5. Semi-Analytical Technique

5.1. Introduction

Semi-Analytical techniques such as Adomian Decomposition Method (ADM), Homotopy Perturbation method (HPM) and Variational iterative method (VIM) [22-27] etc. are powerful tools to solve nonlinear differential equations.

In this chapter, we focus on ADM to solve fuzzy semi-linear dynamical systems. We consider two approaches for solving fuzzy systems using ADM. The First approach is to solve a fuzzy semi linear dynamical system using fuzzy ADM [66] in parametric form. In this approach, we convert fuzzy dynamical systems into the parametric form, taking $\alpha - cut$ and applying fuzzy ADM in that form. For solving fuzzy differential equations (FDE) or systems using the parametric form, convert FDEs into their counter crisp part. In the second approach, we developed Fuzzy Adomian Decomposition Method (FADM) [69] in a complete fuzzy environment. The advantage of this technique is that we solve problem in a complete fuzzy environment instead of bringing it to the crisp form for solving and going back in fuzzy as done conventionally. For establishing this technique, we require some results on fuzzy power series along with its convergence and Fuzzy Taylor's theorem.

In this chapter, we also consider the complexity involved in fuzzy mapping when input and output both are fuzzy and redefine Modified Hukuhara derivative and name it Modified generalized Hukuhara (mgH) derivative. So, all the results related to FADM are proposed under mgH-derivative. In the proposed results we have in general considered any kind of fuzzy numbers, positive as negative. Lastly, we give theorem for convergence of FADM followed by numerical examples.

In the next section, we give a brief introduction of the Adomian Decomposition method in a crisp environment as given in [23].

5.2. Adomian Decomposition Method in a crisp environment (ADM):

Consider a general form of the nonlinear differential equation, given as,

$$\mathbb{L}u + Nu + \mathbb{R}u = g;\ u(0) = u_0,$$

where, $\mathbb{L}$ is the highest linear differentiable operator, N is nonlinear operator, $\mathbb{R}$ is the operator of less order than that of $\mathbb{L}$ and g is source term. Then applying, $\mathbb{L}^{-1}$ - integration operator on both sides we get,

$$\mathbb{L}^{-1}\mathbb{L}u = \mathbb{L}^{-1}g - \mathbb{L}^{-1}(Nu) - \mathbb{L}^{-1}(\mathbb{R}u),$$

which gives,

$$u = u_0 + \mathbb{L}^{-1}g - \mathbb{L}^{-1}(Nu) - \mathbb{L}^{-1}(\mathbb{R}u)$$

By ADM as in [23], we consider a series solution, $u = \sum_{n=0}^{\infty} u_n$. Also, the nonlinear term Nu is decomposed in a series of Adomian polynomials, i.e., $Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, .. u_n)$.

Thus, $\sum_{n=0}^{\infty} u_n = u_0 + \mathbb{L}^{-1}g - \mathbb{L}^{-1}(\sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, .. u_n)) - \mathbb{L}^{-1}(\mathbb{R}\sum_{n=0}^{\infty} u_n)$

where,
$$A_n = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(N\sum_{k=0}^{\infty} u_k \lambda^k\right)\right]_{\lambda=0} \tag{5.1}$$

and λ is the decomposition factor.

Using recurrence relation, we have,

$$u_n = u_0 + \mathbb{L}^{-1}g - \mathbb{L}^{-1}(A_{n-1}) - \mathbb{L}^{-1}\mathbb{R}(u_{n-1}) \tag{5.2}$$

Putting $n = 1,2,3$ ... in equation (5.2) and using equation (5.1) we get,

$$u_1 = u_0 + \mathbb{L}^{-1}g - \mathbb{L}^{-1}(A_0) - \mathbb{L}^{-1}\mathbb{R}(u_0)$$
$$u_2 = u_0 + \mathbb{L}^{-1}g - \mathbb{L}^{-1}(A_1) - \mathbb{L}^{-1}\mathbb{R}(u_1)$$
$$\vdots$$

Thus, the solution of general form of the nonlinear equation given as,

$$u = u_0 + u_1 + u_2 + \cdots$$

In the next section, we present a method that we have adopted to convert FDE into a system of ODE, considering, in general, positive as well as negative fuzzy numbers.

5.2.1. Fuzzy Semi-linear dynamical system

We consider fully fuzzy semi-linear dynamical system as in chapter 4,

$$\dot{\tilde{X}}(t) = \tilde{A} \otimes \tilde{X} \oplus \tilde{f}(t, \tilde{X}); \tilde{X}(0) = \tilde{X}_0 \tag{5.3}$$

Using these notations ${}^{\alpha}\dot{\tilde{f}} = [\dot{\underline{f}}, \dot{\overline{f}}]$, ${}^{\alpha}\dot{\tilde{X}} = [\dot{\underline{X}}, \dot{\overline{X}}]$ and ${}^{\alpha}\tilde{X} = [\underline{X}, \overline{X}]$, writing equation (5.3) in parametric form, as follows,

$$\begin{aligned}[\dot{\underline{X}}, \dot{\overline{X}}] &= [\min(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}), \max(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX})] + [\underline{f}, \overline{f}] \\ [\underline{X}_0, \overline{X}_0] &= \left[\min\left(\underline{X}(0), \overline{X}(0)\right), \max\left(\underline{X}(0), \overline{X}(0)\right)\right]\end{aligned} \tag{5.4}$$

where, $\dot{\underline{f}} = \min \dot{f}(t, \underline{X}, \overline{X})$, $\dot{\overline{f}} = \max \dot{f}(t, \underline{X}, \overline{X})$.

Comparing component wise on both the sides of equation (5.4), we get

$$\dot{\underline{X}} = \min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right) + \underline{f} \;\; ; \; \underline{X}_0 = \min\left(\underline{X}(0), \overline{X}(0)\right) \quad \text{(5.5-a)}$$

$$\dot{\overline{X}} = \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right) + \overline{f} \;\; ; \; \overline{X}_0 = \max\left(\underline{X}(0), \overline{X}(0)\right) \quad \text{(5.5-b)}$$

In the next section, we show that equation (5.3) is equivalent to equation (5.4) by the following lemma.

5.2.2. Lemma

A fuzzy semi-linear dynamical system is given in (5.3) is equivalent to system (5.4) iff ${}^{\alpha}\tilde{f} = \left[\underline{f}, \overline{f}\right]$ is equicontinuous, uniformly bounded and Lipschitz.

Proof: Since, $\left[\underline{f}, \overline{f}\right]$ is equi-continuous and Lipschitz in parametric form,

which gives,

$$\left|\underline{f}\left(t, \underline{X}, \overline{X}\right) - \underline{f}\left(t, \underline{X}_1, \overline{X}_1\right)\right| < \underline{L}\left|\underline{X} - \underline{X}_1\right|$$

and,

$$\left|\overline{f}\left(t, \underline{X}, \overline{X}\right) - \overline{f}\left(t, \underline{X}_1, \overline{X}_1\right)\right| < \overline{L}\left|\overline{X} - \overline{X}_1\right|$$

where, $L = \min(\underline{L}, \overline{L})$ is Lipchitz constant of $\tilde{f}$.

Thus equation (5.4) has a unique solution in parametric form i.e., $\underline{X}(t)$ and $\overline{X}(t)$ are the solution of equations (5.5-a) and (5.5-b) respectively.

Conversely, let ${}^{\alpha}\tilde{X}(t) = \left[\underline{X}(t), \overline{X}(t)\right]$ be the solution of equation (5.4) which implies that $\underline{X}(t)$ and $\overline{X}(t)$ are the solution of equation (5.5-a) and (5.5-b) component wise, also ${}^{\alpha}\tilde{f} = \left[\underline{f}, \overline{f}\right]$ is Lipschitz. So, it guarantees the unique solution of equation (5.4). Now, using the Decomposition theorem as in Klir [70], we can construct the fuzzy solution $\forall\, \alpha \in (0\ 1]$, as follows,

$${}^{\alpha}\tilde{X}(t) = \bigcup_{\alpha \in (0\ 1]} \left[\underline{X}(t), \overline{X}(t)\right],$$

where, ∪ denotes fuzzy union.

Thus, equation (5.3) is equivalent to the system (5.4).

In the next section, we present the Fuzzy Adomian Decomposition method in parametric form.

5.2.3. Fuzzy Adomian Decomposition Method in Parametric form (FADMP)

In Section 5.2, the Adomian Decomposition method is presented in a crisp environment. Now, we propose FADMP for the fuzzy semi-linear dynamical system as in equation (5.4) as follows. Consider equation (5.4),

$$\left[\dot{\underline{X}}, \dot{\overline{X}}\right] = \left[\min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right), \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right)\right] + \left[\underline{f}, \overline{f}\right],$$

with initial condition, $\left[\underline{X}_0, \overline{X}_0\right] = \left[\min\left(\underline{X}(0), \overline{X}(0)\right), \max\left(\underline{X}(0), \overline{X}(0)\right)\right]$.

Using the operator $\mathbb{L}$, equation (5.4) can be written as follows,

$$\left[\mathbb{L}\underline{X}, \mathbb{L}\overline{X}\right] = \left[\min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right), \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right)\right] + \left[\underline{f}, \overline{f}\right].$$

Now, taking $\mathbb{L}^{-1}$on both sides, where $\mathbb{L}^{-1} = \int_0^t (\cdot)dt$

$$\mathbb{L}^{-1}\left[\mathbb{L}\underline{X}, \mathbb{L}\overline{X}\right] = \mathbb{L}^{-1}\left[\min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right), \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right)\right] + \mathbb{L}^{-1}\left[\underline{f}, \overline{f}\right]. \tag{5.6}$$

Denoting, $A_l\underline{X} = \left[\min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right)\right]$ and $A_u\overline{X} = \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{AX}\right)$.

Then equation (5.6) can be represented as,

$$\left[\underline{X}(t), \overline{X}(t)\right] = \left[\underline{X_0}, \overline{X_0}\right] + \left[\int_0^t A_l\underline{X}(t)dt, \int_0^t A_u\overline{X}(t)dt\right] + \mathbb{L}^{-1}\left[\underline{f}, \overline{f}\right] \tag{5.7}$$

Now, using section 5.2, consider the series solutions $\underline{X} = \sum_{n=0}^{\infty}\underline{X}_n, \overline{X} = \sum_{n=0}^{\infty}\overline{X}_n$ in parametric form of equation (5.7) and nonlinear terms, $\underline{f}$ and $\overline{f}$ can be decomposed as a series of Adomian polynomials.

i.e., $\underline{f} = \sum_{n=0}^{\infty}\underline{A}_n\left(\underline{X}_0, \underline{X}_1, \ldots \underline{X}_n\right) = \min\left(\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{k=1}^{\infty}\underline{X}_k\lambda^k\right), \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{k=1}^{\infty}\overline{X}_k\lambda^k\right)\right)$,

$$\overline{f} = \sum_{n=0}^{\infty}\overline{A}_n\left(\overline{X}_0, \overline{X}_1, \ldots \overline{X}_n\right) = \max\left(\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{k=1}^{\infty}\underline{X}_k\lambda^k\right), \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{k=1}^{\infty}\overline{X}_k\lambda^k\right)\right).$$

Thus, the solution of equation (5.4) in parametric form is given as,

$$\left[\underline{X}(t), \overline{X}(t)\right] = \left[\underline{X}_1(t), \overline{X}_1(t)\right] + \left[\underline{X}_2(t), \overline{X}_2(t)\right] + \left[\underline{X}_3(t), \overline{X}_3(t)\right] + \cdots \tag{5.8}$$

In the next section, we give existence condition for solution of equation (5.4) in parametric form.

5.2.4. Theorem

If $[\underline{f}, \overline{f}]$ be an analytic function in equation (5.4), then equation (5.4) has a series solution by fuzzy Adomian decomposition method in parametric form by equation (5.8) as follows,

$$[\underline{X}(t), \overline{X}(t)] = [\underline{X}_1(t), \overline{X}_1(t)] + [\underline{X}_2(t), \overline{X}_2(t)] + [\underline{X}_3(t), \overline{X}_3(t)] + \cdots$$

where,

$$[\underline{X}_1(t), \overline{X}_1(t)] = \left[A_l \int_0^t \underline{X}_0 dt, A_u \int_0^t \overline{X}_0 dt\right] + \mathbb{L}^{-1}[\min(\underline{A}_0, \overline{A}_0), \max(\underline{A}_0, \overline{A}_0)]$$

$$[\underline{X}_2(t), \overline{X}_2(t)] = \left[A_l \int_0^t \underline{X}_1 dt, A_u \int_0^t \overline{X}_1 dt\right] + \mathbb{L}^{-1}[\min(\underline{A}_1, \overline{A}_1), \max(\underline{A}_1, \overline{A}_1)]$$

$$\vdots$$

Proof: Let $[\underline{f}, \overline{f}]$ be an analytic function in equation (5.4), so FADMP is applicable. Using Section 5.2.3, we put, $\underline{X}(t) = \sum_{n=0}^{\infty} \underline{X}_n(t)$, $\overline{X}(t) = \sum_{n=0}^{\infty} \overline{X}_n(t)$ in equation (5.8), we obtain,

$$\left[\sum_{n=0}^{\infty} \underline{X}_n(t), \sum_{n=0}^{\infty} \overline{X}_n(t)\right] = [\underline{X_0}, \overline{X_0}] + \left[A_l \int_0^t \sum_{n=0}^{\infty} \underline{X}_n(t)\, dt, A_u \int_0^t \sum_{n=0}^{\infty} \overline{X}_n(t)\, dt\right]$$
$$+ \mathbb{L}^{-1}\left[\min\left(\sum_{n=0}^{\infty} \underline{A}_n, \sum_{n=0}^{\infty} \overline{A}_n\right), \max\left(\sum_{n=0}^{\infty} \underline{A}_n, \sum_{n=0}^{\infty} \overline{A}_n\right)\right]$$

By Recurrence relation we get,

$$[\underline{X}_n(t), \overline{X}_n(t)] = \left[A_l \int_0^t \underline{X}_{n-1} dt, A_u \int_0^t \overline{X}_{n-1} dt\right] + \mathbb{L}^{-1}[\min(\underline{A}_{n-1}, \overline{A}_{n-1}), \max(\underline{A}_{n-1}, \overline{A}_{n-1})]$$

for $n = 1, 2, 3, \ldots$

Therefore, for different values of n we get,

$$[\underline{X}_1(t), \overline{X}_1(t)] = \left[A_l \int_0^t \underline{X}_0 dt, A_u \int_0^t \overline{X}_0 dt\right] + \mathbb{L}^{-1}[\min(\underline{A}_0, \overline{A}_0), \max(\underline{A}_0, \overline{A}_0)]$$

$$[\underline{X}_2(t), \overline{X}_2(t)] = \left[A_l \int_0^t \underline{X}_1 dt, A_u \int_0^t \overline{X}_1 dt\right] + \mathbb{L}^{-1}[\min(\underline{A}_1, \overline{A}_1), \max(\underline{A}_1, \overline{A}_1)]$$

$$[\underline{X}_3(t), \overline{X}_3(t)] = \left[A_l \int_0^t \underline{X}_2 dt, A_u \int_0^t \overline{X}_2 dt\right] + \mathbb{L}^{-1}[\min(\underline{A}_2, \overline{A}_2), \max(\underline{A}_2, \overline{A}_2)]$$

and so on.

$$[\underline{X}_n(t), \overline{X}_n(t)] = \left[A_l \int_0^t \underline{X}_{n-1} dt, A_u \int_0^t \overline{X}_{n-1} dt\right] + \mathbb{L}^{-1}[\min(\underline{A}_{n-1}, \overline{A}_{n-1}), \max(\underline{A}_{n-1}, \overline{A}_{n-1})]$$

The Adomian polynomials, $[\underline{A_n}, \overline{A_n}]$ can be obtained by the formula as given in equation (5.2). So, the solution of equation (5.4) is given by,

$$[\underline{X}(t), \overline{X}(t)] = [\underline{X}_1(t), \overline{X}_1(t)] + [\underline{X}_2(t), \overline{X}_2(t)] + [\underline{X}_3(t), \overline{X}_3(t)] + \cdots$$

5.2.5. Example

Using the proposed Fuzzy ADM technique, we solve the following fully fuzzy prey-predator model [65] involving the fuzzy parameters and fuzzy initial condition given as,

$$\begin{aligned} \dot{\tilde{x}} &= (\widetilde{0.1} \otimes \tilde{x}) \ominus (\widetilde{0.005} \otimes \tilde{x} \otimes \tilde{y}) \\ \dot{\tilde{y}} &= \ominus (\widetilde{0.4} \otimes \tilde{y}) \oplus (\widetilde{0.008} \otimes \tilde{x} \otimes \tilde{y}) \end{aligned} \tag{5.9}$$

with fuzzy initial conditions, $\tilde{x}_0 = \widetilde{130} = [120, 130, 150]$ and $\tilde{y}_0 = \widetilde{40} = [20,40, 50]$ where,

$$\widetilde{0.1} = [0.05, 0.1, 0.15], \quad \widetilde{0.005} = [0.004, 0.005, 0.006],$$
$$\widetilde{0.4} = [0.3, 0.4, 0.5], \quad \widetilde{0.008} = [0.007, 0.008\ 0.009].$$

Applying the proposed FADM technique, we have

$$\begin{aligned} \mathbb{L}[\underline{x}, \overline{x}] &= ([0.05, 0.15] \cdot [\underline{x}, \overline{x}]) - \left([0.004, 0.006] \cdot [\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}]\right) \\ \mathbb{L}[\underline{y}, \overline{y}] &= -\left([0.3\ ,0.5] \cdot [\underline{y}, \overline{y}]\right) + \left([0.007,0.009] \cdot [\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}]\right) \end{aligned}$$

Applying inverse operator on both sides, we have

$$\begin{aligned} \mathbb{L}^{-1}\mathbb{L}[\underline{x}, \overline{x}] &= \mathbb{L}^{-1}([0.05, 0.15] \cdot [\underline{x}, \overline{x}]) - \mathbb{L}^{-1}\left([0.004, 0.006] \cdot [\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}]\right) \\ \mathbb{L}^{-1}\mathbb{L}[\underline{y}, \overline{y}] &= \mathbb{L}^{-1}\left(-[0.3, 0.5] \cdot [\underline{y}, \overline{y}]\right) + \mathbb{L}^{-1}\left([0.007, 0.009] \cdot [\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}]\right) \end{aligned}$$

Thus, by solving the above equation, we get,

$$\begin{aligned} [\underline{x}(t), \overline{x}(t)] &= [\underline{x}(0), \overline{x}(0)] + \mathbb{L}^{-1}([0.05, 0.15] \cdot [\underline{x}, \overline{x}]) - \mathbb{L}^{-1}\left([0.004, 0.006] \cdot [\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}]\right) \\ [\underline{y}(t), \overline{y}(t)] &= [\underline{y}(0), \overline{y}(0)] + \mathbb{L}^{-1}\left([-0.5, -0.3] \cdot [\underline{y}, \overline{y}]\right) + \mathbb{L}^{-1}\left([0.007, 0.009] \cdot [\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}]\right) \end{aligned} \tag{5.10}$$

Let $[\underline{x}(t), \overline{x}(t)] = \sum_{n=0}^{\infty}[\underline{x}_n(t), \overline{x}_n(t)]$ and $[\underline{y}(t), \overline{y}(t)] = \sum_{n=0}^{\infty}[\underline{y}_n(t), \overline{y}_n(t)]$ be the series solution in the parametric form of equation (5.10) and the nonlinear term be expressed in the form of Adomian polynomial as $\sum_{n=0}^{\infty}[\underline{A}_n(t), \overline{A}_n(t)] = [\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}]$.

So, the equation (5.10) can be written as

$$\sum_{n=0}^{\infty}[\underline{x}_n(t), \overline{x}_n(t)] = [\underline{x}(0), \overline{x}(0)] + \mathbb{L}^{-1}\left([0.05, 0.15] \cdot \sum_{n=0}^{\infty}[\underline{x}_n(t), \overline{x}_n(t)]\right)$$
$$-\mathbb{L}^{-1}\left([0.004, 0.006] \cdot \sum_{n=0}^{\infty}[\underline{A}_n, \overline{A}_n]\right)$$

$$\sum_{n=0}^{\infty}\left[\underline{y}_n(t),\overline{y}_n(t)\right]=\left[\underline{y}(0),\overline{y}(0)\right]+\mathbb{L}^{-1}\left([-0.5,-0.3]\cdot\sum_{n=0}^{\infty}\left[\underline{y}_n(t),\overline{y}_n(t)\right]\right)$$
$$+\mathbb{L}^{-1}\left([0.007,0.009]\cdot\sum_{n=0}^{\infty}\left[\underline{A}_n,\overline{A}_n\right]\right)$$

comparing both the sides, we get

$$\left[\underline{x}_0(t),\overline{x}_0(t)\right]=\left[\underline{x}(0),\overline{x}(0)\right]$$
$$\left[\underline{x}_1(t),\overline{x}_1(t)\right]=\mathbb{L}^{-1}\left([0.05,0.15]\cdot\left[\underline{x}_0(t),\overline{x}_0(t)\right]\right)-\mathbb{L}^{-1}\left([0.004,0.006]\cdot\left[\underline{A}_0,\overline{A}_0\right]\right)$$
$$\left[\underline{x}_2(t),\overline{x}_2(t)\right]=\mathbb{L}^{-1}\left([0.05,0.15]\cdot\left[\underline{x}_1(t),\overline{x}_1(t)\right]\right)-\mathbb{L}^{-1}\left([0.004,0.006]\cdot\left[\underline{A}_1,\overline{A}_1\right]\right)$$
$$\vdots$$
$$\left[\underline{x}_n(t),\overline{x}_n(t)\right]=\mathbb{L}^{-1}\left([0.05,0.15]\cdot\left[\underline{x}_{n-1}(t),\overline{x}_{n-1}(t)\right]\right)$$
$$-\mathbb{L}^{-1}\left([0.004,0.006]\cdot\left[\underline{A}_{n-1},\overline{A}_{n-1}\right]\right)$$

Similarly,

$$\left[\underline{y}_0(t),\overline{y}_0(t)\right]=\left[\underline{y}(0),\overline{y}(0)\right]$$
$$\left[\underline{y}_1(t),\overline{y}_1(t)\right]=\mathbb{L}^{-1}\left([-0.5,-0.3]\cdot\left[\underline{y}_0(t),\overline{y}_0(t)\right]\right)+\mathbb{L}^{-1}\left([0.007,0.009]\cdot\left[\underline{A}_0,\overline{A}_0\right]\right)$$
$$\left[\underline{y}_2(t),\overline{y}_2(t)\right]=\mathbb{L}^{-1}\left([-0.5,-0.3]\cdot\left[\underline{y}_1(t),\overline{y}_1(t)\right]\right)+\mathbb{L}^{-1}\left([0.007,0.009]\cdot\left[\underline{A}_1,\overline{A}_1\right]\right)$$
$$\vdots$$
$$\left[\underline{y}_n,\overline{y}_n\right]=\mathbb{L}^{-1}\left([-0.5,-0.3]\cdot\left[\underline{y}_{n-1},\overline{y}_{n-1}\right]\right)+\left([0.007,0.009]\cdot\left[\underline{A}_{n-1},\overline{A}_{n-1}\right]\right)$$

Now for computation of above iterations, using following values,

$${}^{\alpha}\tilde{x}_0={}^{\alpha}\widetilde{130}=[120+10\alpha,150-20\alpha]\,,$$

and, ${}^{\alpha}\tilde{y}_0={}^{\alpha}\widetilde{40}=[20+20\alpha,50-30\alpha]$.

The Adomian polynomial,

$$\tilde{A}_0=\tilde{x}_0\otimes\tilde{y}_0$$
$${}^{\alpha}\tilde{A}_0={}^{\alpha}\tilde{x}_0\otimes{}^{\alpha}\tilde{y}_0$$

At, $\alpha=0$,

$$=[120,150]\cdot[20,50]$$
$$=[\min(2400,6000,3000,7500),\max(2400,6000,3000,7500)]$$
$$=[2400\ 7500]$$
$$\left[\underline{x}_1,\overline{x}_1\right]=\mathbb{L}^{-1}([0.05,0.15]\cdot[120,150])-\mathbb{L}^{-1}([0.004,0.006]\cdot[2400,7500])$$
$$=\mathbb{L}^{-1}[6,22.5]-\mathbb{L}^{-1}[9.6,45]$$
$$=\mathbb{L}^{-1}[6,22.5]+\mathbb{L}^{-1}[-45,-9.6]$$

$$= [-39t, 12.9t].$$

$$\left[\underline{y_1}, \overline{y_1}\right] = \mathbb{L}^{-1}(-[0.3, 0.4] \cdot [20, 50]) + \mathbb{L}^{-1}([0.007, 0.009] \cdot [2400, 7500])$$

$$= [-8.2t, 61.5t].$$

Similarly, for $n = 2, 3, \ldots$ we can compute the solution of the system.

$$^{\alpha}\tilde{x}(t) = {}^{\alpha}\tilde{x}_0 + {}^{\alpha}\tilde{x}_1 + {}^{\alpha}\tilde{x}_2 + \cdots$$

$$= [120\ 150] + [-39t, 12.9t] + [-32.535t^2, 10.5075t^2]$$

$$+ [-18.2037t^3, 17.481375t^3] + [-11.4075t^4, 15.2055t^4] + \cdots$$

$$^{\alpha}\tilde{x}(t) = {}^{\alpha}\tilde{y}_0 + {}^{\alpha}\tilde{y}_1 + {}^{\alpha}\tilde{y}_2 + \cdots$$

$$= [20, 50] + [-8.2t, 61.5t] + [-29.685t^2, 46.465t^2]$$

$$+ [-33.1782t^3, 29.8129t^3] + [-25.5515t^4, 20.2346t^4] + \cdots$$

The table 5.1 shows number of Prey and Predator as follows,

Table 5.1: Evolution on Predator and Prey data with time

t	Prey-data			Predator-data		
0	*l*-120	*c*-130	*r*-150	l-20	40	50
0.056	117.7297	129.2541	150.7522	19.4470	41.4411	53.5654
0.11	115.2383	128.4609	151.5894	18.6730	42.9199	57.4510
0.17	112.5032	127.6203	152.5345	17.6354	44.4364	61.6943
0.22	109.4991	126.7323	153.6145	16.2855	45.9907	66.3377
0.28	106.1982	125.7971	154.8593	14.5685	47.5827	71.4281
0.33	102.57	124.8144	156.3027	12.4241	49.2124	77.0168
0.38	98.5814	123.7845	157.9817	9.78599	50.8799	83.1600
0.44	94.1968	122.7072	159.9369	6.5821	52.5851	89.9185
0.5	89.3778	121.5825	162.2124	2.7345	54.328	97.3575

The evolution of prey-predator population at different time is as shown in Table 5.1. fig 5.1. and fig. 5.2. shows the evolution if predator population and prey population for time t = 0 to 0.5, respectively.

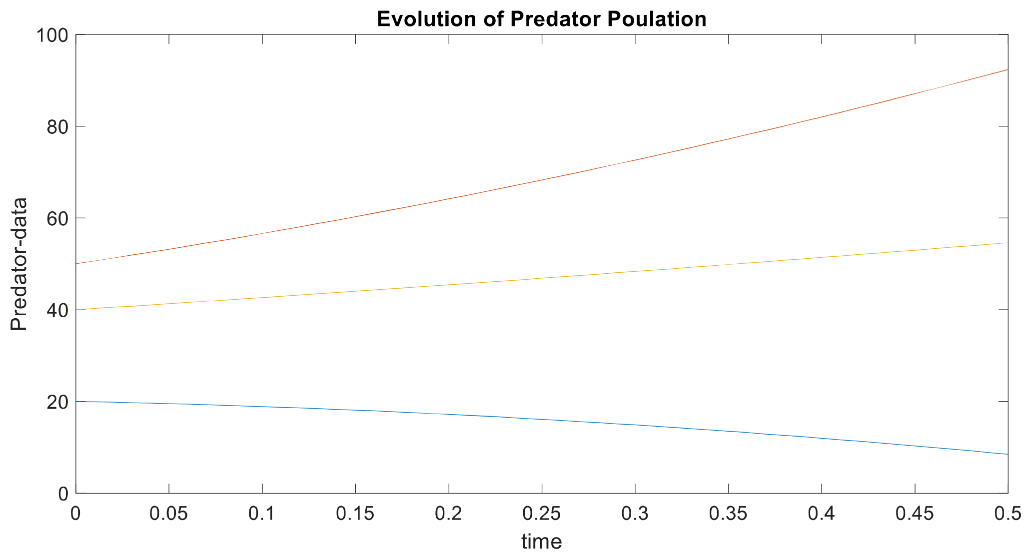

Figure 5.1: Evolution of Predator population for $t = 0$ to $t = 0.5$

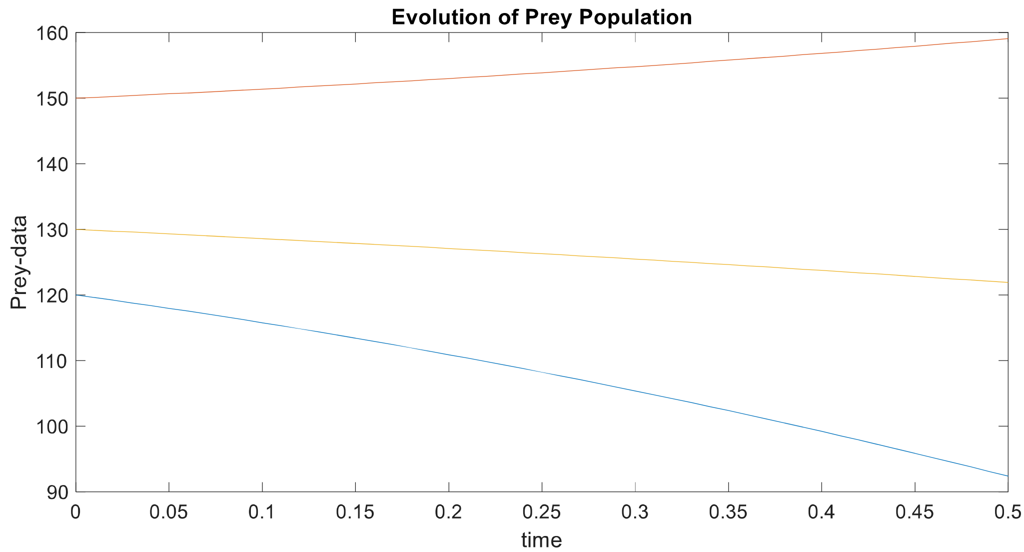

Figure 5.2: Evolution of Prey population for $t = 0$ to $t = 0.5$

In the next section, we develop Fuzzy Adomian Decomposition method (FADM) in a complete fuzzy environment. The benefit of which is mentioned earlier. In the next section, all the results required, in the development of FADM are proposed and proved.

5.3. Fuzzy Adomian Decomposition Method (FADM) in fully fuzzy Environment

For developing the FADM in a complete fuzzy environment, we redefine the definition of fuzzy function and using it, we give modified generalized Hukuhara (mgH) derivative. We propose and prove various other results like the Decomposition theorem, fuzzy power series and its convergence, fuzzy Taylor's theorem with examples under mgH derivative. Also, the convergence of FADM is proved. All the required results are extended for E^n and validated at the core. The advantage of FADM is that it can be applied directly to the differential equations in fuzzy form, instead of converting to crisp. The illustrative examples are solved using this FADM.

We begin the next section with the extension of the Decomposition theorem.

5.3.1. Decomposition theorem

We extend the first Decomposition in Klir [70] for $\tilde{X} \in E^n$.

Theorem: For, every $\tilde{X} \in E^n$,

$$\tilde{X} = \left(\cup_{\alpha\in[0\ 1]}\ {}_{\alpha}X\right) \text{ where } {}_{\alpha}X = \alpha.\ {}^{\alpha}\tilde{X} = \alpha.\left[\underline{X}, \overline{X}\right].$$

Proof: Let $\tilde{X} \in E^n$ is a fuzzy number i.e., $\tilde{X} = \left(\tilde{X}_1, \tilde{X}_2, \ldots \tilde{X}_n\right)$. So ${}^{\alpha}\tilde{X} = \left({}^{\alpha}\tilde{X}_1, {}^{\alpha}\tilde{X}_2, \ldots {}^{\alpha}\tilde{X}_n\right)$ where, ${}^{\alpha}\tilde{X}_1 = \left[\underline{X_1}, \overline{X_1}\right]$, ${}^{\alpha}\tilde{X}_2 = \left[\underline{X_2}, \overline{X_2}\right]$,.... ${}^{\alpha}\tilde{X}_n = \left[\underline{X_n}, \overline{X_n}\right]$.

Now,

$$\alpha.\ {}^{\alpha}\tilde{X} = \left(\alpha.\ {}^{\alpha}\tilde{X}_1, \alpha.\ {}^{\alpha}\tilde{X}_2, \ldots \alpha.\ {}^{\alpha}\tilde{X}_n\right)$$

$$\alpha.\left[\underline{X}, \overline{X}\right] = \left(\alpha.\left[\underline{X_1}, \overline{X_1}\right], \alpha.\left[\underline{X_2}, \overline{X_2}\right], \alpha.\left[\underline{X_3}, \overline{X_3}\right], \ldots\ \alpha.\left[\underline{X_n}, \overline{X_n}\right]\right)$$

Taking fuzzy union on both sides,

$$\bigcup_{\alpha\in[0\ 1]} \alpha.\left[\underline{X}, \overline{X}\right] = \sup\left[\alpha.\underline{X_i}, \alpha.\overline{X_i}\right]$$

$$\left(\bigcup_{\alpha\in[0\ 1]} {}_{\alpha}X\right) = \sup\left[\alpha.\underline{X_i}, \alpha.\overline{X_i}\right]$$

$$\left(\bigcup_{\alpha\in[0\ 1]} {}_{\alpha}X\right) = \max\left[\sup\ {}_{\alpha}X \text{ for } \alpha \in [0\ a], \sup\ {}_{\alpha}X \text{ for} \alpha \in (a\ 1]\right].$$

Hence,

$$\left(\bigcup_{\alpha\in[0\ 1]}\ {}_{\alpha}X\right)(x) = \sup\alpha \ \text{ for } \alpha \in [0\ a].$$

Thus, $\tilde{X} = \left(\bigcup_{\alpha\in[0\ 1]}\ {}_{\alpha}X\right)$,

where, ${}_{\alpha}X = \alpha.\ {}^{\alpha}\tilde{X} = \alpha.\left[\underline{X}, \overline{X}\right]$.

Next, we redefine fuzzy function as follows.

5.3.2. Fuzzy function

Consider a fuzzy valued scalar function with fuzzy argument $\tilde{f}: E \to E$. Its parametric form can be defined as follows,

$${}^{\alpha}\tilde{f}(\tilde{x}) = [\underline{f}(\tilde{x}), \overline{f}(\tilde{x})], \text{ where, } \underline{f}(\tilde{x}) = \min \tilde{f}(\tilde{x}) \text{ and } \overline{f}(\tilde{x}) = \max \tilde{f}(\tilde{x})$$

Further, ${}^{\alpha}\tilde{f}(\tilde{x}) = \left[\underline{\underline{f}}\left(\underline{x}, \overline{x}\right), \overline{\overline{f}}\left(\underline{x}, \overline{x}\right)\right]$,

where, $\underline{\underline{f}}\left(\underline{x}, \overline{x}\right) = \min\left(\underline{f}\left(\underline{x}, \overline{x}\right), \overline{f}\left(\underline{x}, \overline{x}\right)\right)$, $\overline{\overline{f}}\left(\underline{x}, \overline{x}\right) = \max\left(\underline{f}\left(\underline{x}, \overline{x}\right), \overline{f}\left(\underline{x}, \overline{x}\right)\right)$ (5.11)

This definition can be extended to n dimensional fuzzy valued function, by now considering $\tilde{f}$ as, $\tilde{f}: E^n \to E^n$ where, $\tilde{f} = \left\{\tilde{f}_1\left(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n\right), \tilde{f}_2\left(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n\right), \dots \tilde{f}_n\left(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n\right)\right\}$.

where parametric form,

$${}^{\alpha}\tilde{f}_i\left(\tilde{x}_j\right) = \left[\underline{f_i}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n), \overline{f}_i(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)\right] \ \ \forall\, i = 1,2,3, \dots, n.$$

Further, ${}^{\alpha}\tilde{f}_i\left(\tilde{x}_j\right) = \left[\underline{\underline{f_i}}\left(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n, \overline{x}_1, \overline{x}_2, \dots \overline{x}_n\right), \overline{\overline{f}}_i\left(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n, \overline{x}_1, \overline{x}_2, \dots \overline{x}_n\right)\right]$

$\forall i = 1,2,3, .., n.$

where, $\underline{\underline{f_i}}$ and $\overline{\overline{f}}_i$ are similar to as defined in equation (5.11) can be written as,

$$\underline{\underline{f_i}} = \min\left(\underline{f_i}\left(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n\right), \overline{f}_i\left(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n\right)\right), \forall i = 1,2,3, .., n.$$

$$\overline{\overline{f}}_i = \max\left(\underline{f_i}\left(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n\right), \overline{f}_i\left(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n\right)\right), \forall i = 1,2,3, .., n.$$

Based on this definition of the fuzzy function, we redefined the modified Hukuhara derivative in a new manner as follows.

5.3.3. Modified Generalized Hukuhara derivative:

A function $\tilde{f}: E \to E$ is said to be Modified Generalized Hukuhara (mgH) differentiable for an element $\dot{\tilde{f}}(\tilde{x}_0) \in E$, such that for small $h > 0, \tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)$, $\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)$ should exist and

$$\lim_{h\to 0+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)}{h} = \lim_{h\to 0-} \frac{\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)}{h} = \dot{\tilde{f}}(\tilde{x}_0) \tag{5.12}$$

The equivalent parametric form for the first limit is given as,

$$\lim_{h\to 0+} \frac{{}^{\alpha}\tilde{f}(\tilde{x}_0 + h) - {}^{\alpha}\tilde{f}(\tilde{x}_0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\min\left(\underline{f}(\tilde{x}_0 + h) - \underline{f}(\tilde{x}_0), \overline{f}(\tilde{x}_0 + h) - \overline{f}(\tilde{x}_0)\right), \max\left(\underline{f}(\tilde{x}_0 + h) - \underline{f}(\tilde{x}_0), \overline{f}(\tilde{x}_0 + h) - \overline{f}(\tilde{x}_0)\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\min\left(\underline{\underline{f}}\left(\underline{x}_0 + h, \overline{x}_0 + h\right) - \underline{\underline{f}}\left(\underline{x}_0, \overline{x}_0\right), \overline{\overline{f}}\left(\underline{x}_0 + h, \overline{x}_0 + h\right) - \overline{\overline{f}}\left(\underline{x}_0, \overline{x}_0\right)\right), \max\left(\underline{\underline{f}}\left(\underline{x}_0 + h, \overline{x}_0 + h\right) - \underline{\underline{f}}\left(\underline{x}_0, \overline{x}_0\right), \overline{\overline{f}}\left(\underline{x}_0 + h, \overline{x}_0 + h\right) - \overline{\overline{f}}\left(\underline{x}_0, \overline{x}_0\right)\right)\right]$$

Similarly, the second limit in equation (5.12) can be given as,

$$\lim_{h\to 0-} \frac{{}^{\alpha}\tilde{f}(\tilde{x}_0) - {}^{\alpha}\tilde{f}(\tilde{x}_0 - \mathrm{h})}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\min\left(\underline{f}(\tilde{x}_0) - \underline{f}(\tilde{x}_0 - h), \overline{f}(\tilde{x}_0) - \overline{f}(\tilde{x}_0 - h)\right), \max\left(\underline{f}(\tilde{x}_0) - \underline{f}(\tilde{x}_0 - h), \overline{f}(\tilde{x}_0) - \overline{f}(\tilde{x}_0 - h)\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\min\left(\underline{\underline{f}}\left(\underline{x}_0, \overline{x}_0\right) - \underline{\underline{f}}\left(\underline{x}_0 - h, \overline{x}_0 - h\right), \overline{\overline{f}}\left(\underline{x}_0, \overline{x}_0\right) - \overline{\overline{f}}\left(\underline{x}_0 - h, \overline{x}_0 - h\right)\right), \max\left(\underline{\underline{f}}\left(\underline{x}_0, \overline{x}_0\right) - \underline{\underline{f}}\left(\underline{x}_0 - h, \overline{x}_0 - h\right), \overline{\overline{f}}\left(\underline{x}_0, \overline{x}_0\right) - \overline{\overline{f}}\left(\underline{x}_0 - h, \overline{x}_0 - h\right)\right)\right]$$

In the following examples, we compute the derivative of a given function using the definition as in equation (5.12).

5.3.4. Example 1:

Derivative of $\tilde{f}(\tilde{x}) = \tilde{x}^2$ at $\tilde{x}_0$, $\tilde{x}_0 > 0$.

From the definition of derivative as in equation (5.12), we get,

$$
\begin{aligned}
{}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) &= \lim_{h \to 0+} \frac{{}^{\alpha}\tilde{f}(\tilde{x}_0 + h) - {}^{\alpha}\tilde{f}(\tilde{x}_0)}{h} \\
&= \lim_{h \to 0+} \frac{[\underline{x}_0 + h, \overline{x}_0 + h]^2 - [\underline{x}_0, \overline{x}_0]^2}{h} \\
&= \lim_{h \to 0+} \frac{[\underline{x}_0 + h, \overline{x}_0 + h][\underline{x}_0 + h, \overline{x}_0 + h] - [\underline{x}_0, \overline{x}_0][\underline{x}_0, \overline{x}_0]}{h} \\
&= \lim_{h \to 0+} \frac{[x_l, x_u] - [x_{l0}, x_{u0}]}{h} \qquad (5.13)
\end{aligned}
$$

where,

$$x_l = \min\left((\underline{x}_0 + h)^2, (\underline{x}_0 + h)(\overline{x}_0 + h), (\overline{x}_0 + h)^2\right)$$

$$x_u = \max\left((\underline{x}_0 + h)^2, (\underline{x}_0 + h)(\overline{x}_0 + h), (\overline{x}_0 + h)^2\right)$$

$$x_{l0} = \min\left((\underline{x}_0)^2, (\underline{x}_0\overline{x}_0), (\overline{x}_0)^2\right)$$

$$x_{u0} = \max\left((\underline{x}_0)^2, (\underline{x}_0\overline{x}_0), (\overline{x}_0)^2\right)$$

Since, $\tilde{x}_0 > 0$, $x_l = (\underline{x}_0 + h)^2$, $x_u = (\overline{x}_0 + h)^2$, $x_{l0} = (\underline{x}_0)^2$, $x_{u0} = (\overline{x}_0)^2$. (5.14)

Using equations (5.13) and (5.14), we have,

$${}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = \lim_{h \to 0+} \frac{[\min(x_l - x_{l0}, x_u - x_{u0}), \max(x_l - x_{l0}, x_u - x_{u0})]}{h}$$

$${}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = \lim_{h \to 0+} \frac{\left[(\underline{x}_0 + h)^2 - (\underline{x}_0)^2, (\overline{x}_0 + h)^2 - (\overline{x}_0)^2\right]}{h}$$

$${}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = \lim_{h \to 0+} \frac{[2\underline{x}_0 h + h^2, 2\overline{x}_0 h + h^2]}{h}$$

Similarly,

$${}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = \lim_{h \to 0+} \frac{{}^{\alpha}\tilde{f}(\tilde{x}_0) \ominus {}^{\alpha}\tilde{f}(\tilde{x}_0 - \mathrm{h})}{h} = [2\underline{x}_0, 2\overline{x}_0]$$

$$\therefore {}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = [2\underline{x}_0, 2\overline{x}_0]$$

Thus, by the Decomposition theorem as in, Section 5.3.1, $\dot{\tilde{f}}(\tilde{x}_0) = 2\tilde{x}_0$.

Example 2:

Derivative of $\tilde{f}(\tilde{x}) = a\tilde{x}^n, a > 0$ at $\tilde{x}_0$, $\tilde{x}_0 > 0$.

By using definition as in equation (5.12), we have

$$ {}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = \lim_{h\to 0+} \frac{{}^{\alpha}\tilde{f}(\tilde{x}_0 + h) - {}^{\alpha}\tilde{f}(\tilde{x}_0)}{h} $$

$$ = \lim_{h\to 0+} \frac{a[\underline{x}_0 + h, \overline{x}_0 + h]^n - a[\underline{x}_0, \overline{x}_0]^n}{h} $$

where,

$$ x_l = \min\left\{a(\underline{x}_0 + h)^n, a(\overline{x}_0 + h)(\underline{x}_0 + h)^{n-1}, a(\overline{x}_0 + h)^2(\underline{x}_0 + h)^{n-2} \dots, a(\overline{x}_0 + h)^n\right\}, $$

$$ x_u = \max\left\{a(\underline{x}_0 + h)^n, a(\overline{x}_0 + h)(\underline{x}_0 + h)^{n-1}, a(\overline{x}_0 + h)^2(\underline{x}_0 + h)^{n-2} \dots, a(\overline{x}_0 + h)^n\right\}, $$

$$ x_{l0} = \min\{a(\underline{x}_0)^n, a(\underline{x}_0{}^{n-1}\overline{x}_0), \dots, a(\overline{x}_0)^n\}, $$

$$ x_{u0} = \max\{a(\underline{x}_0)^n, a(\underline{x}_0{}^{n-1}\overline{x}_0), \dots, a(\overline{x}_0)^n\}. $$

Since, $\tilde{x}_0 > 0$, $x_l = a(\underline{x}_0 + h)^n$, $x_u = a(\overline{x}_0 + h)^n, x_{l0} = a(\underline{x}_0)^n, x_{u0} = a(\overline{x}_0)^n$. (5.15)

Putting equation (5.15) in equation (5.13).

$$ {}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = \lim_{h\to 0+} \frac{[a(\underline{x}_0 + h)^n - a(\underline{x}_0)^n, a(\overline{x}_0 + h)^n - a(\overline{x}_0)^n]}{h} $$

Also,

$$ {}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = \lim_{h\to 0-} \frac{[a(\underline{x}_0 - h)^n - a(\underline{x}_0)^n, a(\overline{x}_0 - h)^n - a(\overline{x}_0)^n]}{h} $$

$$ {}^{\alpha}\dot{\tilde{f}}(\tilde{x}_0) = [na\underline{x}_0{}^{n-1}, na\overline{x}_0{}^{n-1}] $$

Thus, by the Decomposition theorem as in, Section 3.2, $\dot{\tilde{f}}(\tilde{x}_0) = na\tilde{x}_0{}^{n-1}$.

Using this definition of a fuzzy derivative, we give fuzzy power series along with its convergence results and substantiate it with examples.

5.3.5. Fuzzy power series and its convergence

We know that power series is in crisp form is given as,

$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0)^1 + a_2(x - x_0)^2 + \cdots$.

A power series of fuzzy valued function around the point $\tilde{x}_0$can be given as,

$$ \sum \tilde{a}_n \otimes (\tilde{x} \ominus \tilde{x}_0)^n = \tilde{a}_0 \oplus \tilde{a}_1 \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \tilde{a}_2 \otimes (\tilde{x} \ominus \tilde{x}_0)^2 \oplus \dots \quad (5.16) $$

where, $\tilde{a}_n$ are any fuzzy coefficients and *n* is a positive integer. Its parametric form can be given by,

$$ \sum {}^{\alpha}\tilde{a}_n \otimes {}^{\alpha}(\tilde{x} \ominus \tilde{x}_0)^n $$

$$= [\underline{a}_0, \overline{a}_0]$$
$$\oplus [\underline{a}_1, \overline{a}_1] \otimes \left[[\underline{x}, \overline{x}] - [\underline{x}_0, \overline{x}_0]\right]$$
$$\oplus [\underline{a}_2, \overline{a}_2] \otimes \left[[\underline{x}, \overline{x}] - [\underline{x}_0, \overline{x}_0]\right]^2$$
$$\vdots$$
$$\oplus [\underline{a}_n, \overline{a}_n] \otimes \left[[\underline{x}, \overline{x}] - [\underline{x}_0, \overline{x}_0]\right]^n$$
$$\oplus \ldots$$
$$= [\underline{a}_0, \overline{a}_0]$$
$$\oplus [\underline{a}_1, \overline{a}_1] \otimes [\underline{x} - \underline{x}_0, \overline{x} - \overline{x}_0]$$
$$\oplus [\underline{a}_2, \overline{a}_2] \otimes [\underline{x} - \underline{x}_0, \overline{x} - \overline{x}_0]^2$$
$$\vdots$$
$$\oplus [\underline{a}_n, \overline{a}_n] \otimes [\underline{x} - \underline{x}_0, \overline{x} - \overline{x}_0]^n$$
$$\oplus \ldots$$
$$= [\underline{a}_0, \overline{a}_0]$$
$$\oplus [\underline{a}_1, \overline{a}_1] \otimes [\underline{x} - \underline{x}_0, \overline{x} - \overline{x}_0]$$
$$\oplus [\underline{a}_2, \overline{a}_2] \otimes \left[\min\left\{\left(\underline{x} - \underline{x}_0\right)^2, (\overline{x} - \overline{x}_0)\left(\underline{x} - \underline{x}_0\right), (\overline{x} - \overline{x}_0)^2\right\}, \max\left\{\left(\underline{x} - \underline{x}_0\right)^2, (\overline{x} - \overline{x}_0)\left(\underline{x} - \underline{x}_0\right), (\overline{x} - \overline{x}_0)^2\right\}\right]$$
$$\vdots$$
$$\oplus [\underline{a}_n, \overline{a}_n] \otimes \left[\min\left\{\left(\left(\underline{x} - \underline{x}_0\right)^n, (\overline{x} - \overline{x}_0)\left(\underline{x} - \underline{x}_0\right)^{n-1}, (\overline{x} - \overline{x}_0)^2\left(\underline{x} - \underline{x}_0\right),^{n-2} \ldots (\overline{x} - \overline{x}_0)^n\right), (\overline{x} - \overline{x}_0)^n, (\overline{x} - \overline{x}_0)^{n-1}\left(\underline{x} - \underline{x}_0\right), (\overline{x} - \overline{x}_0)^{n-2}\left(\underline{x} - \underline{x}_0\right),^2 \ldots, \left(\underline{x} - \underline{x}_0\right)^n\right\}, \max\left\{\left(\left(\underline{x} - \underline{x}_0\right)^n, (\overline{x} - \overline{x}_0)\left(\underline{x} - \underline{x}_0\right)^{n-1}, (\overline{x} - \overline{x}_0)^2\left(\underline{x} - \underline{x}_0\right),^{n-2} \ldots (\overline{x} - \overline{x}_0)^n\right), (\overline{x} - \overline{x}_0)^n, (\overline{x} - \overline{x}_0)^{n-1}\left(\underline{x} - \underline{x}_0\right), (\overline{x} - \overline{x}_0)^{n-2}\left(\underline{x} - \underline{x}_0\right),^2 \ldots, \left(\underline{x} - \underline{x}_0\right)^n\right\}\right]$$
$$\oplus ..$$

Since, $\tilde{x}_0 > 0$, then the above series expansion is given as follows,

$$\sum {}^{\alpha}\tilde{a}_n \otimes {}^{\alpha}(\tilde{x} \ominus \tilde{x}_0)^n$$

$$= [\underline{a}_0, \overline{a}_0]$$
$$\oplus [\underline{a}_1, \overline{a}_1] \otimes [\underline{x} - \underline{x}_0, \overline{x} - \overline{x}_0]$$
$$\oplus [\underline{a}_2, \overline{a}_2] \otimes \left[\left(\underline{x} - \underline{x}_0\right)^2, (\overline{x} - \overline{x}_0)^2\right]$$
$$\vdots$$
$$\oplus [\underline{a}_n, \overline{a}_n] \otimes \left[\left(\underline{x} - \underline{x}_0\right)^n, (\overline{x} - \overline{x}_0)^n\right]$$
$$\oplus \ldots$$
$$= [\underline{a}_0, \overline{a}_0]$$
$$\oplus [\min\{\underline{a}_1 (\underline{x} - \underline{x}_0), \underline{a}_1(\overline{x} - \overline{x}_0), \overline{a}_1(\underline{x} - \underline{x}_0), \overline{a}_1(\overline{x} - \overline{x}_0)\}, \max\{\underline{a}_1 (\underline{x} - \underline{x}_0), \underline{a}_1(\overline{x} - \overline{x}_0), \overline{a}_1(\underline{x} - \underline{x}_0), \overline{a}_1(\overline{x} - \overline{x}_0)\}]$$
$$\oplus \left[\min\left\{\underline{a}_1 \left(\underline{x} - \underline{x}_0\right)^2, \underline{a}_1(\overline{x} - \overline{x}_0)^2, \overline{a}_1\left(\underline{x} - \underline{x}_0\right)^2, \overline{a}_1(\overline{x} - \overline{x}_0)^2\right\}, \max\left\{\underline{a}_1 \left(\underline{x} - \underline{x}_0\right)^2, \underline{a}_1(\overline{x} - \overline{x}_0)^2, \overline{a}_1\left(\underline{x} - \underline{x}_0\right)^2, \overline{a}_1(\overline{x} - \overline{x}_0)^2\right\}\right]$$
$$\vdots$$
$$\oplus [\min\{\underline{a}_1 \left(\underline{x} - \underline{x}_0\right)^n, \underline{a}_1(\overline{x} - \overline{x}_0)^n, \overline{a}_1\left(\underline{x} - \underline{x}_0\right)^n, \overline{a}_1(\overline{x} - \overline{x}_0)^n\}, \max\{\underline{a}_1 \left(\underline{x} - \underline{x}_0\right)^n, \underline{a}_1(\overline{x} - \overline{x}_0)^n, \overline{a}_1\left(\underline{x} - \underline{x}_0\right)^n, \overline{a}_1(\overline{x} - \overline{x}_0)^n\}]$$
$$\oplus \ldots$$
$$= [\underline{a}_0, \overline{a}_0]$$
$$\oplus [\underline{a}_1(\underline{x} - \underline{x}_0), \overline{a}_1(\overline{x} - \overline{x}_0)]$$
$$\oplus \left[\underline{a}_2\left(\underline{x} - \underline{x}_0\right)^2, \overline{a}_2(\overline{x} - \overline{x}_0)^2\right]$$
$$\vdots$$
$$\oplus \left[\underline{a}_n\left(\underline{x} - \underline{x}_0\right)^n, \overline{a}_n(\overline{x} - \overline{x}_0)^n\right]$$
$$\oplus \ldots$$

With this representation of fuzzy power series, in the next section, we give result of the radius of convergence for fuzzy power series.

5.3.6. Radius of Convergence

If $\widetilde{\boldsymbol{a}_n} \neq \mathbf{0},$ **the** radius of convergence $\tilde{R}$ given by $\lim_{n\to\infty}\left|\frac{\widetilde{a_n}}{\tilde{a}_{n+1}}\right|$ of fuzzy power series can be defined as in parametric form, ${}^{\alpha}\tilde{R} = [\underline{R}, \overline{R}]$, where,

$$\underline{R}=\min[\lim_{n\to\infty}\left|\frac{\underline{a}_n}{\underline{a}_{n+1}}\right|, \lim_{n\to\infty}\left|\frac{\underline{a}_n}{\overline{a}_{n+1}}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_n}{\underline{a}_{n+1}}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_n}{\overline{a}_{n+1}}\right|]$$

$$\overline{R}=\max[\lim_{n\to\infty}\left|\frac{\underline{a}_n}{\underline{a}_{n+1}}\right|, \lim_{n\to\infty}\left|\frac{\underline{a}_n}{\overline{a}_{n+1}}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_n}{\underline{a}_{n+1}}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_n}{\overline{a}_{n+1}}\right|]$$

The fuzzy power series $\sum \tilde{a}_n \otimes (\tilde{x} \ominus \tilde{x}_0)^n$ with radius of convergence $\tilde{R}$ and the set of the points from an interval at which fuzzy series is convergent, known as the interval of convergence such that $d(\tilde{x}, \tilde{x}_0) < \tilde{R}$.

The parametric form of $d({}^{\alpha}\tilde{x}, {}^{\alpha}\tilde{x}_0) = [d(\underline{x}, \underline{x}_0), d(\overline{x}, \overline{x}_0)]$ and ${}^{\alpha}\tilde{R} = [\underline{R}, \overline{R}]$,

This assertion can be simplified in the following manner,

$$d({}^{\alpha}\tilde{x}, {}^{\alpha}\tilde{x}_0) = [d(\underline{x}, \underline{x}_0), d(\overline{x}, \overline{x}_0)] < [\underline{R}, \overline{R}]$$

Thus,

$$d(\underline{x}, \underline{x}_0) < \underline{R}$$

$$-\underline{R} < (\underline{x} - \underline{x}_0) < \underline{R}$$

$$(\underline{x}_0 - \underline{R}) < \underline{x} < (\underline{x}_0 + \underline{R})$$

Similarly,

$$d(\overline{x}, \overline{x}_0) < \overline{R}$$

$$-\overline{R} < (\overline{x} - \overline{x}_0) < \overline{R}$$

$$(\overline{x}_0 - \overline{R}) < \overline{x} < (\overline{x}_0 + \underline{R})$$

Thus, when we combine the result of radius of convergence in parametric form, we obtain the following condition for convergence,

$$\max d(\underline{x}, \underline{x}_0) < \underline{x} < \overline{x} < \min d(\overline{x}, \overline{x}_0) \quad (5.17)$$

This is the interval based on $\alpha - cut$ in which fuzzy power series converges.

Using the result as in equation 5.17, we proceed to find the radius of convergence for the power series representation of function.

5.3.7. Lemma

If $\underline{L} = \frac{1}{\underline{R}} = \min[\lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\overline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\overline{a}_n}\right|]$,

$$\overline{L} = \frac{1}{\overline{R}} = \max[\lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\overline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\overline{a}_n}\right|]$$

and $\underline{L} < 1$ and $\overline{L} < 1$ then fuzzy power series in parametric form converges absolutely.

Proof: $\underline{L} = \min[\lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\overline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\overline{a}_n}\right|]$,

Let $\frac{b_{n+1}}{b_n}$ be the minimum value of $\min[\lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\underline{a}_{n+1}}{\overline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\underline{a}_n}\right|, \lim_{n\to\infty}\left|\frac{\overline{a}_{n+1}}{\overline{a}_n}\right|]$,

and $\left|\frac{b_{n+1}}{b_n}\right| < 1$

Now,

$$|b_{n+1}| < |rb_n|$$

$|b_{n+i}| < |r^i b_n|$ for all n> N and $i = 1,2,3, \ldots\ldots$

Now,

$$\sum_{i=N+1}^{\infty} |b_{n+i}| < \sum_{i=1}^{\infty} |r^i b_n|$$

The right hand of the above equation is convergent absolutely.

Similarly, series converges for $\overline{L} < 1$.

In parametric form, we can write,

$$d({}^{\alpha}\tilde{x}, {}^{\alpha}\tilde{x}_0) = [d(\underline{x}, \underline{x}_0), d(\overline{x}, \overline{x}_0)] < [\underline{R}, \overline{R}]$$

So, by the Decomposition theorem as in Section 5.3.1, the fuzzy power series $\sum \tilde{a}_n \otimes (\tilde{x} \ominus \tilde{x}_0)^n$ converges absolutely for $d(\tilde{x}, \tilde{x}_0) < \tilde{R}$.

In the next section, we give an example for the expansion of a fuzzy power series along with its convergence.

5.3.8. Example

The fuzzy power series for $\sum_{n=0}^{\infty} \widetilde{-1} \otimes \tilde{x}^n$ can be obtained as follows,

where, $\widetilde{-1} = (-2, -1, 1)$.

$$\sum_{n=0}^{\infty} {}^{\alpha}\widetilde{-1} \otimes {}^{\alpha}\tilde{x}^n$$

$$= \sum_{n=1}^{\infty} [\alpha - 2, 1 - 2\alpha] \otimes [\underline{x^n}, \overline{x^n}]$$

$$= \sum_{n=0}^{\infty} [\min\{(\alpha - 2) \otimes \underline{x^n}, (\alpha - 2) \otimes \overline{x^n}, (1 - 2\alpha) \otimes \underline{x^n}, (1 - 2\alpha) \otimes \overline{x^n}\}, \max\{(\alpha - 2) \otimes \underline{x^n}, (\alpha - 2) \otimes \overline{x^n}, (1 - 2\alpha) \otimes \underline{x^n}, (1 - 2\alpha) \otimes \overline{x^n}\}] \quad (5.18)$$

Putting different values of n in equation (5.18), we can obtain the expression of the fuzzy power series. The first three terms in this expansion can be written as,

$$\sum_{n=0}^{\infty} {}^{\alpha}\widetilde{-1} \otimes {}^{\alpha}\tilde{x}^{n}$$

$$= [(\alpha - 2), (1 - 2\alpha)]$$
$$\oplus \big[\min\{(\alpha - 2) \otimes \underline{x}^1, (\alpha - 2) \otimes \overline{x}^1, (1 - 2\alpha) \otimes \underline{x}^1, (1 - 2\alpha) \otimes \overline{x}^1\},$$
$$\max\{(\alpha - 2) \otimes \underline{x}^1, (\alpha - 2) \otimes \overline{x}^1, (1 - 2\alpha) \otimes \underline{x}^1, (1 - 2\alpha) \otimes \overline{x}^1\}\big]$$
$$\oplus \Big[\min\{(\alpha - 2) \otimes \underline{x^2}, (\alpha - 2) \otimes \overline{x^2}, (\alpha - 2) \otimes \underline{x} \otimes \overline{x}, (1 - 2\alpha) \otimes \underline{x^2}, (1 - 2\alpha)$$
$$\otimes \overline{x^2}, (1 - 2\alpha) \otimes \underline{x}$$
$$\otimes \overline{x}\}, \max\{(\alpha - 2) \otimes \underline{x^2}, (\alpha - 2) \otimes \underline{x} \otimes \overline{x}\ (\alpha - 2) \otimes \overline{x^2}, (1 - 2\alpha)$$
$$\otimes \underline{x^2}, (1 - 2\alpha) \otimes \overline{x^2}, (1 - 2\alpha) \otimes \underline{x} \otimes \overline{x}\}\Big]$$
$$\oplus \ldots \tag{5.19}$$

The radius of convergence for $\sum_{n=0}^{\infty} \widetilde{-1} \otimes \tilde{x}^n$ is defined as,

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\tilde{a}_n}{\tilde{a}_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\tilde{1}}{\tilde{1}}\right| = \tilde{1}.$$

And, the region of convergence by lemma 5.3.7 is $|\tilde{x}| < \tilde{1}$.

The following section contains an example of radius of convergence for fuzzy power series.

5.3.9. Example 1:

Find the radius of convergence of $\sum_{n=0}^{\infty} \tilde{2} \otimes \tilde{x}^n$, where $\tilde{2} = (1,2,3)$.

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\tilde{a}_n}{\tilde{a}_{n+1}}\right| = \lim_{n\to\infty} \frac{\tilde{2}}{\tilde{2}}.$$

Let ${}^{\alpha}\tilde{2} = [1 + \alpha, 3 - \alpha]$ then the radius of convergence is defined as,

$$\underline{R} = \min[\lim_{n\to\infty} \frac{\underline{a}_n}{\underline{a}_{n+1}}, \lim_{n\to\infty} \frac{\underline{a}_n}{\overline{a}_{n+1}}, \lim_{n\to\infty} \frac{\overline{a}_n}{\underline{a}_{n+1}}, \lim_{n\to\infty} \frac{\overline{a}_n}{\overline{a}_{n+1}}]$$
$$= \min\left[1, \frac{(1+\alpha)}{(3-\alpha)}, \frac{(3-\alpha)}{(1+\alpha)}, 1\right] = \frac{(1+\alpha)}{(3-\alpha)}$$
$$\overline{R} = \max[\lim_{n\to\infty} \frac{\underline{a}_n}{\underline{a}_{n+1}}, \lim_{n\to\infty} \frac{\underline{a}_n}{\overline{a}_{n+1}}, \lim_{n\to\infty} \frac{\overline{a}_n}{\underline{a}_{n+1}}, \lim_{n\to\infty} \frac{\overline{a}_n}{\overline{a}_{n+1}}]$$
$$= \max\left[1, \frac{(1+\alpha)}{(3-\alpha)}, \frac{(3-\alpha)}{(1+\alpha)}, 1\right] = \frac{(3-\alpha)}{(1+\alpha)}$$

Therefore, the fuzzy radius of convergence is $\tilde{1}$, and in parametric form $\left[\frac{(1+\alpha)}{(3-\alpha)}, \frac{(3-\alpha)}{(1+\alpha)}\right]$. When we put $\alpha = 1$, the radius of convergence is 1 which is the same as its counter crisp problem.

The given fuzzy series is convergent for $|\tilde{x}| < \tilde{1}$, where $\tilde{1} = \left(\frac{1}{3}, 1,3\right)$.

Example 2:

Consider the fuzzy power series $\sum \frac{n}{\tilde{5}^{n-1}} \otimes \left(\tilde{x} \oplus \tilde{2}\right)^n$.

Here,

$$\tilde{a}_n = \frac{n}{\tilde{5}^{n-1}}$$

$$\tilde{a}_{n+1} = \frac{(n+1)}{\tilde{5}^n}$$

The radius of convergence is given by,

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\tilde{a}_n}{\tilde{a}_{n+1}}\right|$$

$$\tilde{R} = \tilde{5}$$

Thus, series is convergent for those $\tilde{x}$ which satisfies this condition $\left|\tilde{x} \oplus \tilde{2}\right| < \tilde{5}$

i.e., $-\tilde{7} < \tilde{x} < \tilde{3}$.

The convergence of $\left|\tilde{x} \oplus \tilde{2}\right| < \tilde{5}$ can be visualized in crisp form by using the parametric form.

Let ${}^{\alpha}\tilde{x} = \left[\underline{x}\ \overline{x}\right]$, ${}^{\alpha}\tilde{2} = [1+\alpha,\ 3-\alpha]$ and ${}^{\alpha}\tilde{5} = [4+\alpha, 6-\alpha]$, then the inequality becomes

$\left|\left[\underline{x}\ \overline{x}\right] + [1+\alpha, 3-\alpha]\right| < |4+\alpha, 6-\alpha|$

That is,

$$\left|\underline{x} + 1 + \alpha\right| < 4 + \alpha$$

which implies,

$$-(4+\alpha) < \left(\underline{x} + 1 + \alpha\right) < 4 + \alpha$$

Thus,

$$-(5+2\alpha) < \underline{x} < 3$$

Similarly, $|\overline{x} + 3 - \alpha| < 6 - \alpha$

$$\Rightarrow -(6-\alpha) < (\overline{x} + 3 - \alpha) < 6 - \alpha$$

$$\Rightarrow (-9+2\alpha) < \overline{x} < 3$$

By 5.3.6, we can combine the results in crisp form,

$$[(-9+2\alpha), -(5+2\alpha)] < \left[\underline{x}, \overline{x}\right] < [3,3].$$

And by the Decomposition theorem given in Section 5.3.1, we can write that the given series converges for the interval $-\tilde{7} < \tilde{x} < \tilde{3}$.

The above-mentioned theories, fuzzy power series and its convergence are required for proving Fuzzy Taylor's series.

In the next section, Fuzzy Taylor's series is proposed and proved.

5.3.10. Fuzzy Taylor's theorem

If a fuzzy valued function, $\tilde{f}(x): E^n \to E^n$ is, n times modified generalized Hukuhara (mgH) differentiable, then fuzzy Taylor's expansion is given as,

$\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \frac{\ddot{\tilde{f}}(\tilde{x}_0)}{2!} \otimes (\tilde{x} \ominus \tilde{x}_0)^2 \oplus \frac{\dddot{\tilde{f}}(\tilde{x}_0)}{3!} \otimes (\tilde{x} \ominus \tilde{x}_0)^3 \oplus$
$\frac{\tilde{f}^4(\tilde{x}_0)}{4!} \otimes (\tilde{x} \ominus \tilde{x}_0)^4 \oplus \frac{\tilde{f}^5(\tilde{x}_0)}{5!} \otimes (\tilde{x} \ominus \tilde{x}_0)^5 \oplus \ldots \oplus \frac{\tilde{f}^n(\tilde{x}_0)}{n!} \otimes (\tilde{x} \ominus \tilde{x}_0)^n + \cdots.$

Proof: We know that fuzzy valued function $\tilde{f}$ is modified generalized Hukuhara (mgH) differentiable.

$$\dot{\tilde{f}}(\tilde{x}_0) = \lim_{\tilde{x} \to \tilde{x}_0} \frac{\tilde{f}(\tilde{x}) \ominus \tilde{f}(\tilde{x}_0)}{(\tilde{x} \ominus \tilde{x}_0)}$$

The above expression gets approximated as,

$$\tilde{f}(\tilde{x}) \ominus \tilde{f}(\tilde{x}_0) = \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0)$$

Thus, the linear approximation of $\tilde{f}$ at $\tilde{x}$ in neighborhood of $\tilde{x}_0$ is,

$$\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0)$$

Denote the approximation of $\tilde{f}(\tilde{x})$ by $\tilde{p}_1(\tilde{x})$. That is,

$$\tilde{p}_1(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0)$$

Then this polynomial of first degree satisfies, $\tilde{p}_1(\tilde{x}_0) = \tilde{f}(\tilde{x}_0), \dot{\tilde{p}}_1(\tilde{x}_0) = \dot{\tilde{f}}(\tilde{x}_0)$.

Linear approximation of $\tilde{f}$ is well defined at $\tilde{x}_0$, if $\tilde{f}$ has a constant slope. However, if $\tilde{f}$ has curvature near the point $\tilde{x}_0$ then, it requires quadratic approximation. For quadratic approximation, we add one more term denoting it by $\tilde{p}_2(\tilde{x})$.

Thus,

$$\tilde{p}_2(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \tilde{c} \otimes (\tilde{x} \ominus \tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \quad (5.20)$$

To determine a new term $\tilde{c}$, $\tilde{p}_2(x)$ must be a good approximation to $\tilde{f}$, near the point $\tilde{x}_0$.

This requires,

$$\tilde{p}_2(\tilde{x}_0) = \tilde{f}(\tilde{x}_0), \dot{\tilde{p}}_2(\tilde{x}_0) = \dot{\tilde{f}}(\tilde{x}_0) \text{ and } \ddot{\tilde{p}}_2(\tilde{x}_0) = \ddot{\tilde{f}}(\tilde{x}_0)$$

Differentiating, equation (5.20), we get

$$\dot{\tilde{p}}_2(\tilde{x}) = \dot{\tilde{f}}(\tilde{x}_0) \oplus 2\,\tilde{c} \otimes (\tilde{x} \ominus \tilde{x}_0)$$

Again, differentiating the above equation, we get

$$\ddot{\tilde{p}}_2(\tilde{x}) = 2\,\tilde{c} \Rightarrow \tilde{c} = \frac{\ddot{\tilde{p}}_2(x)}{2}$$

At, $\tilde{x} = \tilde{x}_0,\ \ddot{\tilde{p}}_2(\tilde{x}_0) = \ddot{\tilde{f}}(\tilde{x}_0)$

So, equation (5.20) becomes,

$$\tilde{p}_2(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \frac{\ddot{\tilde{f}}(\tilde{x}_0)}{2} \otimes (\tilde{x} \ominus \tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0)$$

Following a similar way, we get the third approximation as,

$$\tilde{p}_3(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \frac{\ddot{\tilde{f}}(\tilde{x}_0)}{2} \otimes (\tilde{x} \ominus \tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \frac{\dddot{\tilde{f}}(\tilde{x}_0)}{6} \otimes (\tilde{x} \ominus \tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0)$$

And the n^{th} degree approximation to $\tilde{f}$, can be given as,

$$\tilde{p}_n(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \frac{\ddot{\tilde{f}}(\tilde{x}_0)}{2!} \otimes (\tilde{x} \ominus \tilde{x}_0)^2 \oplus \frac{\dddot{\tilde{f}}(\tilde{x}_0)}{3!} \otimes (\tilde{x} \ominus \tilde{x}_0)^3 \oplus \frac{\tilde{f}^4(\tilde{x}_0)}{4!} \otimes (\tilde{x} \ominus \tilde{x}_0)^4 \oplus \frac{\tilde{f}^5(\tilde{x}_0)}{5!} \otimes (\tilde{x} \ominus \tilde{x}_0)^5 \oplus \dots \oplus \frac{\tilde{f}^n(\tilde{x}_0)}{n!} \otimes (\tilde{x} \ominus \tilde{x}_0)^n \oplus \dots.$$

Thus, the series approximation of $\tilde{f}$, can be written as,

$$\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \frac{\ddot{\tilde{f}}(\tilde{x}_0)}{2!} \otimes (\tilde{x} \ominus \tilde{x}_0)^2 \oplus \frac{\dddot{\tilde{f}}(\tilde{x}_0)}{3!} \otimes (\tilde{x} \ominus \tilde{x}_0)^3 \oplus \frac{\tilde{f}^4(\tilde{x}_0)}{4!} \otimes (\tilde{x} \ominus \tilde{x}_0)^4 \oplus \frac{\tilde{f}^5(\tilde{x}_0)}{5!} \otimes (\tilde{x} \ominus \tilde{x}_0)^5 \oplus \dots \frac{\tilde{f}^n(\tilde{x}_0)}{n!} \otimes (\tilde{x} \ominus \tilde{x}_0)^n \oplus \dots.$$

Also, this expansion is true for all x contained in the radius of convergence.

In the next section, we give the fuzzy Taylor series for some fuzzy functions.

5.4. Illustrative examples for Fuzzy Taylor series expansions of fuzzy functions

5.4.1. Fuzzy Taylor series for $\tilde{f}(\tilde{x}) = e^{\tilde{x}}$ about $\tilde{x} = \tilde{0} = (-1,0,1)$.

Now for all α, ${}^{\alpha}\tilde{x} = {}^{\alpha}\tilde{0} = (\alpha - 1, 1 - \alpha)$,

$$\dot{\tilde{f}}(\tilde{x}) = e^{\tilde{x}},$$

$$\dot{\tilde{f}}(\tilde{0}) = e^{\tilde{0}} = e^{(\alpha-1,1-\alpha)} = [min(e^{\alpha-1}, e^{1-\alpha}), max(e^{\alpha-1}, e^{1-\alpha})] = [e^{1-\alpha}, e^{\alpha-1}]$$

$$\ddot{\tilde{f}}(\tilde{x}) = e^{\tilde{x}}, \ddot{\tilde{f}}(\tilde{0}) = e^{\tilde{0}} = [e^{1-\alpha}, e^{\alpha-1}]$$

$$\vdots$$

$$\vdots$$

$$\tilde{f}^n(\tilde{x}) = e^{\tilde{x}}, \widetilde{f^n}(\tilde{0}) = e^{\tilde{0}} = [e^{1-\alpha}, e^{\alpha-1}]$$

Then,

$$e^{\tilde{x}} = e^{\tilde{0}} \oplus e^{\tilde{0}} \otimes \left(\tilde{x} \ominus \tilde{0}\right) \oplus \frac{e^{\tilde{0}}}{2} \otimes \left(\tilde{x} \ominus \tilde{0}\right)^2 \oplus \frac{e^{\tilde{0}}}{3!} \otimes \left(\tilde{x} \ominus \tilde{0}\right)^3 \oplus \frac{e^{\tilde{0}}}{4!} \otimes \left(\tilde{x} \ominus \tilde{0}\right)^4 \oplus \frac{e^{\tilde{0}}}{5}$$

$$\otimes \left(\tilde{x} \ominus \tilde{0}\right)^5 \oplus \ldots\ldots\ldots\ldots\ldots\ldots. \frac{e^{\tilde{0}}}{n!} \otimes \left(\tilde{x} \ominus \tilde{0}\right)^n \oplus \ldots$$

$$e^{\tilde{x}} = e^{\tilde{0}} \otimes [1 \oplus \left(\tilde{x} \ominus \tilde{0}\right) \oplus \frac{(\tilde{x}\ominus\tilde{0})^2}{2!} \oplus \frac{(\tilde{x}\ominus\tilde{0})^3}{3!} \oplus \frac{(\tilde{x}\ominus\tilde{0})^4}{4!} \oplus \frac{(\tilde{x}\ominus\tilde{0})^5}{5} \oplus \ldots \oplus \frac{(\tilde{x}\ominus\tilde{0})^n}{n!} \oplus ..$$

After taking alpha – cut,

$${}^{\alpha}e^{\tilde{x}} = [e^{1-\alpha}, e^{\alpha-1}]\left[1 \oplus \left(\tilde{x} \ominus \tilde{0}\right) \oplus \frac{(\tilde{x}\ominus\tilde{0})^2}{2!} \oplus \frac{(\tilde{x}\ominus\tilde{0})^3}{3!} \oplus \ldots \oplus \frac{(\tilde{x}\ominus\tilde{0})^n}{n!} \oplus \ldots\right]$$

The radius of convergence for $e^{\tilde{x}}$, as defined in section 5.3.7,

$$\tilde{R} = \lim_{n\to\infty}\left|\frac{\tilde{a}_n}{\tilde{a}_{n+1}}\right|$$

$$\tilde{R} = \lim_{n\to\infty}\left|\frac{\frac{\left(\tilde{x} \ominus \tilde{0}\right)^n}{n!}}{\frac{\left(\tilde{x} \ominus \tilde{0}\right)^{n+1}}{n+1!}}\right|$$

$$\tilde{R} = \lim_{n\to\infty}\left|\frac{(n+1)}{\left(\tilde{x} \ominus \tilde{0}\right)}\right|$$

So as n tends to infinity, the radius of convergence becomes infinite. Thus $e^{\tilde{x}}$ is convergent everywhere.

At core, $\alpha = 1$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

That is the same as Taylor's expansion for real-valued function.

5.4.2. Fuzzy Taylor series for $\tilde{f}(\tilde{x}) = sin\tilde{x}$ about $\tilde{x} = \tilde{0}$.

$\dot{\tilde{f}}(\tilde{x}) = cos(\tilde{x}),\quad \dot{\tilde{f}}\left(\tilde{0}\right) = cos\tilde{0}$

$\ddot{\tilde{f}}(\tilde{x}) = -sin\tilde{x},\quad \ddot{\tilde{f}}\left(\tilde{0}\right) = -sin\tilde{0}$

$\dddot{\tilde{f}}(\tilde{x}) = -cos\tilde{x},\quad \dddot{\tilde{f}}\left(\tilde{0}\right) = -cos\tilde{0}$

$\tilde{f}^4(\tilde{x}) = sin\tilde{x},\quad \tilde{f}^4(\tilde{x}) = sin\tilde{0}$

:

:

Now Using Fuzzy Taylor's series as in Section 5.3.10, we have,

$$sin\tilde{x} = \tilde{f}(\tilde{0}) \oplus \dot{\tilde{f}}(\tilde{0}) \otimes (\tilde{x} \ominus \tilde{0}) \oplus \frac{\ddot{\tilde{f}}(\tilde{0})}{2!} \otimes (\tilde{x} \ominus \tilde{0})^2 \oplus \frac{\dddot{\tilde{f}}(\tilde{0})}{3!} \otimes (\tilde{x} \ominus \tilde{0})^3 \oplus \frac{\tilde{f}^4(\tilde{0})}{4!}$$

$$\otimes (\tilde{x} \ominus \tilde{0})^4 \oplus \frac{\tilde{f}^5(\tilde{0})}{5!} \otimes (\tilde{x} \ominus \tilde{0})^5 \oplus \dots\dots\dots\dots\dots\dots. \frac{\tilde{f}^n(\tilde{0})}{n!} \otimes (\tilde{x} \ominus \tilde{0})^n$$

$$\oplus \dots$$

$$sin\tilde{x} = sin\tilde{0} \oplus cos\tilde{0} \otimes (\tilde{x} \ominus \tilde{0}) \ominus \frac{\sin(\tilde{0})}{2} \otimes (\tilde{x} \ominus \tilde{0})^2 \ominus \frac{\cos(\tilde{0})}{3!} \otimes (\tilde{x} \ominus \tilde{0})^3 \oplus \frac{\sin(\tilde{0})}{4!} \otimes$$

$$(\tilde{x} \ominus \tilde{0})^4 \oplus \frac{\cos(\tilde{0})}{5!} \otimes (\tilde{x} \ominus \tilde{0})^5 \oplus \dots$$

The radius of convergence for $sin\tilde{x}$, as defined in section 5.3.7,

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\tilde{a}_n}{\tilde{a}_{n+1}}\right|$$

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\frac{(\tilde{x} \ominus \tilde{0})^n}{n!} \sin\left(\tilde{x} + \frac{n\pi}{2}\right)}{\frac{(\tilde{x} \ominus \tilde{0})^{n+1}}{n+1!} \sin\left(\tilde{x} + \frac{(n+1)\pi}{2}\right)}\right|$$

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{(n+1)}{(\tilde{x} \ominus \tilde{0})} \frac{\sin\left(\tilde{x} + \frac{n\pi}{2}\right)}{\sin\left(\tilde{x} + \frac{(n+1)\pi}{2}\right)}\right|$$

So as n tends to infinity, this term $\frac{\sin\left(\tilde{x}+\frac{n\pi}{2}\right)}{\sin\left(\tilde{x}+\frac{(n+1)\pi}{2}\right)}$ remains finite and the radius of convergence becomes infinite. Thus $sin\tilde{x}$ is convergent everywhere.

At core, $\alpha = 1$,

$$sin = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

5.4.3. Fuzzy Taylor series for $\tilde{f}(\tilde{x}) = cos\tilde{x}$ about $\tilde{x} = \tilde{0}$.

$$\dot{\tilde{f}}(\tilde{x}) = -sin\tilde{x}, \qquad \dot{\tilde{f}}(\tilde{0}) = -sin\tilde{0}$$

$$\ddot{\tilde{f}}(\tilde{x}) = -cos\tilde{x}, \qquad \ddot{\tilde{f}}(\tilde{0}) = -cos\tilde{0}$$

$$\dddot{\tilde{f}}(\tilde{x}) = sin\tilde{x}, \qquad \dddot{\tilde{f}}(\tilde{0}) = sin\tilde{0}$$

$$\tilde{f}^4(\tilde{x}) = cos\tilde{x}, \qquad \tilde{f}^4(\tilde{x}) = cos\tilde{0}$$

:

:

Now Using Fuzzy Taylor's series as in Section 5.3.10, we have,

$cos\tilde{x} = cos\tilde{0} \ominus sin\tilde{0} \otimes (\tilde{x} \ominus \tilde{0}) \ominus \frac{\cos\tilde{0}}{2!} \otimes (\tilde{x} \ominus \tilde{0})^2 \oplus \frac{sin\tilde{0}}{3!} \otimes (\tilde{x} \ominus \tilde{0})^3 \oplus \frac{cos\tilde{0}}{4!} \otimes (\tilde{x} \ominus \tilde{0})^4 \ominus \frac{\sin\tilde{0}}{5!} \otimes (\tilde{x} \ominus \tilde{0})^5 \oplus \dots$

The radius of convergence for $sin\tilde{x}$, as defined in section 5.3.7,

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\tilde{a}_n}{\tilde{a}_{n+1}}\right|$$

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\frac{(\tilde{x} \ominus \tilde{0})^n}{n!} \cos\left(\tilde{x} + \frac{n\pi}{2}\right)}{\frac{(\tilde{x} \ominus \tilde{0})^{n+1}}{n+1!} \cos\left(\tilde{x} + \frac{(n+1)\pi}{2}\right)}\right|$$

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{(n+1)}{(\tilde{x} \ominus \tilde{0})} \frac{\cos\left(\tilde{x} + \frac{n\pi}{2}\right)}{\cos\left(\tilde{x} + \frac{(n+1)\pi}{2}\right)}\right|$$

So as n tends to infinity, this term $\frac{\cos\left(\tilde{x}+\frac{n\pi}{2}\right)}{\cos\left(\tilde{x}+\frac{(n+1)\pi}{2}\right)}$ remains finite and radius of convergence becomes infinite. Thus $cos\tilde{x}$ is convergent everywhere.

At core, $\alpha = 1$,

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

In the next section, we establish Fuzzy Adomian Decomposition Method in a fully fuzzy environment by using the above proposed results.

5.5. Fuzzy Adomian Decomposition Method

Consider general nonlinear fuzzy differential equation as in Section 5.1. in fuzzy environment. As already defined, $\mathbb{L}$ is a linear differentiable operator of the highest order, N is nonlinear operator, $\mathbb{R}$ is the operator of less order than that of $\mathbb{L}$ and g is source term.

$$\mathbb{L}\tilde{u} \oplus N\tilde{u} \oplus \mathbb{R}\tilde{u} = \tilde{g} \qquad (5.21)$$

Taking $\mathbb{L}^{-1}$ on both sides of the above equation,

$$\tilde{u} = \mathbb{L}^{-1}\tilde{g} \ominus \mathbb{L}^{-1}(N\tilde{u} \oplus \mathbb{R}\tilde{u}) \qquad (5.22)$$

The method allows us to express the unknown fuzzy function in the form of series,

That is, $\tilde{u} = \tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \dots$

$$\tilde{u} = \sum_{n=1}^{\infty} \tilde{u}_n \tag{5.23}$$

where, the fuzzy components $\tilde{u}_n$ are estimated from recurrence relation as described below. The method decomposes the nonlinear fuzzy term by a series of Adomian polynomials,

$$N(\tilde{u}) = \sum_{n=0}^{\infty} \tilde{A}_n \, (\tilde{u}_0, \tilde{u}_1, \tilde{u}_2 \ldots \ldots \ldots . \tilde{u}_n) \tag{5.24}$$

Using (5.23) and (5.24) in equation (5.22).

$$\sum_{n=1}^{\infty} \tilde{u}_n = \mathbb{L}^{-1}\tilde{g} \ominus \mathbb{L}^{-1}(\sum_{n=0}^{\infty} \tilde{A}_n \, (\tilde{u}_0, \tilde{u}_1, \tilde{u}_2 \ldots \ldots \ldots . \tilde{u}_n)) \ominus \mathbb{L}^{-1}(\mathbb{R}(\sum_{n=1}^{\infty} \tilde{u}_n))$$

Expanding the series on both sides, we have,

$$\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \tilde{u}_4 \ldots = \mathbb{L}^{-1}\tilde{g} \ominus \mathbb{L}^{-1}\big(\tilde{A}_0(\tilde{u}_0) \oplus \tilde{A}_1(\tilde{u}_0, \tilde{u}_1) \oplus$$
$$\tilde{A}_2(\tilde{u}_0, \tilde{u}_1, \tilde{u}_2) \oplus \tilde{A}_3(\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \oplus \ldots\big) \ominus \; \mathbb{L}^{-1}(\mathbb{R}(\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \tilde{u}_4 \ldots)$$

By comparing both the sides,

$$\tilde{u}_0 = \mathbb{L}^{-1}\tilde{g}$$
$$\tilde{u}_1 = \ominus \, \mathbb{L}^{-1}\left(\tilde{A}_0(\tilde{u}_0)\right) \ominus \mathbb{L}^{-1}\big(\mathbb{R}(\tilde{u}_0)\big)$$
$$\tilde{u}_2 = \ominus \, \mathbb{L}^{-1}\left(\tilde{A}_1(\tilde{u}_0, \tilde{u}_1)\right) \ominus \mathbb{L}^{-1}\big(\mathbb{R}(\tilde{u}_1)\big)$$
$$\vdots$$

In general, the recurrence relation is,

$$\tilde{u}_n = \ominus \, \mathbb{L}^{-1}\left(\tilde{A}_{n-1}(\tilde{u}_0, \tilde{u}_1, \tilde{u}_2 \ldots \ldots \ldots . \tilde{u}_n)\right) \ominus \mathbb{L}^{-1}\big(\mathbb{R}(\tilde{u}_{n-1})\big)$$

Now, we expand the nonlinear part $N(\tilde{u})$ in equation (5.21) by fuzzy Taylor's series as in 5.3.10.

$$N(\tilde{u}) = N(\tilde{u}_0) \oplus \dot{N}(\tilde{u}_0) \otimes (\tilde{u} \ominus \tilde{u}_0) \oplus \frac{\ddot{N}(\tilde{u}_0)}{2} \otimes (\tilde{u} \ominus \tilde{u}_0)^2$$
$$\oplus \frac{\dddot{N}(\tilde{u}_0)}{3!} \otimes (\tilde{u} \ominus \tilde{u}_0)^3 \oplus \frac{N^4(\tilde{u}_0)}{4!} \otimes (\tilde{u} \ominus \tilde{u}_0)^4 \oplus \frac{N^5(\tilde{u}_0)}{5} \otimes (\tilde{u} \ominus \tilde{u}_0)^5 \oplus \ldots \oplus$$
$$\frac{N^n(\tilde{u}_0)}{n!} \otimes (\tilde{u} \ominus \tilde{u}_0)^n \oplus \ldots$$

Using the series form of $\tilde{u}$ in above equation, we get,

$$N(\tilde{u}) = N(\tilde{u}_0) \oplus \dot{N}(\tilde{u}_0) \otimes (\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \tilde{u}_4 \ldots \ominus \tilde{u}_0)$$
$$\oplus \frac{\ddot{N}(\tilde{u}_0)}{2} \otimes (\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \tilde{u}_4 \ldots \ldots \ominus \tilde{u}_0)^2$$
$$\oplus \frac{\dddot{N}(\tilde{u}_0)}{3!} \otimes (\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \tilde{u}_4 \ldots \ldots \ominus \tilde{u}_0)^3 \tag{5.25}$$

$\oplus N^4 \otimes (\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \tilde{u}_4 \ldots \ominus \tilde{u}_0)^4 \oplus \ldots$

$\oplus \frac{N^n(\tilde{u}_0)}{n!} \otimes \ (\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \tilde{u}_4 \ldots \ldots \ominus \tilde{u}_0)^n \oplus \ldots$

Fuzzy Adomian polynomials are constructed in such a manner $\tilde{A}_1$ contains all terms of order 1 in an expansion of equation (5.25). Similarly, $\tilde{A}_2$ contains all term of order 2 and so on.
Some fuzzy Adomian polynomials are given below,

$$\tilde{A}_0 = N(\tilde{u}_0)$$

$$\tilde{A}_1 = \dot{N}(\tilde{u}_0) \otimes \tilde{u}_1$$

$$\tilde{A}_2 = \dot{N}(\tilde{u}_0) \otimes \tilde{u}_2 \oplus \frac{\ddot{N}(\tilde{u}_0)}{2} \otimes \tilde{u}_1^2$$

$$\tilde{A}_3 = \dot{N}(\tilde{u}_0) \otimes \tilde{u}_3 \oplus \frac{\ddot{N}(\tilde{u}_0)}{2} \otimes (2\tilde{u}_1 \otimes \tilde{u}_2) \oplus \frac{\dddot{N}(\tilde{u}_0)}{3!} \otimes \tilde{u}_1^3$$

$$\tilde{A}_4 = \dot{N}(\tilde{u}_0) \otimes \tilde{u}_4 \oplus \frac{\ddot{N}(\tilde{u}_0)}{2} \otimes (\tilde{u}_2^2 \oplus 2\tilde{u}_1 \otimes \tilde{u}_3) \oplus \frac{\dddot{N}(\tilde{u}_0)}{3!} \otimes (3\tilde{u}_1^2 \otimes \tilde{u}_2) \oplus \frac{N^4(\tilde{u}_0)}{4!} \otimes \tilde{u}_1^4$$

And so on….
In the next section, we prove the convergence of FADM.

5.5.1. Convergence of FADM

Let, $\tilde{u}$ be the fuzzy valued function from convex complete fuzzy metric space E^n to itself.
i.e. $\tilde{u}: E^n \rightarrow E^n$
Consider a nonlinear equation as follows,

$$\tilde{u} \ominus N\tilde{u} = \tilde{g} \tag{5.26}$$

According to FADM, we take, $\tilde{u} = \sum_{n=0}^{\infty} \tilde{u}_n$ and the nonlinear term can be represented as a series of Adomian polynomials,

$$N\left(\sum_{n=1}^{\infty} \tilde{u}_n\right) = \sum_{n=0}^{\infty} \tilde{A}_n \, (\tilde{u}_0, \tilde{u}_1, \tilde{u}_2 \ldots \ldots \ldots . \tilde{u}_n) \tag{5.27}$$

Using the series form of $\tilde{u}$ and equation (5.27) in equation (5.26), we get,

$$\sum_{n=0}^{\infty} \tilde{u}_n \ominus \sum_{n=0}^{\infty} \tilde{A}_n = \tilde{g}$$

$$\sum_{n=0}^{\infty} \tilde{u}_n = \tilde{g} \oplus \sum_{n=0}^{\infty} \tilde{A}_n$$

Comparing both sides, we get,

$$\tilde{u}_0 = \tilde{g}$$

$$\tilde{u}_{n+1} = \tilde{A}_n \quad (5.28)$$

Now, using (5.28) in (5.27) and comparing the term from both sides of equation (5.27), we get,

$$N(\tilde{u}_0) = \tilde{A}_0$$

$$N(\tilde{u}_0 \oplus \tilde{u}_1) = \tilde{A}_0 \oplus \tilde{A}_1$$

$$N(\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2) = \tilde{A}_0 \oplus \tilde{A}_1 \oplus \tilde{A}_2$$

$$\vdots$$

$$N(\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \ldots \oplus \tilde{u}_k) = \tilde{A}_0 \oplus \tilde{A}_1 \oplus \tilde{A}_2 \oplus \ldots \oplus \tilde{A}_k \text{ , for } k \geq 0. \quad (5.29)$$

Let $\tilde{S}_n$ be the partial sum of $\tilde{u}_1, \tilde{u}_2 \ldots . \tilde{u}_n$ i.e., $\tilde{S}_n = \tilde{u}_1 \oplus \tilde{u}_2 \oplus \ldots \oplus \tilde{u}_n$

Using equations (5.28) and (5.29), we have,

$$N(\tilde{u}_0 \oplus \tilde{S}_n) = N(\tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \ldots \oplus \tilde{u}_{n+1})$$

$$= (\tilde{A}_0 \oplus \tilde{A}_1 \oplus \tilde{A}_2 \oplus \ldots \oplus \tilde{A}_n)$$

$$= \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 \oplus \ldots \oplus \tilde{u}_{n+1}$$

That is, $N(\tilde{u}_0 \oplus \tilde{S}_n) = \tilde{S}_{n+1}$ (5.30)

In the following section, the convergence of FADM is proposed and proved.

5.5.2. Theorem

Let *N* is contraction, then the sequence $\tilde{S}_n$ given by $N_n(\tilde{u}_0 \oplus \tilde{S}_n) = \tilde{S}_{n+1}, n = 0,1,2,3, \ldots$ converges to $\tilde{S}$ and in general this convergent limit $\tilde{S}$ is the solution of $N(\tilde{u}_0 \oplus \tilde{S}) = \tilde{S}$.

Proof: $N: E^n \to E^n$ is a contraction mapping with $d(N(\tilde{x}), N(\tilde{y})) \leq cd(\tilde{x}, \tilde{y})$ where $\tilde{x}, \tilde{y} \in E^n$, $c < 1$. E^n be a complete fuzzy metric space.

Consider a sequence $\tilde{S}_n$, we will prove it is Cauchy's sequence. Consider an iterative sequence $\tilde{S}_0$, $\tilde{S}_1 = N(\tilde{u}_0 \oplus \tilde{S}_0)$, $\tilde{S}_2 = N(\tilde{u}_0 \oplus \tilde{S}_1)$, $\tilde{S}_3 = N(\tilde{u}_0 \oplus \tilde{S}_2)$,….

$$d(\tilde{S}_m, \tilde{S}_{m+1}) = d\left(N(\tilde{u}_0 \oplus \tilde{S}_{m-1}), N(\tilde{u}_0 \oplus \tilde{S}_m)\right)$$

$$\leq c\, d(\tilde{S}_{m-1}, \tilde{S}_m)$$

(by the definition of contraction mapping)

$$\vdots$$

$$\leq c^{\,m} d(\tilde{S}_1, \tilde{S}_0)$$

Hence, by triangle inequality,

$$d(\tilde{S}_m, \tilde{S}_n) \leq d(\tilde{S}_m, \tilde{S}_{m+1}) \oplus d(\tilde{S}_{m+1}, \tilde{S}_{m+2}) \oplus \ldots d(\tilde{S}_{n-1}, \tilde{S}_n),$$

$$\leq (c^{m} + c^{m+1} + c^{m+2} + \cdots c^{n-1}) d(\tilde{S}_1, \tilde{S}_0),$$
$$\leq c^{m}(1 + c + c^2 + \cdots c^{n-m-1}) d(\tilde{S}_1, \tilde{S}_0),$$
$$\leq c^{m} \frac{(1-c^{n-m})}{1-c} d(\tilde{S}_1, \tilde{S}_0).$$

Now, $d(\tilde{S}_1, \tilde{S}_0)$ is finite value and as we take m very large then this expression $c^{m} \frac{(1-c^{n-m})}{1-c}$ tends to zero.

So, this sequence $\tilde{S}_n$ is a Cauchy sequence and converges to $\tilde{S}$. By the triangle inequality,

$$d(\tilde{S}, N(\tilde{u}_0 \oplus \tilde{S})) \leq d(\tilde{S}, \tilde{S}_n) \oplus d(\tilde{S}_n, N(\tilde{u}_0 \oplus \tilde{S}))$$
$$\leq d(\tilde{S}, \tilde{S}_n) \oplus c d(\tilde{S}_{n-1}, \tilde{u}_0 \oplus \tilde{S})$$

We can make R.H.S of above inequality smaller as the value of m increases. So,

$$N(\tilde{u}_0 \oplus \tilde{S}) = \tilde{S}.$$

In the next section, we solve some numerical examples by the proposed technique.

5.5.3. Example

$$\frac{d\tilde{u}}{dt} = \tilde{u}^2;\ \widetilde{u(0)} = \tilde{1}$$

It can be written as,

$$\mathbb{L}\tilde{u} = \tilde{u}^2,$$

Taking $\mathbb{L}^{-1}$ on both sides of the above equation,

$$\mathbb{L}^{-1}\mathbb{L}\tilde{u} = \mathbb{L}^{-1}(\tilde{u}^2),$$

which gives,

$$\tilde{u}(t) = \tilde{u}_0 \oplus \mathbb{L}^{-1}(\tilde{u}^2)$$

Using the Fuzzy Adomian Decomposition method

$$\sum_{n=0}^{\infty} \tilde{u}_n = \tilde{u}_0 \oplus \mathbb{L}^{-1}\left(\sum_{n=0}^{\infty} \tilde{A}_n\right) \text{ and } \tilde{u}_0 = \tilde{1}.$$

Now, using Section 5.5,

$$\tilde{A}_0 = {\tilde{u}_0}^2,$$

which gives,

$$\tilde{u}_1 = \mathbb{L}^{-1}(\tilde{A}_0) = \mathbb{L}^{-1}({\tilde{u}_0}^2) = \tilde{1} \otimes \tilde{1}t,$$

Using $\tilde{u}_1$, we have $\tilde{A}_1$ and continuing this process as in Section 5.5,

$$\tilde{u}_2 = \mathbb{L}^{-1}(\tilde{A}_1) = \mathbb{L}^{-1}(2\,\tilde{1} \otimes \tilde{1} \otimes \tilde{1}t) = \tilde{1} \otimes \tilde{1} \otimes \tilde{1}t^2$$

$$\tilde{u}_3 = \mathbb{L}^{-1}(2\,\tilde{1}\,\otimes \tilde{1}\,\otimes \tilde{1} \otimes \tilde{1}t^2 \oplus \frac{2}{2}(\tilde{1} \otimes \tilde{1}t)^2)$$

$$= (2\,\tilde{1}\,\otimes \tilde{1}\,\otimes \tilde{1} \otimes \tilde{1} \oplus \tilde{1} \otimes \tilde{1} \otimes \tilde{1} \otimes \tilde{1})\frac{t^3}{3}$$

$$= \tilde{1} \otimes \tilde{1} \otimes \tilde{1} \otimes \tilde{1}t^3$$

$$\vdots$$

So, the solution,

$$\tilde{u} = \tilde{u}_0 \oplus \tilde{u}_1 \oplus \tilde{u}_2 \ldots$$

$$\tilde{u} = \tilde{1} \oplus \tilde{1} \otimes \tilde{1}t \oplus \tilde{1}\,\otimes \tilde{1} \otimes \tilde{1}t^2 \oplus \tilde{1} \otimes \tilde{1} \otimes \tilde{1} \otimes \tilde{1}t^3 \oplus \ldots = (\tilde{1} \ominus t)^{-1}$$

Above series is convergent for $0 < t < 1$,

Let, ${}^{\alpha}\tilde{1} = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$ then lower and upper alpha cut of the above equation is,

$$[\underline{u}(t), \overline{u}(t)] = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha] + [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]^2 t + [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]^3 t^2 + [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]^4 t^3 + \cdots$$

At core, $\alpha = 1$,

$$u(t) = 1 + t + t^2 + t^3 + \cdots = (1 - t)^{-1}.$$

Which is the same as the solution in crisp environment.

5.5.4. Example

A fully fuzzy Prey-Predator model, given below, is solved by proposed FADM

$$\left.\begin{aligned} \dot{\tilde{x}} &= (\widetilde{0.1} \otimes \tilde{x}) \ominus (\widetilde{0.002} \otimes \tilde{x} \otimes \tilde{y}) \\ \dot{\tilde{y}} &= \ominus (\widetilde{0.2} \otimes \tilde{y}) \oplus (\widetilde{0.0025} \otimes \tilde{x} \otimes \tilde{y}) \end{aligned}\right\} \quad (5.31)$$

with $\tilde{x}_0 = \widetilde{80}$ and $\tilde{y}_0 = \widetilde{20}$

Solution:

Using Section 5.5, the proposed scheme (FADM), which calculates Adomian polynomials, is obtained as follows.

$$\tilde{A}_0 = (\tilde{x}_0 \otimes \tilde{y}_0)$$

$$\tilde{A}_1 = \tilde{x}_1 \otimes \tilde{y}_0 \oplus \tilde{x}_0 \otimes \tilde{y}_1$$

$$\tilde{A}_2 = \tilde{x}_2 \otimes \tilde{y}_0 \oplus \tilde{x}_1 \otimes \tilde{y}_1 \oplus \tilde{x}_2 \otimes \tilde{y}_2$$

$$\vdots$$

By using Adomian polynomial $\widetilde{A_0} = \widetilde{80} \otimes \widetilde{20}$, we obtain $\tilde{x}_1, \tilde{y}_1$ as follows,

$$\tilde{x}_1 = \widetilde{0.1} \otimes \widetilde{80}\, t \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20}\, t$$

$$\tilde{x}_1 = (\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20})t$$

$$\tilde{y}_1 = \ominus \int_0^t \tilde{c} \otimes \tilde{y}_0 dt \oplus \tilde{d} \otimes \int_0^t (\tilde{A}_0) dt$$

$$\tilde{y}_1 = \ominus \widetilde{0.2} \otimes \widetilde{20}\, t \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20}\, t$$

$$\tilde{y}_1 = (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20})\, t$$

Now using $\tilde{x}_1$ and $\tilde{y}_1$, next Adomian polynomial $\tilde{A}_1$ can be obtained as follows,

$$\tilde{A}_1 = \tilde{x}_1 \otimes \tilde{y}_0 \oplus \tilde{x}_0 \otimes \tilde{y}_1$$

$$\tilde{A}_1 = (\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20})t \otimes \widetilde{20} \oplus \widetilde{80} \otimes (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20})\, t.$$

Similarly, using $\tilde{A}_1$, we obtain $\tilde{x}_2$ and $\tilde{y}_2$ and continue the process for other values

$$\tilde{x}_2 = \int_0^t \tilde{a} \otimes \tilde{x}_1 dt \ominus \tilde{b} \otimes \int_0^t (\tilde{A}_1) dt$$

$$\tilde{x}_2 = \widetilde{0.1} \otimes (\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20}) \frac{t^2}{2} \ominus \widetilde{0.002} \otimes ((\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20}) \otimes \widetilde{20} + \widetilde{80} \otimes (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20})) \frac{t^2}{2}.$$

$$\tilde{y}_2 = \ominus \int_0^t \tilde{c} \otimes \tilde{y}_1 dt \oplus \tilde{d} \otimes \int_0^t (\tilde{A}_1) dt$$

$$\tilde{y}_2 = -\widetilde{0.2} \otimes (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20}) \frac{t^2}{2} \oplus \widetilde{0.0025} \otimes ((\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20}) \otimes \widetilde{20} \oplus \widetilde{80} \otimes (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20})) \frac{t^2}{2}.$$

Thus, series solution of system (5.31) is given as,

$$\tilde{x} = \widetilde{80} \oplus (\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20})t \oplus \widetilde{0.1} \otimes (\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20}) \frac{t^2}{2} \ominus \widetilde{0.002} \otimes ((\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20}) \otimes \widetilde{20} \oplus \widetilde{80} \otimes (\ominus \widetilde{0.2} \otimes \widetilde{20} + \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20})) \frac{t^2}{2} \oplus \ \ldots \quad (5.32)$$

$$\tilde{y} = \widetilde{20} \oplus (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20})\, t \ominus \widetilde{0.2} \otimes (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20}) \frac{t^2}{2} \oplus \widetilde{0.0025} \otimes ((\widetilde{0.1} \otimes \widetilde{80} \ominus \widetilde{0.002} \otimes \widetilde{80} \otimes \widetilde{20}) \otimes \widetilde{20} \oplus \widetilde{80} \otimes (\ominus \widetilde{0.2} \otimes \widetilde{20} \oplus \widetilde{0.0025} \otimes \widetilde{80} \otimes \widetilde{20})) \frac{t^2}{2} \oplus \ldots \quad (5.33)$$

Equations (5.32) and (5.33) are the required fuzzy solution of the fully fuzzy Prey-Predator model as given by equation (5.31).

5.6. Conclusion

In this chapter, we have discussed the semi-analytical technique FADM in two ways. First, we have used it traditionally i.e., applying on a problem in parametric form. Secondly, we have developed the whole technique in a fuzzy environment. For that, we have proved many results under new derivative, the Modified generalized Hukuhara derivative. The advantage of this technique, we can solve the directly fuzzy differential equation without converting it into the system of ordinary differential equation. It is directly applicable to fuzzy differential equations. We have also solved examples by both techniques. All the results are validated at the core.