3. Numerical Techniques

3.1. Introduction

In this chapter, we propose and establish results for the solution of fuzzy systems using numerical techniques. We give two numerical techniques [62-63] to solve such systems. First, numerical technique is based on discretization of Hukuhara derivative [4] and the other is on Improved Euler method [17]. We extend both numerical techniques for the system of fuzzy differential equations. We also give convergence analysis of both numerical techniques, which is based on complete error analysis. At the end the illustrative numericals are substantiated.

To apply numerical techniques, we consider the fuzzy system,

$$\dot{\tilde{X}}(t) = \tilde{f}(t, \tilde{X}), \tilde{X}(0) = \tilde{X}_0$$

where, $\tilde{f} : I \times E^n \to E^n$,

i.e., $\tilde{f} = \begin{bmatrix} \tilde{f}_1(t, \tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n) \\ \tilde{f}_2(t, \tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n) \\ \vdots \\ \tilde{f}_n(t, \tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n) \end{bmatrix}, \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}$ and $\tilde{X}_0 = \begin{bmatrix} \tilde{x}_{10} \\ \tilde{x}_{20} \\ \vdots \\ \tilde{x}_{n0} \end{bmatrix}$.

Here, each $\tilde{f}_1, \tilde{f}_2, \dots \tilde{f}_n$ are Hukuhara differentiable.

If $\tilde{f}$ can be put in the form $\tilde{f}(t, \tilde{X}) = \tilde{A} \otimes \tilde{X} \oplus \tilde{B},$ then it is linear otherwise nonlinear,

where, $\tilde{A}_{n\times n} = \begin{bmatrix} \tilde{a}_{11} & \dots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \dots & \tilde{a}_{nn} \end{bmatrix}$ and $\tilde{B}_{n\times 1} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$ contains fuzzy elements.

In the next section, we propose the first numerical technique that is based on the discretization of Hukuhara derivative.

3.2. Numerical Scheme using the Hukuhara Difference

We propose numerical technique by approximating the derivative using H-difference for linear as well as nonlinear systems. In next subsection, a numerical scheme for a fuzzy linear dynamical system is explained in detail.

3.2.1. $\tilde{f}(t,\tilde{X})$ is linear function

We consider the fully fuzzy linear dynamical system as follows,

$$\dot{\tilde{X}}(t) = \tilde{A} \otimes \tilde{X} \oplus \tilde{B}\ ;\ \ \tilde{X}(0) = \tilde{X}_0 \tag{3.1}$$

Taking $\alpha - cut$ on both sides of equation (3.1) we get,

$$^{\alpha}\dot{\tilde{X}}(t) = {}^{\alpha}\tilde{A} \otimes {}^{\alpha}\tilde{X} \oplus {}^{\alpha}\tilde{B}\ ;\ \ {}^{\alpha}\tilde{X}(0) = {}^{\alpha}\tilde{X}_0.$$

Writing the above equation in parametric form gives,

$$\left[\dot{\underline{X}}, \dot{\overline{X}}\right] = [\underline{A}, \overline{A}] \otimes [\underline{X}, \overline{X}] \oplus [\underline{B}, \overline{B}];\ \ {}^{\alpha}\tilde{X}_0 = [\underline{X}_0, \overline{X}_0].$$

Using fuzzy multiplication and addition as in Section 1.2.5 of chapter 1, we have

$$\begin{gathered}\left[\dot{\underline{X}}, \dot{\overline{X}}\right] = \left[\min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\overline{X}\right), \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\overline{X}\right)\right] \\ \oplus \left[\min\left(\underline{B}\,, \overline{B}\right), \max\left(\underline{B}\,, \overline{B}\right)\right]\end{gathered} \tag{3.2}$$

For notation, let $A_l = \min\left(\underline{A}\,, \overline{A}\right)$, $A_u = \max\left(\underline{A}\,, \overline{A}\right)$, $B_l = \min\left(\underline{B}\,, \overline{B}\right)$ and $B_u = \max\left(\underline{B}\,, \overline{B}\right)$. Using the above notations and comparing components of both sides, we get equations

$$\dot{\underline{X}} = A_l\,\underline{X} + B_l \tag{3.3}$$

$$\dot{\overline{X}} = A_u\overline{X} + B_u \tag{3.4}$$

with initial condition, ${}^{\alpha}\tilde{X}_0 = \left[\underline{X}_0, \overline{X}_0\right]^T$.

Equation (3.3) and (3.4) can be written as matrix form,

$$\begin{bmatrix}\dot{\underline{X}}(t) \\ \dot{\overline{X}}(t)\end{bmatrix} = \begin{bmatrix}A_l & 0 \\ 0 & A_u\end{bmatrix}\begin{bmatrix}\underline{X} \\ \overline{X}\end{bmatrix} + \begin{bmatrix}B_l \\ B_u\end{bmatrix}; \tilde{X}_0 = \begin{bmatrix}\underline{X}_0 \\ \overline{X}_0\end{bmatrix}. \tag{3.5}$$

Now replacing derivative, by Hukuhara difference as defined in [5], of the left side of equation (3.5) at $t = t_k$, we get,

$$\begin{bmatrix}\underline{X}_{k+1} \\ \overline{X}_{k+1}\end{bmatrix} = \begin{bmatrix}\underline{X}_k \\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}A_l & 0 \\ 0 & A_u\end{bmatrix}\begin{bmatrix}\underline{X}_k \\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}B_l \\ B_u\end{bmatrix};\ \tilde{X}_0 = \begin{bmatrix}\underline{X}_0 \\ \overline{X}_0\end{bmatrix}. \tag{3.6}$$

where, $k = 0, 1, 2, \ldots$

If ${}^{\alpha}\tilde{A} = [A_l, A_u]$ contains some negative fuzzy elements then their place in A_l and A_u will be interchanged in equation (3.5), as in [30] and [72].

In the next section, we give the convergence of solution by the proposed numerical technique.

3.2.2. Theorem

Let $\underline{X}(t), \overline{X}(t)$ be the exact solution of equation (3.5) and $\underline{X}_{k+1}(t), \overline{X}_{k+1}(t)$ be the numerical solutions defined by the equations (3.6) converges to the exact solution of equation (3.5) For sufficiently small h, the determinant value of $(I + hA_l)$ and $(I + hA_u)$ is less than 1.
i.e., $|I + hA_l| < 1$ and $|I + hA_u| < 1$.

Proof: It is sufficient to show,

$$\lim_{k\to\infty} \underline{X}_{k+1}(t) = \underline{X}(t)$$

$$\lim_{k\to\infty} \overline{X}_{k+1}(t) = \overline{X}(t)$$

From equation (3.5), writing the equivalent discrete form,

$$\begin{bmatrix}\underline{X}(t)\\ \overline{X}(t)\end{bmatrix} = \begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} + h\begin{bmatrix}A_l & 0\\ 0 & A_u\end{bmatrix}\begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} + h\begin{bmatrix}B_l\\ B_u\end{bmatrix}; \tilde{X}_0 = \begin{bmatrix}\underline{X}_0\\ \overline{X}_0\end{bmatrix} \quad (3.7)$$

By subtracting equation (3.7) from equation (3.6), we get

$$\begin{bmatrix}\underline{X}_{k+1} - \underline{X}\\ \overline{X}_{k+1} - \overline{X}\end{bmatrix} = \begin{bmatrix}\underline{X}_k - \underline{X}\\ \overline{X}_k - \overline{X}\end{bmatrix} + h\begin{bmatrix}A_l & 0\\ 0 & A_u\end{bmatrix}\begin{bmatrix}\underline{X}_k - \underline{X}\\ \overline{X}_k - \overline{X}\end{bmatrix}$$

Taking $\underline{E}_{k+1} = \underline{X}_{k+1} - \underline{X}$ and $\overline{E}_{k+1} = \overline{X}_{k+1} - \overline{X}$ as the errors at $(k+1)^{th}$ iteration, we can write,

$$\underline{E}_{k+1} = E_k + hA_l\, \underline{E}_k \quad (3.8\text{-a})$$

$$\overline{E}_{k+1} = \overline{E}_k + hA_u\, \overline{E}_k \quad (3.8\text{-b})$$

Now by backward substitution, equation (3.8-a) can be written as,

$$\underline{E}_{k+1} = (I + hA_l)^{(k+1)} E_0$$

Since, A_l is a nonsingular matrix, $(I + hA_l)$ is also nonsingular so the solution of the system exists.

For the convergence of equation (3.6),

$$|(I + hA_l)| < 1$$

Take $E_0 = 0, E_{k+1} \to 0$,

i.e.,

$$\lim_{k\to\infty} \underline{X}_{k+1}(t) = \underline{X}(t)$$

Similarly,

$$\lim_{k\to\infty} \overline{X}_{k+1}(t) = \overline{X}(t)$$

In the next section, we give a numerical scheme and its convergence for the system of nonlinear fuzzy differential equations.

3.2.3. $\tilde{f}$ is nonlinear function

Instead of representation (3.1) of fuzzy differential system, we consider more general form as,

$$\dot{\tilde{X}}(t) = \tilde{f}(t,\tilde{X}), \tilde{X}(0) = \tilde{X}_0$$

By using variables in parametric form,

$$^{\alpha}\dot{\tilde{X}} = \left[\dot{\underline{X}},\dot{\overline{X}}\right], \; ^{\alpha}\tilde{X} = [\underline{X}, \; \overline{X}], \; ^{\alpha}\tilde{f} = \left[\underline{f}, \; \overline{f}\right].$$

where, $\underline{f} = \min f(t,\underline{X},\overline{X}), \overline{f} = \max f(t,\underline{X},\overline{X})$.

The parametric form of the above equation can be written as,

$$\left[\dot{\underline{X}},\dot{\overline{X}}\right] = \left[\underline{f},\overline{f}\right]; \; ^{\alpha}\tilde{X}_0 = [\underline{X}_0,\overline{X}_0]$$

Now, comparing the component,

$$\dot{\underline{X}}(t) = \underline{f}(t,\underline{X},\overline{X}) \tag{3.9-a}$$

$$\dot{\overline{X}}(t) = \overline{f}(t,\underline{X},\overline{X}) \tag{3.9-b}$$

with fuzzy initial conditions $^{\alpha}\tilde{X}_0 = [\underline{X}_0,\overline{X}_0]$.

Now by the proposed scheme,

$$\underline{X}_{k+1} = \underline{X}_k + h\,\underline{f}(t_k,\underline{X}_k,\overline{X}_k) \tag{3.10-a}$$

$$\overline{X}_{k+1} = \overline{X}_k + \; h\overline{f}(t_k,\underline{X}_k,\overline{X}_k) \tag{3.10-b}$$

with fuzzy initial condition, $^{\alpha}\tilde{X}_0 = [\underline{X}_0,\overline{X}_0]$.

By using the Decomposition theorem as in Klir [70], the equations (3.10-a) and (3.10-b) is as follows,

$$\tilde{X}_{k+1} = \tilde{X}_k + h\tilde{f}(t_k,\tilde{X}_k);\tilde{X}_0.$$

In the next section, we give the result for convergence of solution.

3.2.4. Theorem

For systems (3.9-a) and (3.9-b), let $\underline{f}(t,\underline{X},\overline{X})$, $\overline{f}(t,\underline{X},\overline{X}) \in E^n$, their partial derivatives $\frac{\partial \underline{f}(t,\underline{X},\overline{X})}{\partial \underline{X}}$ and $\frac{\partial \overline{f}(t,\underline{X},\overline{X})}{\partial \overline{X}}$ bounded and Lipschitz in E^n. Let $\underline{X}$, $\overline{X}$ both are the exact solution of equations

(3.9-a) and (3.9-b) and $\underline{X}_k$, $\overline{X}_k$ both are the numerical solution of equations (3.10-a) and (3.10-b) at $t = t_k$ then numerical solution converges to the exact solution uniformly.

Proof: Define error in n^{th} term for $\underline{X}$ as,

$$\underline{e}_n = \underline{X} - \underline{X}_n, \overline{e}_n = \overline{X} - \overline{X}_n$$

Thus, next error term will be,

$$\underline{e}_{n+1} = \underline{X} - \underline{X}_{n+1}, \overline{e}_{n+1} = \overline{X} - \overline{X}_{n+1}$$

Considering, the first expression and replacing $\underline{X}_{n+1}$ by equation (3.10-a).

$$\underline{e}_{n+1} = \underline{X} - \underline{X}_{n+1}$$

By using Taylor's expansion and neglecting higher terms,

$$\underline{e}_{n+1} = \left[\underline{X} + h\underline{X'} + \frac{h^2}{2}\underline{X''}(t_n + \theta h) - \left(\underline{X}_n + h\underline{f}\left(t_n, \underline{X}_n, \overline{X}_n\right)\right)\right]$$

$$\underline{e}_{n+1} = \left(\underline{X} - \underline{X}_n\right) + h(\underline{X'} - \underline{f}\left(t_n, \underline{X}_n, \overline{X}_n\right) + \frac{h^2}{2}\underline{X''}(t_n + \theta h)$$

By taking modulus $\left|\underline{X''}(t_n + \theta h)\right| \le M, 0 \le \theta \le 1$ and using Lipschitz condition for function $\underline{f}$ with Lipschitz constant $\underline{L}$.

$$\left|\underline{e}_{n+1}\right| \le \left|\underline{e}_n\right|\left(1 + h\underline{L}\right) + M\frac{h^2}{2}$$

Computing in a backward manner we get at kth step, $k < n$,

$$\left|\underline{e}_n\right| \le \left|\underline{e}_{n-1}\right| + h\,\underline{L}\left|\underline{e}_{n-1}\right| + M\frac{h^2}{2}$$

$$\left|\underline{e}_{k-1}\right| \le \left[\left|\underline{e}_{k-2}\right|\left(1 + h\,\underline{L}\right) + M\frac{h^2}{2}\right]\left(1 + h\underline{L}\right) + M\frac{h^2}{2}$$

$$\vdots$$

$$\left|\underline{e}_1\right| \le \left|\underline{e}_0\right| + h\,\underline{L}\left|\underline{e}_0\right| + M\frac{h^2}{2}$$

Applying lemma as in [10] we get,

$$\left|\underline{e}_n\right| \le \left|\underline{e}_0\right|\left(1 + h\underline{L}\right)^n + M\frac{h^2}{2}\frac{((1 + h\underline{L})^n - 1)}{h\underline{L}}$$

$$\left|\underline{e}_n\right| \le \left|\underline{e}_0\right|\left(1 + h\underline{L}\right)^n + M\frac{h^2}{2}\frac{((1 + h\underline{L})^n - 1)}{\left(1 + h\underline{L} - 1\right)}$$

$$\left|\underline{e}_n\right| \le \left|\underline{e}_0\right|\left(1 + h\underline{L}\right)^n + M\frac{h^2}{2}\frac{((1+h\underline{L})^n - 1)}{h\underline{L}},$$

Assuming the initial error $\underline{e}_0 = 0$ (obviously) and $\left(1 + h\underline{L}\right) \le e^{h\underline{L}}$.

$$\left|\underline{e}_n\right| \leq M\frac{h^2}{2}\frac{(\mathrm{e}^{nh\underline{L}}-1)}{h\underline{L}}$$

$$\left|\underline{e}_n\right| \leq M\frac{h^2}{2}\frac{(\mathrm{e}^{nh\underline{L}}-1)}{h\underline{L}}$$

which is valid for $0 \leq t_n = nh \leq T \rightarrow$ stability, finally,

$$\left|\underline{e}_n\right| \leq M\frac{h^2}{2}\frac{(\mathrm{e}^{T\underline{L}}-1)}{h\underline{L}}$$

as $h \rightarrow 0$, we have $\left|\underline{e}_n\right| \rightarrow 0$,

$$\left|\underline{e}_n\right| \leq M\frac{h^2}{2}\frac{(\mathrm{e}^{T\underline{L}}-1)}{h\underline{L}} = M\frac{h}{2}\frac{(\mathrm{e}^{T\underline{L}}-1)}{\underline{L}} = O(h).$$

Similarly, we can show linear convergence for error term $|\overline{e}_n|$, using $\overline{f}(t,\underline{X},\overline{X})$ with Lipschitz constant $\overline{L}$.

Let $L = \min(\underline{L},\overline{L})$ be Lipschitz constant for function $\tilde{f}$, then this will establish the linear convergence of the proposed iterative method.

In the next section, we propose and establish results for another numerical technique Improved Euler method for fuzzy systems.

3.3. Improved Euler Method

We again consider a system of fully fuzzy differential equations,

$$\dot{\tilde{X}}(t) = \tilde{f}(t,\tilde{X}), \tilde{X}(0) = \tilde{X}_0$$

Again, putting in the parametric form as in equations (3.9-a) and (3.9-b) we get

$$\underline{\dot{X}}(t) = \underline{f}(t,\underline{X},\overline{X})$$

$$\dot{\overline{X}}(t) = \overline{f}(t,\underline{X},\overline{X})$$

with fuzzy initial conditions ${}^{\alpha}\tilde{X}_0 = [\underline{X}_0,\overline{X}_0]$.

Now approximating the Hukuhara derivative using the first order derivative, as in Section 3.2.3, we propose the numerical scheme as follows,

$$\tilde{X}^*{}_{k+1} = \tilde{X}_k + h\,\tilde{f}\,(t_k,\tilde{X}_k)$$

$$\tilde{X}_{k+1} = \tilde{X}_k + \frac{h}{2}\{\tilde{f}\,(t_k,\tilde{X}_k) + \tilde{f}\,(t_{k+1},\tilde{X}^*{}_{k+1})\} \quad (3.11)$$

where, $k = 0,1,2,3,\ldots$

After taking $\alpha - cut$ of equation (3.11),

$${}^{\alpha}\tilde{X}^*{}_{k+1} = {}^{\alpha}\tilde{X}_k + h\;{}^{\alpha}\tilde{f}(t_k,\,{}^{\alpha}\tilde{X}_k)$$

$$ {}^{\alpha}\tilde{X}_{k+1} = {}^{\alpha}\tilde{X}_k + \frac{h}{2}\{ {}^{\alpha}\tilde{f}\left(t_k, {}^{\alpha}\tilde{X}_k\right) + {}^{\alpha}\tilde{f}\left(t_{k+1}, {}^{\alpha}\tilde{X}^*{}_{k+1}\right)\} $$

The parametric form of equation (3.11) is as follows,

$$ [\underline{X}^*{}_{k+1}, \overline{X}^*{}_{k+1}] = [\underline{X}_k, \overline{X}_k] + h\,[\underline{f}\,(t_k, \underline{X}_k, \overline{X}_k), \overline{f}(t_k, \underline{X}_k, \overline{X}_k)] $$

$$ [\underline{X}_{k+1}, \overline{X}_{k+1}] = [\underline{X}_k, \overline{X}_k] + \frac{h}{2}\Big\{\left[\underline{f}\,(t_k, \underline{X}_k, \overline{X}_k), \overline{f}(t_k, \underline{X}_k, \overline{X}_k)\right] + \left[\underline{f}\,\left(t_{k+1}, \underline{X}^*{}_{k+1}, \overline{X}^*{}_{k+1}\right), \overline{f}\,\left(t_{k+1}, \underline{X}^*{}_{k+1}, \overline{X}^*{}_{k+1}\right)\right]\Big\} $$

$$ [\underline{X}_{k+1}, \overline{X}_{k+1}] = [\underline{X}_k, \overline{X}_k] + h[\underline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k), \overline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k)] \qquad (3.12) $$

where,

$$ \underline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k) = \frac{1}{2}\left\{\underline{f}\,(t_k, \underline{X}_k, \overline{X}_k) + \underline{f}\,\left(t_{k+1}, \underline{X}^*{}_{k+1}, \overline{X}^*{}_{k+1}\right)\right\} $$

$$ \overline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k) = \frac{1}{2}\left\{\overline{f}(t_k, \underline{X}_k, \overline{X}_k) + \overline{f}\,\left(t_{k+1}, \underline{X}^*{}_{k+1}, \overline{X}^*{}_{k+1}\right)\right\}. $$

Comparing the component of equation (3.12),

$$ \underline{X}_{k+1} = \underline{X}_k + h\underline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k) \qquad (3.13\text{-a}) $$

$$ \overline{X}_{k+1} = \overline{X}_k + h\overline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k) \qquad (3.13\text{-b}) $$

In the following section, we give results for existence and convergence of numerical solution of equations (3.9-a) and (3.9-b) by the proposed numerical scheme. For the purpose following lemma is required.

3.3.1. Lemma

${}^{\alpha}\tilde{\emptyset} = [\underline{\emptyset}, \overline{\emptyset}]$ defined in equation (3.12) is Lipchitz if ${}^{\alpha}\tilde{f} = [\underline{f}, \overline{f}]$ is Lipschitz.

Proof: Let ${}^{\alpha}\tilde{X} = [\underline{X}, \overline{X}]$ and ${}^{\alpha}\tilde{U} = [\underline{U}, \overline{U}]$ are in E^n.

$$ \left|\underline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k) - \underline{\emptyset}(t_k, \underline{U}_k, \overline{U}_k)\right| $$

$$ = \left|\frac{1}{2}\left\{\underline{f}\,(t_k, \underline{X}_k, \overline{X}_k) + \underline{f}\,\left(t_{k+1}, \underline{X}^*{}_{k+1}, \overline{X}^*{}_{k+1}\right)\right\} - \frac{1}{2}\left\{\underline{f}\,(t_k, \underline{U}_k, \overline{U}_k) + \underline{f}\,\left(t_{k+1}, \underline{U}^*{}_{k+1}, \overline{U}^*{}_{k+1}\right)\right\}\right| $$

$$ \leq \frac{1}{2}\left[\left|\underline{f}\,(t_k, \underline{X}_k, \overline{X}_k) - \underline{f}\,(t_k, \underline{U}_k, \overline{U}_k)\right| + \left|\underline{f}\,\left(t_{k+1}, \underline{X}^*{}_{k+1}, \overline{X}^*{}_{k+1}\right) - \underline{f}\,\left(t_{k+1}, \underline{U}^*{}_{k+1}, \overline{U}^*{}_{k+1}\right)\right|\right] $$

We know that $\underline{f}$ is Lipschitz continuous function with $\underline{L}$ as a Lipschitz constant and value of $\underline{U}^*_{k+1}, \underline{X}^*_{k+1}$ are calculated by Euler's method. So, we have,

$$\leq \underline{L}\left|\underline{X}_k - \underline{U}_k\right| + \frac{1}{2}\underline{L}h^2\left|\underline{X}_k - \underline{U}_k\right|$$

$$\leq \left(\left|\underline{X}_k - \underline{U}_k\right|\left(\underline{L} + \frac{1}{2}\underline{L}h^2\right)\right).$$

Thus, $\underline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k)$ is Lipchitz continuous function with constant $\underline{L'} = \left(\underline{L} + \frac{1}{2}\underline{L}h^2\right)$.

Similarly, $\overline{\emptyset}(t_k, \underline{X}_k, \overline{X}_k)$ is also Lipchitz continuous function with constant $\overline{L'} = \left(\overline{L} + \frac{1}{2}\overline{L}h^2\right)$.

Hence, ${}^{\alpha}\tilde{\emptyset} = [\underline{\emptyset}, \overline{\emptyset}]$ is Lipchitz function with Lipschitz constant $L = \min\left(\underline{L'}, \overline{L'}\right)$.

In the next section, we establish convergence result for the proposed scheme.

3.3.2. Theorem

Let $[\underline{X}, \overline{X}]$ be the exact solution of equations (3.9-a) and (3.9-b) and $[\underline{X}_n, \overline{X}_n]$ be the numerical solution of equations (3.13-a) and (3.13-b) and $[\underline{\emptyset}, \overline{\emptyset}]$ is Lipschitz in equations (3.13-a) and (3.13-b) then numerical solution converges to exact solution if ${}^{\alpha}\tilde{f} = [\underline{f}, \overline{f}]$ is differentiable.

Proof: We prove convergence of proposed technique as in Section 3.3 as follows,

Define the error at $(n+1)^{th}$ term,

$$\left|\underline{e}_{n+1}\right| = \left|\underline{X} - \underline{X}_{n+1}\right|$$

Using the value of $\underline{X}_{n+1}$ is from equation (3.13-a) and expand $\underline{X}$ by Taylor's expansion, we have,

$$\left|\underline{e}_{n+1}\right| = \left|\underline{X} + h\underline{X'} + \frac{h^2}{2!}\underline{X''} + \frac{h^2}{3!}\underline{X'''} + O(h^4) - \left(\underline{X}_n + h\underline{\emptyset}(t_n, \underline{X}_n, \overline{X}_n)\right)\right| \tag{3.14}$$

We know that from Section 3.3, the value of $\underline{\emptyset}(t_n, \underline{X}_n, \overline{X}_n)$ is as follows,

$$\underline{\emptyset}(t_n, \underline{X}_n, \overline{X}_n) = \frac{1}{2}\left\{\underline{f}\left(t_n, \underline{X}_n, \overline{X}_n\right) + \underline{f}\left(t_{n+1}, \underline{X}^*_{n+1}, \overline{X}^*_{n+1}\right)\right\}$$

Using value of $\underline{\emptyset}(t_n, \underline{X}_n, \overline{X}_n)$ in equation (3.14), we get,

$$|\underline{e}_{n+1}| = \left|\underline{X} + h\underline{X}' + \frac{h^2}{2!}\underline{X}'' + \frac{h^2}{3!}\underline{X}''' + 0(h^4) - \left(\underline{X}_n + \frac{h}{2}\left\{\underline{f}\left(t_n, \underline{X}_n, \overline{X}_n\right) + \underline{f}\left(t_{n+1}, \underline{X}^*_{n+1}, \overline{X}^*_{n+1}\right)\right\}\right)\right| \tag{3.15}$$

Expanding this term $\underline{f}\left(t_{n+1}, \underline{X}^*_{n+1}, \overline{X}^*_{n+1}\right)$ for $\left(t_{n+1}, \underline{X}^*_{n+1}\right)$ by Taylor's theorem, we get,

$$\underline{f}\left(t_{n+1}, \underline{X}^*_{n+1}, \overline{X}^*_{n+1}\right) = \underline{f}\left(t_n, \underline{X}^*_n\right) + h\underline{f}_t + g\underline{f}_{\underline{X}_n} + h^2\underline{f}_{tt} + g^2\underline{f}_{\underline{X}_n\underline{X}_n} + 2hg\underline{f}_{t\underline{X}_n} + O(h^3) \tag{3.16}$$

We know that,

$$\underline{X}' = \underline{f}\left(t_n, \underline{X}_n, \overline{X}_n\right), \underline{X}'' = \underline{f}\,\underline{f}_{\underline{X}_n} + \underline{f}_t, \ \underline{X}''' = \underline{f}_{tt} + 2\underline{f}\,\underline{f}_{t\underline{X}_n} + \underline{f}_t\underline{f}_{\underline{X}_n} + \underline{f}^2\,\underline{f}_{\underline{X}_n\underline{X}_n} + \underline{f}\,\underline{f}^2_{\underline{X}_n}$$

The above-mentioned partial derivatives are bounded because $\underline{\emptyset}\left(t_n, \underline{X}_n, \overline{X}_n\right) = \frac{1}{2}\left\{\underline{f}\left(t_n, \underline{X}_n, \overline{X}_n\right) + \underline{f}\left(t_{n+1}, \underline{X}^*_{n+1}, \overline{X}^*_{n+1}\right)\right\}$ is Lipschitz continuous.

Using above expression and equation (3.16) in equation (3.15), we obtain,

$$|\underline{e}_{n+1}| = h^3\left(\frac{-1}{12}\left(\underline{f}_{tt} + 2\underline{f}\,\underline{f}_{t\underline{X}_n} + \underline{f}^2\,\underline{f}_{\underline{X}_n\underline{X}_n} + \underline{f}^2\,\underline{f}_{t\underline{X}_n}\right) + \frac{1}{6}\left(\underline{f}_t\underline{f}_{\underline{X}_n} + \underline{f}\,\underline{f}^2_{\underline{X}_n}\right)\right) + O(h^4)$$

As, $h \to 0$, the error term $|\underline{e}_{n+1}|$ becomes zero.

Similarly, we can show, the error term $|\overline{e}_{n+1}|$ becomes zero.

Thus, the numerical solution converges to the exact solution.

3.4. Application (Spread of Infectious Disease Model)

The mathematical model of the spread of infectious disease model is taken from [74].

Consider the population of people, P_0 and a certain contagious disease that infects the people. This population is divided into three parts, $x(t)$ denotes susceptible persons, $y(t)$ represents those presently infected and may spread the disease, $z(t)$ denotes the number of persons, already dead due to infection, recovered or cannot spread the disease. So, we will consider the dynamics for $x(t), y(t)$ only.

The rate of disease transmission from x into y is directly proportional to xy.

i.e., $$\dot{x}(t) = -kx(t)y(t).$$

The rate of transfer into y comes from x and the rate of transfer out of y goes to z which is proportional to y.

i.e., $$\dot{y}(t) = kx(t)y(t) - cy(t),$$

where k and c are positive constants and population $P_0 = x(t) + y(t) + z(t)$.

These parameters k and c depend on many factors like season, the severity of disease, the behavior of people etc. So, variation in parameters result following fuzzy model,

$$\dot{\tilde{x}} = \ominus\tilde{k} \otimes \tilde{x} \otimes \tilde{y}$$

$$\dot{\tilde{y}} = \tilde{k} \otimes \tilde{x} \otimes \tilde{y} \ominus \tilde{c} \otimes \tilde{y}$$

with initial conditions, $\tilde{x}_0$,$\tilde{y}_0$

where, the values of parameters are given as,

$\tilde{k} = (0.003, 0.005, 0.007), \tilde{c} = (0.6, 0.9, 1.2)$

$\tilde{x}_0 = (920, 950, 980), \tilde{y}_0 = (20, 50, 80)$

The parametric form of above-mentioned parameters, is given as,

${}^{\alpha}\tilde{k} = {}^{\alpha}\widetilde{0.005} = (0.003 + 0.002\alpha, 0.007 - 0.002\alpha)$,

${}^{\alpha}\tilde{c} = {}^{\alpha}\widetilde{0.9} = (0.6 + 0.3\alpha, 1.2 - 0.3\alpha)$,

${}^{\alpha}\tilde{x}_0 = {}^{\alpha}\widetilde{950} = (920 + 30\alpha, 980 - 30\alpha)$,

${}^{\alpha}\tilde{y}_0 = {}^{\alpha}\widetilde{50} = (20 + 30\alpha, 80 - 30\alpha)$ and $P_0 = 1000$.

Applying the first numerical technique as given in section 3.2, we have obtained the fuzzy solution of the above disease model, as given in fig. 3.1 and fig. 3.2.

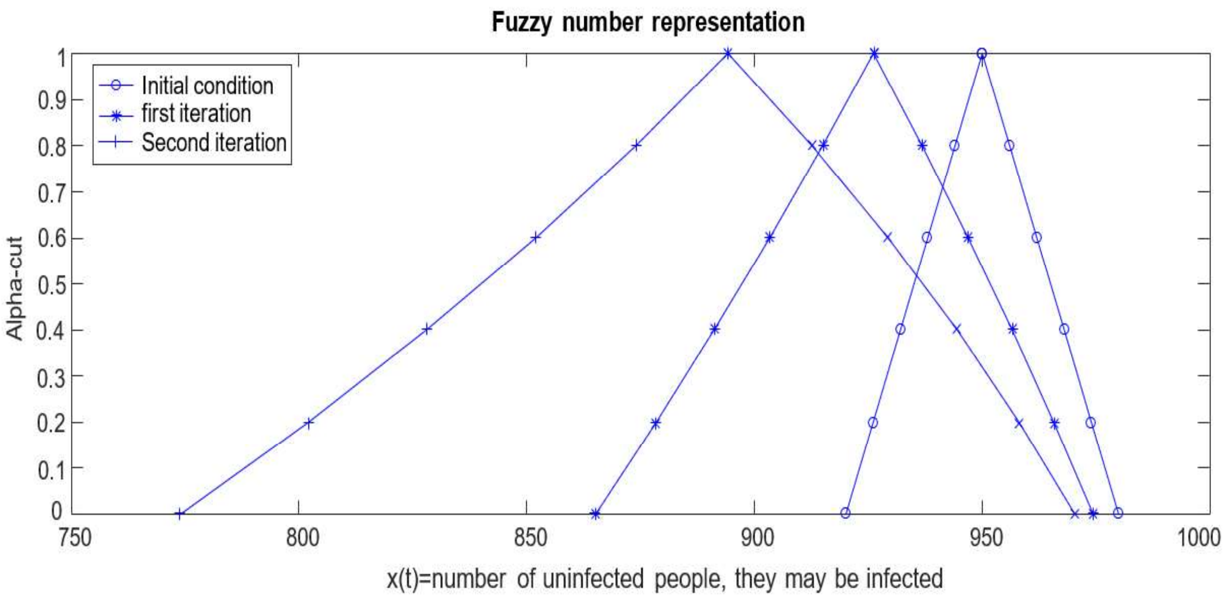

Figure 3.1: Number of uninfected people those may be infected

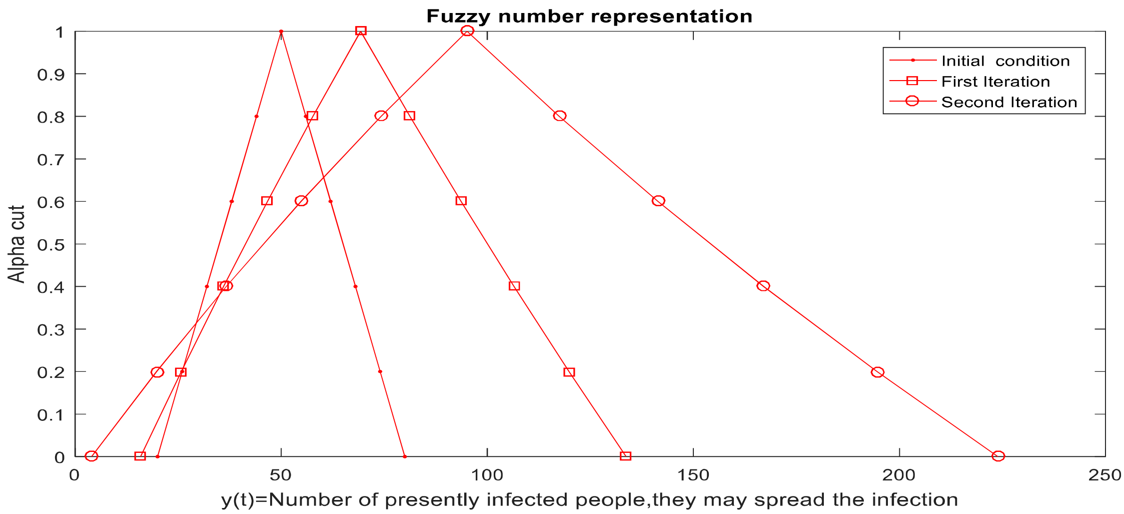

Figure 3.2: Number of infected people those may spread infection

From fig. 3.1 and fig 3.2, we have observed that solution remains fuzzy under the proposed scheme and support i.e., the number of susceptible people and presently the number of infected people increase with time.

Applying the second numerical technique on Section 3.4 as given in Section 3.3, we obtained the fuzzy solution as in fig 3.3 and fig 3.4.

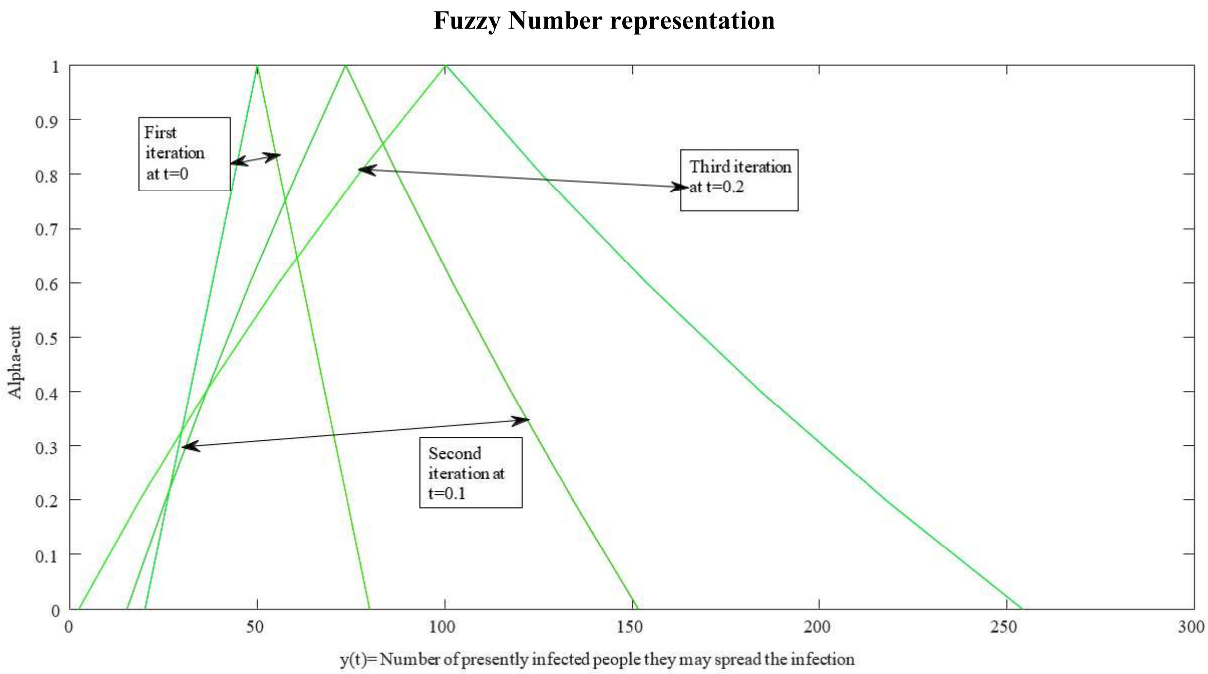

Figure 3.3: Number of infected people they may spread infection by the second technique

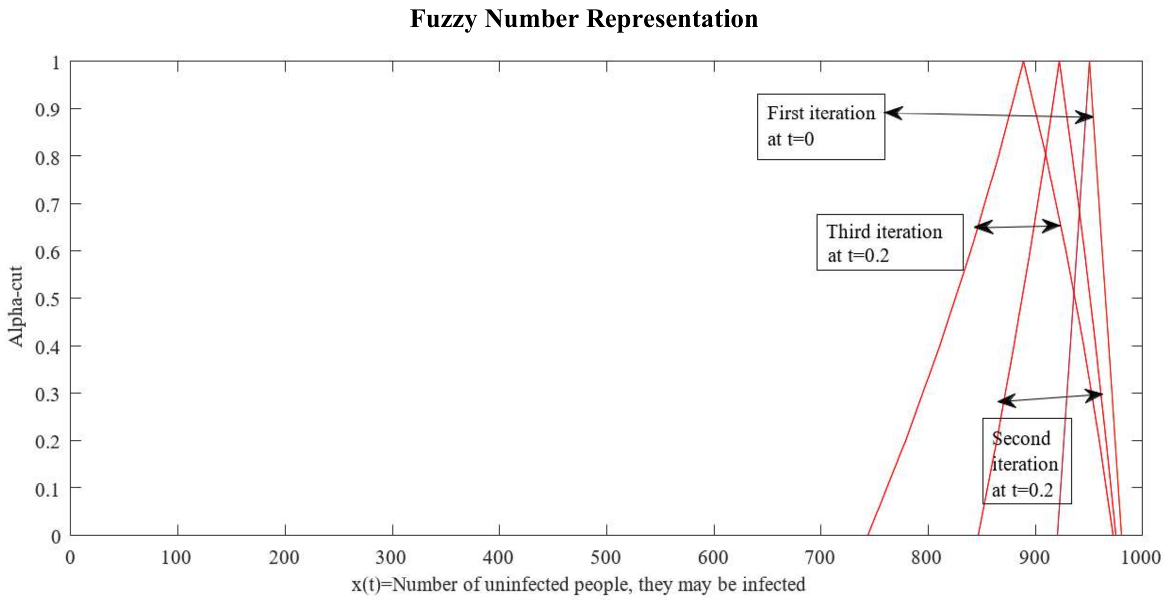

Figure 3.4: Number of uninfected people who may be infected by the second technique

From the above graphs, we can see solution remains fuzzy under the Improved Euler method and supports increases with time. In both proposed techniques, fuzzy solution matches with crisp solution at the core.

3.5. Conclusion

In this chapter, we have proposed numerical techniques to solve fully fuzzy dynamical systems. We have proved convergence for the proposed schemes based on complete error analysis and applied it to real-world problem and compared the results with crisp solution. A fuzzy scenario helps us in estimating the more realistic value of the variable so that we can apply the treatment accordingly.