times e^{-(0.145t)} + 10.44 \times e^{-(0.555t)} + 0.99 \times 45 \times 10^6 \times e^{-(0.145t)} - 0.93 \times e^{-(0.145t)} + 0.93 \times e^{-(0.555t)},$$

$$\overline{C} = -49.62 + 39.10 \times e^{-(0.145t)} + 10.44 \times e^{-(0.555t)} + 0.99 \times 49 \times 10^6 \times e^{-(0.145t)} - 0.93 \times e^{-(0.145t)} + 0.93 \times e^{-(0.555t)}.$$

The pictorial representation is given in the following graphs,

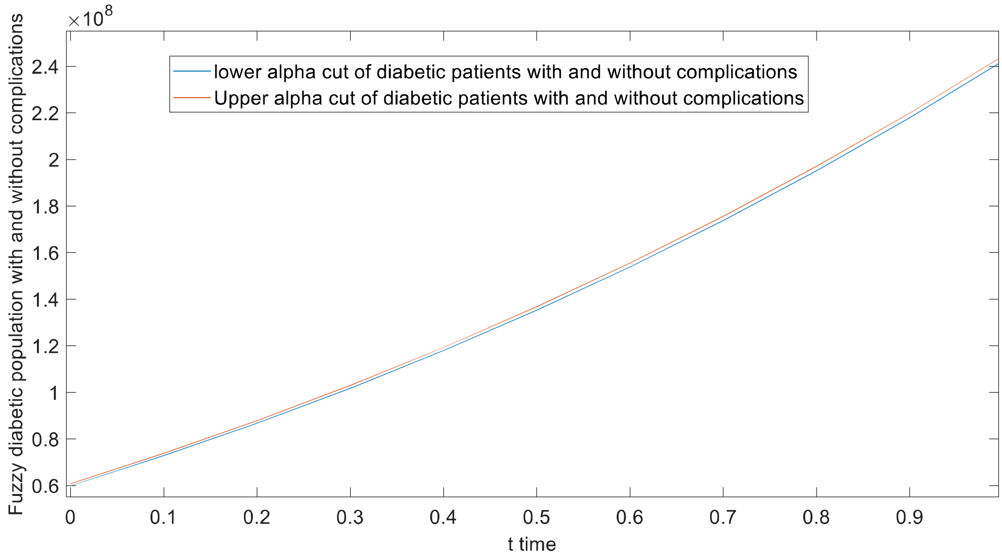


**Figure 2.4: Evolution of Fuzzy Population of diabetes patients with and without complications**

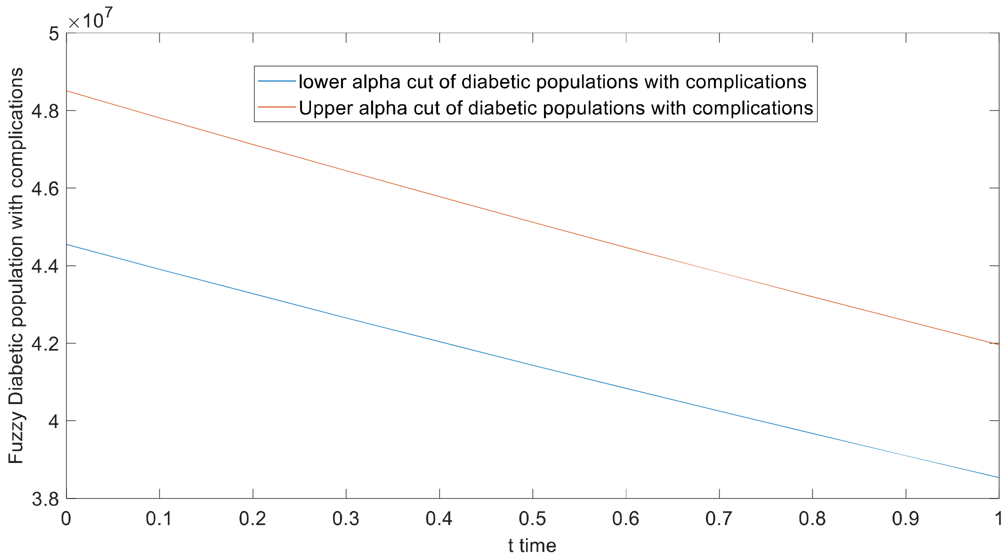


**Figure 2.5: Evolution of Fuzzy Population of diabetes patients with complications**

System such as given by (2.12) is stable as shown below.

Jacobian of (2.12) is given as follows,

$$\begin{bmatrix} -0.68 & 0.66 \\ -0.1 & -0.02 \end{bmatrix}$$

Eigenvalues of the above matrix, are $-0.5571$ and $-0.1429$. A sum of the eigenvalues, $-0.7$ and product of eigenvalues, $0.0796$. Hence, the system is stable.

## 2.6. Conclusion

Here, we have discussed the approximate solution of the Prey Predator model with a fuzzy initial condition. We obtained a closed-form solution of the model by using the proposed scheme. We have also discussed about the stability of both numerical examples. For fuzzy solution, from graphs of both illustrations, we have observed that support remains bounded as time increases.