# Fully Fuzzy Semi-linear Dynamical System Solved by Fuzzy Laplace Transform Under Modified Hukuhara Derivative

**Purnima Pandit and Payal Singh**

**Abstract** Semi-linear dynamical systems draw attention in many useful real world problems like population model, epidemic model, etc., they also occur in various applications involving parabolic equations. Now, when the modelling of such applications has inbuilt possibilistic uncertainty, it can be efficiently realized using fuzzy numbers. In this paper, we modify the existing Hukuhara derivative and give the pertaining results for it. We also redefine the Fuzzy Laplace Transform (FLT) and use it to solve such fully fuzzy semi-linear dynamical system.

**Keywords** Fuzzy semi-linear dynamical system · Fuzzy differential equation (FDE) · Fuzzy Laplace transform (FLT) · Modified Hukuhara derivative (mH-derivative) · Fuzzy convolution theorem

## 1 Introduction

Differential equations play a significant role to model the problem of science, physics, engineering, finance, economics, etc. While modelling, there may be possibilistic uncertainty in measuring or stating a parameter value that is involved. Such uncertainty can be depicted using fuzzy numbers, giving rise to fuzzy differential dynamical system.

The fuzzy theory was introduced by Zadeh in his explanatory paper in 1965 [1]. The fuzzy derivative was first given by Chang and Zadeh [2] and followed up by Dubois and Prade [3], they used extension principle to define fuzzy derivative.

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# Population Dynamic Model of Two Species Solved by Fuzzy Adomian Decomposition Method

**Purnima Pandit, Prani Mistry, and Payal Singh**

**Abstract** Adomian decomposition method (ADM) is powerful method to solve nonlinear functional equations. This method also solves nonlinear differential equations and provides the solution in series form, which effectively and accurately converges very fast to the exact solution, if it exists. In this work, we propose Fuzzy Adomian Decomposition Method (FADM) in parametric form to compute the solution for nonlinear dynamical system. We propose theorem for existence of FADM in parametric form for such system. Population dynamics model of two species; i.e., prey-predator model, involving fuzzy parameters and fuzzy initial condition is solved using proposed method, and results are compared at core.

**Keywords** Fuzzy Set · Fuzzy Adomian Decomposition Method (FADM) · Non-linear differential equation · Population Dynamics · Semi-Analytic solution

## 1 Introduction

Prey-predator dynamics is based on Lotka-Volterra model. This model is of lot practical importance and it fits into in many areas like biological model, financial problem, environmental problem, etc. We propose to consider Prey-Predator dynamics in fuzzy setup because there may be manual or machine error in estimating parameters or/and initial condition. So, modelling in fuzzy setup gives more realistic results.

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# Chapter 55
# Mathematical Modeling of Air Heating Solar Collectors with Fuzzy Parameters

**Purnima Pandit, Prani R. Mistry, and Payal P. Singh**

**Abstract** Limited fuel and fossil energy have compelled the world to look forward for other renewable energy sources. Solar energy is one such vital energy resources that finds the application in various industrial as well as residential processes. Heat from this source is useful for increasing the temperature of air used for blow drying processes. The mathematical model for such phenomenon with imprecise parameter and/or initial condition leads to a fuzzy nonlinear dynamical model. In this paper, we propose Fuzzy Adomian Decomposition Method to obtain solution of this system. This solution obtained is compared at core.

**Keywords** Heat transfer · Solar air collector · Fuzzy parameters · Fuzzy Adomian Decomposition Method

**MSC** 34A07

## Nomenclatures

| | |
|---|---|
| $\delta x$ | An element in $x$ direction (m) |
| $\eta$ | Thermal efficiency |
| $\tau$ | Transmittance |
| $\alpha$ | Absorbance |
| $\varepsilon$ | Emissivity |
| $G_t$ | Solar radiation (W/m$^2$) |

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# Prey Predator Model with Fuzzy Initial Conditions

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***Abstract* - In this paper we consider a fuzzy differential equation describing a prey predator model with fuzzy initial condition. Prey predator model in the fuzzy setup is more realistic depiction of the phenomena, here the initial conditions are considered as fuzzy because the initial population estimates may not be precisely known in the real life situation. The results for the existence of the solution are discussed in paper.**

***Keywords* - Fuzzy differential equation, Prey predator model, Fuzzy initial condition.**

## I. INTRODUCTION

In the population model, consider two species of animals which are part of food chain, predator eats prey and prey depends on other food, the prey are assumed to have unlimited food supply and to reproduce exponentially unless subject to predation, this exponential growth is represented by the term ax. The rate of predation upon the prey is assumed to be proportional to the rate at which prey and predator interact, represented as *bxy*. Secondly, the growth of predator is proportional to *xy* with the proportionality constant *d* and *cy* represents the loss rate of the predator due to natural death or absence of prey.

Thus, the two species population model can be represented as a system of two first order nonlinear differential equations which is also known as Lotka-Volterra equation.

$$\dot{x}(t) = ax - bxy;$$
$$\dot{y}(t) = -cy + dxy; \quad (1)$$

with initial condition $x(0) = x_0$ and $y(0) = y_0$, where $a, b, c$ and $d$ are positive constants as described above, $x(t)$ denotes the population of prey species and $y(t)$ denotes the population of predator species, $x_0$ and $y_0$ is the initial estimates of the species.

For the system as given by (1) it may not be possible to have the exact estimates of initial population, then such a scenario fits into fuzzy setup where the initial estimates are represented by fuzzy numbers, the concept of fuzzy sets was proposed by L. A. Zadeh [8].

System (1) with fuzzy initial condition is given by:

$$\dot{x}(t) = ax - bxy;$$
$$\dot{y}(t) = -cy + dxy; \quad (2)$$

with $x(0) = \tilde{x}_0$ and $y(0) = \tilde{y}_0$.

For such system as given by (2), Muhammad Zaini Ahmad, Bernard De Baest [9] proposed the numerical solution by generalized numerical method and Omer Akin, Omer Oruc [10] proposed the solution by strongly generalized derivative concept.

System (2) can be written in compact form as

$$\dot{X} = AX + f(X(t))$$

Where, $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $A = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ and $f = \begin{pmatrix} -bxy \\ dxy \end{pmatrix}$

In our paper, we adopt analytical approach to solve the fuzzy prey predator model, which gives the estimate of number of prey and number of predator at time *t*. To get the approximate solution first we linearize the equation about equilibrium point by Taylor's expansion then for this linearized system the solution is obtained which satisfies the fuzzy initial condition. The paper is organized in the following manner in the next section the preliminaries are listed, in the section 3 we describe the technique to obtain crisp solution followed by the fuzzy solution. Illustrative example is given at the end.

## II. PRELIMINARIES

### A. *α-cut*

An $\alpha$-cut or $\alpha$-level set of a fuzzy set $A \subseteq X$ is an ordinary set $A^\alpha \subseteq X$, such that $A^\alpha = \mu_A(X) \geq \alpha, \forall x \in X$.

### B. *Fuzzy Number*

A fuzzy set is said to be fuzzy number if it qualifies following condition:

- $A$ is a convex fuzzy set, i.e. $A(r \curlywedge + (1 - \curlywedge)s) \geq \min [A(r), A(s)]$, $\curlywedge \in [0,1]$ and $r, s \in X$;
- $A$ is normal, i.e. $\exists x_0 \in X$ with $(A(x_0) = 1$;
- $A$ is upper semi-continuous i.e. $A(x_0) \geq \lim_{x \to x_0^{\pm}} A(x)$;
- $[A]^0 = \overline{\sup p(A)} = \overline{\{x \in R | A(x) \geq 0\}}$ is compact, where $\bar{A}$ denotes the closure of $A$.

### C. *Fuzzy Operation*

For $u, v \in R_f$ and $\curlywedge \in R$ the sum $u + v$ and the product $\curlywedge . u$ is defined as

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha = [\overline{u}\ \underline{u}] + [\overline{v}\ , \underline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}]$$

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& $[\lambda.u]^\alpha = \lambda\,[\overline{u}\ \underline{u}] = [\lambda\,\overline{u}, \lambda\,\underline{u}]$, for all $\alpha \in [0,1]$.

### D. H-Difference

Let $u, v \in \mathcal{F}$. If there exist $w \in \mathcal{F}$ such that $u = v \oplus w$ then $w$ is called the *H-difference of* $u$ and $v$ is denoted by $u \ominus v$.

### E. Hukuhara Derivative

Consider a fuzzy mapping $F: (a,b) \to \mathcal{F}$ and $t_0 \in (a\,,b)$. We say that $F$ is differentiable at $t_0 \in (a,b)$ if there exist an element $F'(t_0) \in \mathcal{F}$ such that for all $h > 0$ sufficiently small $\exists F(t_0+h) \ominus F(t_0), F(t_0) \ominus F(t_0-h)$ and the limits

$$\lim_{h\to 0+} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0-} \frac{F(t_0) \ominus F(t_0-h)}{h}$$

exist and are equal to $F'(t_0)$.

## III. CRISP PREY-PREDATOR MODEL

We first obtain the equilibrium point by solving

$$AX + f(X(t)) = 0$$

That is $\quad \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -bxy \\ dxy \end{pmatrix} = 0$

Giving $\quad \begin{pmatrix} x_e \\ y_e \end{pmatrix} = \begin{pmatrix} c/d \\ a/b \end{pmatrix}$

as the equilibrium point of the system.

To obtain the solution of such system we linearize the system (1) about the equilibrium point $(x_e, y_e)$ with the help of Taylor's expansion considering the first order term and neglecting the higher order terms, we get,

$$\dot{X}(t) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix} \qquad (3)$$

That is, $\quad \dot{X}(t) = CX + B$

where C is 2x2 matrix and given as $\begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$ and B is 2x1 vector and given as $\begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix}$.

The eigen values of C are $\lambda_1 = i\sqrt{ac}$ and $\lambda_2 = -\,i\sqrt{ac}$. We construct fundamental matrix $\Psi(t)$ with the columns as the linearly independent eigen vectors corresponding to these complex eigen values. The solution of system (3) is then given by:

$$X(t) = \Psi(t)\,\Psi^{-1}(0)\,X(0) + \Psi(t)\int_0^t \Psi^{-1}(s)\,B\,ds$$

## IV. FUZZY PREY-PREDATOR MODEL

The Prey-Predator model with fuzzy initial condition in the linearized form is given as

$$\dot{X}(t) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix} \qquad (4)$$

with $X(0) = \tilde{X}_0$ and $Y(0) = \tilde{Y}_0$

The solution of system (4) is given by,

$$\tilde{X}(t) = \Psi(t)\,\Psi^{-1}(0)\tilde{X}(0) + \Psi(t)\int_0^t \Psi^{-1}(s)\,B\,ds$$

Taking $\alpha - cut$ on both sides we get,

$$[\overline{X}, \underline{X}] = \Psi(t)\,\Psi^{-1}(0)[\overline{X}_0, \underline{X}_0] + \Psi(t)\int_0^t \Psi^{-1}(s)\,B\,ds$$

Comparing the elements of interval we get

$$\overline{X} = \Psi(t)\,\Psi^{-1}(0)\,\overline{X}_0 + \Psi(t)\int_0^t \Psi^{-1}(s)\,B\,ds \qquad (5)$$

$$\underline{X} = \Psi(t)\,\Psi^{-1}(0)\,\underline{X}_0 + \Psi(t)\int_0^t \Psi^{-1}(s)\,B\,ds \qquad (6)$$

The state vector $\tilde{X}(t)$ can be constructed from (5) and (6) using First decomposition theorem, Klir [11].

The solution $\tilde{X}(t)$ obtained using (5) and (6) will be fuzzy solution if for $t > 0$.

(i) $\forall \alpha \in [0,1],\ {}^{\alpha}\underline{x}_i(t) \le {}^{\alpha}\overline{x}_i(t)$

(ii) $\forall \alpha, \beta \in [0,1], \alpha \le \beta$

$${}^{\alpha}\underline{x}_i(t) \le {}^{\beta}\underline{x}_i(t) \le {}^{\beta}\overline{x}_i(t) \le {}^{\alpha}\overline{x}_i(t)$$

Hence, $\forall \alpha \in [0,1],\ \ \tilde{x}_i(t) = \left[{}^{\alpha}\underline{x}_i(t), {}^{\alpha}\overline{x}_i(t)\right]$, for $i = 1,2$ where,

- ${}^{\alpha}\underline{x}_i$ is a bounded left continuous non-decreasing function over $[0,1]$.
- ${}^{\alpha}\overline{x}_i$ is a bounded left continuous non-increasing function over $[0,1]$.

## V. NUMERICAL ILLUSTRATIVE

Consider the following example of Prey-Predator Model in crisp set up.

$$(x(\dot{t})) = 0.1x - 0.005xy;$$

$$(y(\dot{t})) = -0.4y + 0.008xy;$$

$$X(0) = 130; \ \ Y(0) = 40$$

The system has two critical points, the trivial one is origin and the other is (50, 20). First, we linearize this problem at (50, 20) by Taylor's expansion and get,

$$\dot{X}(t) = -0.25y + 5$$

$$\dot{Y}(t) = 0.16x - 8 \qquad (7)$$

The linearized system has 2x2 coefficient matrix C which is given as $\begin{bmatrix} 0 & -0.25 \\ 0.16 & 0 \end{bmatrix}$ and B is 2x1 vector which is given as $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$.

Eigen values of this matrix are $-0.2i$ and $0.2i$ and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -0.8i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0.8i \end{bmatrix}$ respectively and the fundamental matrix is given by

$$\Psi(t) = \begin{bmatrix} \cos(0.2t) & \sin(0.2t) \\ 0.8\sin(0.2t) & -0.8\cos(0.2t) \end{bmatrix}$$

Thus, the solution of system (7) is given by:

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$$X(t) = \Psi(t)\Psi^{-1}(0)X(0) + \Psi(t)\int_0^t \Psi^{-1}(s)B\,ds$$

That is,

$$X(t) = \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} 130 \\ 40 \end{bmatrix} + \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \int_0^t \left( \begin{bmatrix} cos(0.2s) & -\frac{sin(0.2s)}{8i} \\ sin(0.2s) & \frac{cos(0.2s)}{8i} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) ds$$

And we get,

$$x(t) = 50 + 80\cos(0.2t) - 25\sin(0.2t)$$

$$y(t) = 20 + 20\,cos(0.2t) + 64\,sin(0.2t)$$

The evolution of system in small time interval is as shown in Fig. (1).

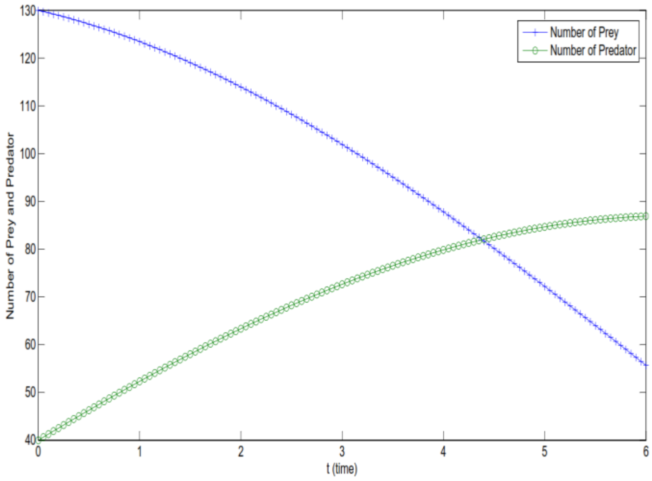


**Fig. 1:** Evolution of Crisp Prey Predator Population

In system (8), when we consider fuzzy initial condition as below,

$$\widetilde{X_0} = \begin{Bmatrix} \frac{x-120}{10} & 120 < x \le 130 \\ \frac{150-x}{20} & 130 < x \le 150 \end{Bmatrix}$$

$$\widetilde{Y_0} = \begin{Bmatrix} \frac{y-20}{10} & 20 < y \le 40 \\ \frac{50-y}{10} & 40 < y \le 50 \end{Bmatrix} \qquad (8)$$

Then, $(\widetilde{X_0})^{\alpha} = [10\alpha + 120, 150 - 20\alpha]$

$(\widetilde{Y_0})^{\alpha} = [20\alpha + 20, 50 - 10\alpha]$

where $\alpha \in [0,1]$.

The solution is now obtained as,

$$\tilde{X}(t) = \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} 10\alpha + 120, 150 - 20\alpha \\ 20\alpha + 20, 50 - 10\alpha \end{bmatrix} + \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \int_0^t \left( \begin{bmatrix} cos(0.2s) & -\frac{sin(0.2s)}{8i} \\ sin(0.2s) & \frac{cos(0.2s)}{8i} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) ds$$

Now, comparing the components, we get

$\underline{X} = 70\,cos(0.2t) + 50$ and $\underline{Y} = 20 + 56\sin(0.2t)$

$$\overline{X} = 50 + 100\,cos(0.2t) - 37.5\sin(0.2t) \text{ and}$$

$$\overline{Y} = 20 + 80\,sin(0.2t) + 30\,cos(0.2t).$$

The evolution of Prey and Predator population in system (8), in small time interval are as shown in Fig. (2) and Fig. (3) respectively.

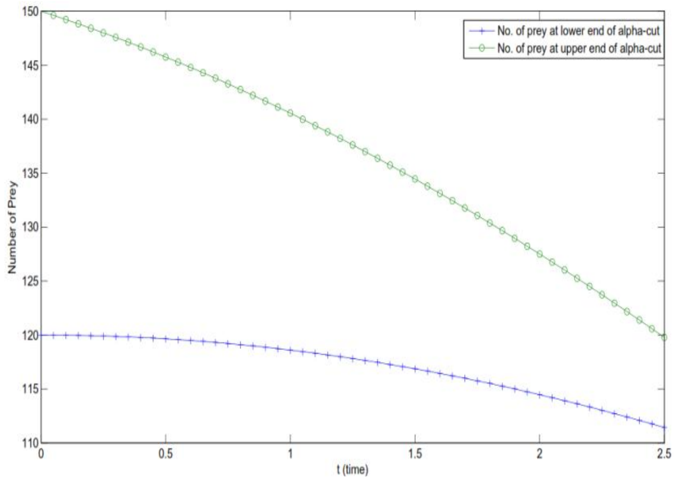


**Fig. 2:** Evolution of Fuzzy Prey Population

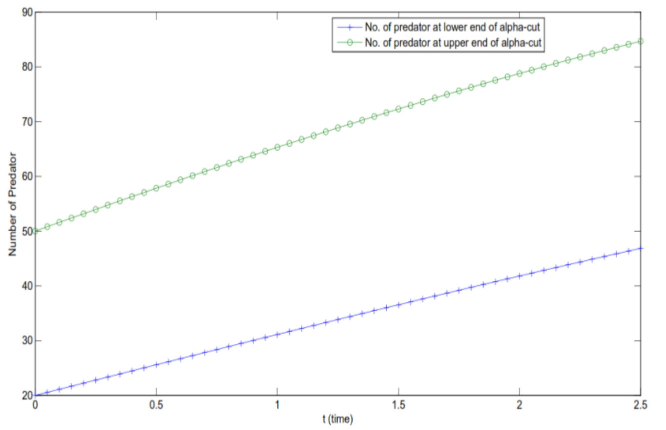


**Fig. 3:** Evolution of Fuzzy Predator Population

## VI. CONCLUSION

Here, we discuss approximate solution of Prey Predator model with fuzzy initial condition over small time interval. In future, for the same we can study the case with the entries of $A$ represented as fuzzy numbers.

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# Numerical technique to solve dynamical system involving fuzzy parameters

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## Abstract

***This paper investigates system of differential equations with fuzzy parameters and fuzzy initial condition i.e. given by*** $\tilde{X}(t) = \tilde{f}(t, \tilde{X})$, $\tilde{X}(t_0) = \tilde{X}_0$ ***for numerical solution. Here*** $\tilde{f}(t, \tilde{X})$ ***can be nonlinear or it can be linear of the form*** $\tilde{A} \otimes \tilde{X} + \tilde{B}$, ***where*** $\tilde{A}_{n\times n}$ ***and*** $\tilde{B}_{n\times 1}$ ***be some matrices with all entries as fuzzy number. In this paper, we propose the numerical technique which is based on approximation of Hukuhara difference, for both kind of dynamical systems (linear and nonlinear). In dynamical system, uncertainty of possibilistic type can be realized efficiently using fuzzy parameters and such systems are mathematical models for various application in varied domains. For such systems, we get the scheme for existence of solution and its convergence. Lastly, illustrative examples are solved by using proposed scheme and compared with crisp solution.***

**Keywords**: Fuzzy parameters, Fuzzy number,Hukuhara differentiability, Linear and nonlinear fuzzy dynamical system.

## 1.INTRODUCTION

Since Zadeh's first paper on fuzzy set in 1965, there has been lots of development in various field of fuzzy set theory. Most of the real-world problem can be modelled as a dynamical system but while modelling such problems, uncertainty mayoccur in estimating parameters or/and initial condition, so to reduce such kind of possibilistic uncertainties in dynamical system, fuzzy theory is widely used.

The concept of fuzzy derivative was first introduced by Chang and Zadeh[1] in 1972. Dubois and Prade[2] defined derivative based on extension principle in 1982.Puri and Ralescue[3] introduced H-derivative of fuzzy-number-valued function.Seikkala[4] and Kaleva [5] first simultaneously solved the fuzzy initial value problem with fuzzy initial condition.Kandel and Byatt applied this theory to fuzzy dynamical system. The basic and most popular approach to solve Fuzzy Differential Equation (FDE) is Hukuhara Differentiability. But Hukuhara differentiability has disadvantage that,FDE requires increasing length of support. To deal with this problem Hüllermeier[6]gave FDE as family of differential inclusions but the main problem with differential inclusion is not having derivative of fuzzy number valued functions. Bede and Gal [7]proposed the alternative of Hukuhara differentiability, i.e. strongly generalized differentiability (GH-differentiability).This concept is based on four forms of lateral derivatives. The disadvantage of this differentiability is that FDE can have several solutions, locally have two, so choose that one which reflects practically possible solution for modelled problem.

At times, it may be difficult or not possible to find exact solution of FDE, also the real-life applications may have only the observations for the dynamical processes involving imprecisionthen for solving such problems use of numerical methods becomes inevitable. Numerical method for solving FDE with fuzzy initial condition given by many authors[8],

[9],[10],[11],[12],[13],[14],[15],[16],[17],[18], [19], [20], [21] and analytical technique used in [33]. In [25], [26], [27], [28], [29] authors solved system of fuzzy differential equations with/or fuzzy parameters and fuzzy initial condition.

In this paper, we propose numerical scheme by approximating the Hukuhara difference and proposed scheme is verified by solving two examples, first example is based on toxic discharge in river by mills [31] and second is based on modelling of spread of infection disease [32].

The paper contains 5 sections,insection 2, basic definitions regarding fuzzy set theory are given, in section 3, numerical scheme is proposed and in last sections examples and conclusion are given.

## 2.PRELIMINARIES

Let $\mathcal{P}(R^n)$ denote the family of all nonempty compact convex subsets of $R^n$ and define the addition and scalar multiplication in $\mathcal{P}(R^n)$ as usual. Let $A, B \in R^n$. The distance between $A$ and $B$ is defined by Housdorff metric.

$$d(A,B) = max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

$\|\cdot\|$ denotes usual norm in $R^n$. $(\mathcal{P}(R^n), d)$ is a complete metric space.

Let,
$E^n = \{\tilde{u}: R^n \to [0\ 1] /\ \tilde{u}$ satisfies following properties$\}$

- $\tilde{u}$ is normal i.e there exist $x_0 \in R^n \tilde{u}(x_0) = 1$.
- $\tilde{u}$is a fuzzy convex.
- $\tilde{u}$is upper semicontinuous.
- $[\tilde{u}]^0 = \overline{\text{supp}\,(\tilde{u}) \quad = \{x \in R^n / \tilde{u}(x) \geq 0\}}$ is compact.

For $0 < \alpha \leq 1$, denote$\tilde{u}^\alpha = \{ x \in R^n / \tilde{u}(x) \geq \alpha\}$ then from the above properties follows, $\alpha$ level sets $\tilde{u}^\alpha \in \mathcal{P}(R^n) \forall\, \alpha \in [0\ 1]$

### Fuzzy number in parametric form

A fuzzy number in parametric form is an order pair of the form $\tilde{u}^\alpha = (\underline{u}(\alpha), \overline{u}(\alpha)) = [\underline{u}\overline{u}]$
where$0 \leq \alpha \leq 1$ satisfying following condition:

- $\underline{u}$is bounded left continuous increasing function in $[0\ 1]$.
- $\overline{u}$ is bounded right continuous increasing function in$[0\ 1]$.
- $\underline{u} \leq \overline{u}$

### Triangular Fuzzy Number

Triangular fuzzy number is defined as with three points $(l, m, n)$,

$$\tilde{u}(x) = \begin{cases} \dfrac{x-l}{m-l} & l \leq x \leq m \\ \dfrac{n-x}{n-m} & m \leq x \leq n \\ 0 & x \geq n \end{cases}$$

$\tilde{u}(x)$ is membership function.

### Hausdorff Distance

Let $d: E^n \times E^n \to R_+ \cup \{0\}$,

$$d(\tilde{u}, \tilde{v}) = \sup_{\alpha\in[0\ 1]} \max\left|\underline{u} - \underline{v}, \overline{u} - \overline{v}\right|$$

is Hausdorff distance between two fuzzy numbers *u* and *v*. $(E^n, d)$ is a complete metric space[3].

### Fuzzy Operation

For$\tilde{u}, \tilde{v} \in E^n$ and $\lambda \in R$ the sum $\tilde{u} + \tilde{v}$and the product $\lambda \otimes \tilde{u}$ is defined as

$$[\tilde{u} + \tilde{v}]^\alpha = [\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}]\ , [\lambda \otimes u]^\alpha = \lambda[\underline{u}, \overline{u}] = [\lambda.\overline{u}, \lambda.\underline{u}] \forall\, \alpha \in [0\ 1].$$

$$[\tilde{u} \otimes \tilde{v}]^\alpha = \left[\min(\underline{u}\overline{v}, \underline{v}\overline{u}, \underline{u}\underline{v}, \overline{u}\overline{v}), \max(\underline{u}\overline{v}, \underline{v}\overline{u}, \underline{u}\underline{v}, \overline{u}\overline{v})\right] \forall\, \alpha \in [0\ 1]$$

### Continuity of Fuzzy function

If $f: R \times E^n \to E^n$then $f$is continuous at point $(t_0, x_0)$ provided that for any fixed number $\alpha \in [0\ 1]$ and any $\epsilon > 0, \exists\, \delta(\epsilon, \alpha) s.t\ d([\tilde{f}(t, \tilde{x})]^\alpha, [\tilde{f}(t_0, \tilde{x}_0)]^\alpha) < \epsilon$ whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([\tilde{x}]^\alpha, [\tilde{x}_0]^\alpha) < \delta(\epsilon, \alpha) \forall\, t \in R$ and$\tilde{x} \in E^n$.(Song and Wu as in [22])

### Hukuhara Derivative

As in [3], for a fuzzy mapping$\tilde{f}: I \times E^n \to E^n$ and $t_0 \in I$, $\tilde{f}$ is said to be differentiable at $t_0 \in I$, if there exist an element $\tilde{f}'(t_0) \in E^n$such that for all $h > 0$ sufficiently small$\exists\, \tilde{f}(t_0 + h) \ominus \tilde{f}(t_0), \tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h\to 0-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \tilde{f}'(t_0)$$

Let $x, y \in E^n$. if there exists $z \in E^n$such that $x = y + z$, then $z$ is called the H-difference of $x$ and $y$ and it is denoted by $x \ominus y$. $x \ominus y \neq x + (-1)y$. In this paper "$\ominus$" stands for always H-difference.

### Seikkala Derivative

This derivative follows if $[\underline{\dot{x}}(t), \dot{\overline{x}}(t)]$ are the $\alpha\ cut$ of any fuzzy numbers then $SD\widetilde{X^\alpha}(t)$ exists and is defined as, [4]

$$SD\widetilde{X^\alpha}(t) = [\underline{\dot{x}}(t), \dot{\overline{x}}(t)]$$

## 3.Fully Fuzzy Dynamical System

Consider a system of ODE in $E^n$ with fuzzy coefficient,

$$\dot{\tilde{X}}(t) = \tilde{f}(t, \tilde{X}), \tilde{X}(t_0) = \tilde{X}_0 \quad (1)$$

where, $\tilde{f}: [t_0\ T] \times E^n \to E^n$ and

$$\tilde{f} = \begin{bmatrix} \widetilde{f_1} \\ \widetilde{f_2} \\ \vdots \\ \widetilde{f_n} \end{bmatrix}, \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}, \tilde{X}_0 = \begin{bmatrix} \tilde{x}_{10} \\ \tilde{x}_{20} \\ \vdots \\ \tilde{x}_{n0} \end{bmatrix}$$

Here each $\widetilde{f_i}$ is Hukuhara differentiable, $\forall\, i = 1,2,3 \ldots ..$ $\tilde{f}$can be linear or nonlinear so we consider both cases.

***Case 1:*** $\tilde{f}$ *is linear function*

System (1) is given as,

$$\dot{\tilde{X}}(t) = \tilde{A} \otimes \tilde{X} + \tilde{B} \tilde{X}(t_0) = \widetilde{X_0} \quad t \in [t_0\ T] \quad (2)$$

Where,$\tilde{A} = (\widetilde{a_{ij}}) \in E$and $\tilde{B} = (\widetilde{b_i}) \in E$, where $i = 1,2,3 \ldots \ldots \ldots, j = 1,2,3 \ldots \ldots ..$
$\alpha -$ cut of $\tilde{A}, \tilde{X}$, $\tilde{B}$ and $\tilde{X}_0$ are given as,

$$\tilde{A}^\alpha = [\underline{A}\overline{A}], \tilde{X}^\alpha = [\underline{X}\overline{X}], \tilde{B}^\alpha = [\underline{B}\overline{B}] \text{ and } \widetilde{X_0}^\alpha = [\underline{X_0}\overline{X_0}].$$

After taking $\alpha\ cut$ of system (2), it is converted into system of ordinary differential equations, Refer [15,23,24].

$$[\underline{\dot{X}}\dot{\overline{X}}] = [\underline{A}\overline{A}][\underline{X}\overline{X}] + [\underline{B}\overline{B}]; [\underline{X_0}\overline{X_0}]$$

$$\underline{\dot{X}}(t) = \min(\underline{AX}, \underline{A}\overline{X}, \underline{X}\overline{A}, \overline{AX}) + \underline{BX}(0) = \underline{X_0}$$

$$\dot{\overline{X}}(t) = \max(\underline{AX}, \underline{A}\overline{X}, \underline{X}\overline{A}, \overline{AX}) + \overline{B}\,\overline{X}(0) = \overline{X_0}$$

Above system can be written in matrix form as given below,

$$\begin{bmatrix} \underline{\dot{X}}(t) \\ \dot{\overline{X}}(t) \end{bmatrix} = \begin{bmatrix} \underline{A} & 0 \\ 0 & \overline{A} \end{bmatrix} \begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \overline{B} \end{bmatrix}; \quad (3)$$

$\underline{X}(0) = \underline{X_0}$ and $\overline{X}(0) = \overline{X_0}$

Now by approximating Hukuhara difference of the left side of system (3) at $t = t_k$,

$$\begin{bmatrix}\underline{X}_{k+1}\\ \overline{X}_{k+1}\end{bmatrix} = \begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{B}\\ \overline{B}\end{bmatrix}; \quad (4)$$

$$\underline{X}(0) = \underline{X_0} \text{ and } \overline{X}(0) = \overline{X_0}$$

where $k = 0, 1, 2, \ldots\ldots\ldots$

For solving purpose, system (4) can be given as component wise,

$$\begin{bmatrix}\underline{x}^1{}_{k+1}\\ \underline{x}^2{}_{k+1}\\ \vdots\\ \underline{x}^n{}_{k+1}\end{bmatrix} = \begin{bmatrix}(I + h\underline{a}_{11}) & \cdots & \underline{a}_{1n}\\ \vdots & \ddots & \vdots\\ \underline{a}_{n1} & \cdots & (I + h\underline{a}_{nn})\end{bmatrix}\begin{bmatrix}\underline{x}^1{}_k\\ \underline{x}^2{}_k\\ \vdots\\ \underline{x}^n{}_k\end{bmatrix} + h\begin{bmatrix}\underline{b}_1\\ \underline{b}_2\\ \vdots\\ \underline{b}_n\end{bmatrix}$$

$$\begin{bmatrix}\overline{x}^1{}_{k+1}\\ \overline{x}^2{}_{k+1}\\ \vdots\\ \overline{x}^n{}_{k+1}\end{bmatrix} = \begin{bmatrix}(I + h\overline{a}_{11}) & \cdots & \overline{a}_{1n}\\ \vdots & \ddots & \vdots\\ \overline{a}_{n1} & \cdots & (I + h\overline{a}_{nn})\end{bmatrix}\begin{bmatrix}\overline{x}^1{}_k\\ \overline{x}^2{}_k\\ \vdots\\ \overline{x}^n{}_k\end{bmatrix} + h\begin{bmatrix}\overline{b}_1\\ \overline{b}_2\\ \vdots\\ \overline{b}_n\end{bmatrix}$$

Existence and uniqueness of solution for system such as (4) is given in [30].

**Theorem (1):**
Let $\underline{X}(t), \overline{X}(t)$ be the exact solution of system (3) and $\underline{X}_{k+1}(t), \overline{X}_{k+1}(t)$ be the sequences of numerical solution defined by the system (4) converges to the exact solution of system (3).

**Proof:** It is sufficient to show,

$$\lim_{k\to\infty} \underline{X}_{k+1}(t) = \underline{X}(t)$$
$$\lim_{k\to\infty} \overline{X}_{k+1}(t) = \overline{X}(t)$$

By proposed scheme in system (4) is given as,

$$\begin{bmatrix}\underline{X}_{k+1}\\ \overline{X}_{k+1}\end{bmatrix} = \begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix} + h\begin{bmatrix}\underline{B}\\ \overline{B}\end{bmatrix}$$

with initial conditions, $\underline{X}(0) = \underline{X_0}$ and $\overline{X}(0) = \overline{X_0}$,

From system (3), writing the equivalent discrete form,

$$\begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} = \begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}\\ \overline{X}\end{bmatrix} + h\begin{bmatrix}\underline{B}\\ \overline{B}\end{bmatrix} (5)$$

By subtractingSystem (5) from system (4),

$$\begin{bmatrix}\underline{X}_{k+1} - \underline{X}\\ \overline{X}_{k+1} - \overline{X}\end{bmatrix} = \begin{bmatrix}\underline{X}_k - \underline{X}\\ \overline{X}_k - \overline{X}\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{X}_k - \underline{X}\\ \overline{X}_k - \overline{X}\end{bmatrix}$$

By using Error term,

$$\begin{bmatrix}\underline{E}_{k+1}\\ \overline{E}_{k+1}\end{bmatrix} = \begin{bmatrix}\underline{E}_k\\ \overline{E}_k\end{bmatrix} + h\begin{bmatrix}\underline{A} & 0\\ 0 & \overline{A}\end{bmatrix}\begin{bmatrix}\underline{E}_k\\ \overline{E}_k\end{bmatrix}$$

For solving purpose, we can write above system as following,

$$\underline{E}_{k+1} = \underline{E}_k + h\underline{A}\underline{E}_k$$
$$\overline{E}_{k+1} = \overline{E}_k + h\overline{A}\overline{E}_k$$

Now by backward substitution,

$$\underline{E}_{k+1} = (I + h\underline{A})^{(k+1)}\underline{E}_0$$

$\underline{A}$ is nonsingular matrix so $(I + h\underline{A})$ is also nonsingular matrix so the solution of system exists.

For the convergence of system (4),

$$|(I + h\underline{A})| < 1$$

Take $\underline{E}_0 = 0, \underline{E}_{k+1} \to 0$

That is,

$$\lim_{k\to\infty} \underline{X}_{k+1}(t) = \underline{X}(t)$$

Similarly,

$$\lim_{k\to\infty} \overline{X}_{k+1}(t) = \overline{X}(t)$$

***Case 2:*** $\tilde{\boldsymbol{f}}$ *is nonlinear function*

Consider a system of ODE in $E^n$ with fuzzy coefficient,

$$\dot{\tilde{X}}(t) = \tilde{f}(t, \tilde{X}), \tilde{X}(t_0) = \tilde{X}_0 \quad (6)$$

where, $\tilde{f}: [t_0 \ T] \times E^n \to E^n$ and

$$\tilde{f} = \begin{bmatrix}\tilde{f}_1\\ \tilde{f}_2\\ \vdots\\ \tilde{f}_n\end{bmatrix}, \tilde{X} = \begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \vdots\\ \tilde{x}_n\end{bmatrix}, \tilde{X}_0 = \begin{bmatrix}\tilde{x}_{10}\\ \tilde{x}_{20}\\ \vdots\\ \tilde{x}_{n0}\end{bmatrix}$$

$\tilde{X}(t)$ is Hukuhara differentiable or Seikkala differentiable then,

$$\dot{\tilde{X}}^{\alpha}(t) = [\dot{\underline{X}}\dot{\overline{X}}], \tilde{f}^{\alpha} = [\underline{f}\overline{f}] \text{ and } \widetilde{X_0}^{\alpha} = [\underline{X_0}\overline{X_0}]$$

Taking $\alpha$ *cut* of system (6) and we get,

$$[\dot{\underline{X}}\dot{\overline{X}}] = [\underline{f}(t, \underline{X}, \overline{X}), \overline{f}(t, \underline{X}, \overline{X})]$$

with initial condition $[\underline{X_0}\overline{X_0}]$

Above system can be presented in form of system of ordinary differential equations [15,23,24]

$$\left.\begin{aligned}\dot{\underline{X}}(t) &= \underline{f}(t, \underline{X}, \overline{X});\ \underline{X}(0) = \underline{X_0}\\ \dot{\overline{X}}(t) &= \overline{f}(t, \underline{X}, \overline{X}); \overline{X}(0) = \overline{X_0}\end{aligned}\right\} \quad (7)$$

Now by the proposed scheme,

$$\left.\begin{aligned}\underline{X}_{k+1} &= \underline{X}_k + h\overline{f}(t, \underline{X}_k, \overline{X}_k); \underline{X}(0) = \underline{X_0}\\ \overline{X}_{k+1} &= \overline{X}_k + h\overline{f}(t, \underline{X}_k, \overline{X}_k); \overline{X}(0) = \overline{X_0}\end{aligned}\right\} \quad (8)$$

**Lemma 1:** (Refer [8]), Letthe sequence of numbers $\{e_n\}_{n=0}^{N}$ satisfy

$$|e_{n+1}| \le A|e_n| + B, \qquad 0 \le n \le N - 1$$

For the given positive constants $A$ and $B$. Then

$$|e_n| \le A^n|e_0| + B\frac{A^n - 1}{A - 1},$$

**Theorem (2):** $\underline{f}(t, \underline{X}, \overline{X})$, $\overline{f}(t, \underline{X}, \overline{X}) \in E^n$, their partial derivatives $\frac{\partial \underline{f}(t,\underline{X},\overline{X})}{\partial \underline{X}}$ and $\frac{\partial \overline{f}(t,\underline{X},\overline{X})}{\partial \overline{X}}$ bounded and Lipschitz in $E^n$. Let $\underline{X}$, $\overline{X}$ both are exact solution of system (7) and $\underline{Y}$, $\overline{Y}$ both are numerical solution of system (8) then numerical solution converges to the exact solution uniformly.

**Proof:** Define error in $nth$ term,

$$e_n = \underline{X}(t_n) - \underline{Y}(t_n)$$
$$e_{n+1} = \underline{X}(t_{n+1}) - \underline{Y}(t_{n+1})$$

By using Taylor's expansion and neglecting higher terms,

$$e_{n+1} = [\underline{X}(t_n) + h\underline{X}'(t_n) + \frac{h^2}{2}\underline{X}''(t_n + \theta h) - \left(\underline{Y}(t_n) + h\underline{f}\left(t, \underline{Y}(t_n), \overline{Y}(t_n)\right)\right)]$$

$$e_{n+1} = \left(\underline{X}(t_n) - \underline{Y}(t_n)\right) + h(\underline{X}'(t_n) - \underline{f}\left(t, \underline{Y}(t_n), \overline{Y}(t_n)\right) + \frac{h^2}{2}\underline{X}''(t_n + \theta h)$$

By taking modulus and using Lipschitz condition,

$$|e_{n+1}| \le |e_n|(1 + hL) + N\frac{h^2}{2}$$

Computing in backward manner we get at $k^{th}$ step, $k < n$

$$|e_n| \le |e_{n-1}| + hL|e_{n-1}| + N\frac{h^2}{2}$$
$$|e_{k-1}| \le \left[|e_{k-2}|(1 + hL) + N\frac{h^2}{2}\right](1 + hL) + N\frac{h^2}{2}$$
$$|e_1| \le |e_1| + hL|e_1| + N\frac{h^2}{2}$$

Applying lemma (1) we get,

$$|e_n| \le |e_0|(1 + hL)^n + N\frac{h^2}{2}\frac{((1 + hL)^n - 1)}{hL}$$

Let $A = 1 + hL$ and $B = N\frac{h^2}{2}$

$$|e_n| \le |e_0|(A)^n + B\frac{((A)^n - 1)}{A - 1}$$
$$|e_n| \le |e_0|(1 + hL)^n + N\frac{h^2}{2}\frac{((1 + hL)^n - 1)}{hL}$$

Suppose $e_0 = 0$ and $(1 + hL) \le (\exp)^{hL}$

$$|e_n| \le N\frac{h^2}{2}\frac{((\exp)^{nhL} - 1)}{hL}$$
$$|e_n| \le N\frac{h^2}{2}\frac{((\exp)^{nhL} - 1)}{hL}$$

which is valid for $0 \le t_n = nh \le T \rightarrow$ stability, finally,

$$\le N\frac{h^2}{2}\frac{((\exp)^{TL} - 1)}{hL}$$

as $h \rightarrow 0$,

$$|e_n| \rightarrow 0$$
$$|e_n| \le N\frac{h^2}{2}\frac{((\exp)^{TL} - 1)}{hL} = N\frac{h}{2}\frac{((\exp)^{TL} - 1)}{L} = O(h)$$

This establishes the linear convergence of proposed the iterative method.

Similarly, we can show linear convergence for $\overline{f}(t, \underline{X}, \overline{X})$.

## 4. NUMERICAL EXAMPLE

In this section, two examples to illustrate the use of proposed scheme from environmental science and biological science.

**Water Pollution by Industrial Wastage**

Contamination of lakes and rivers is common problem. The problem of predicting the impact of a toxic waste discharge into lakes and rivers is important for environmental sciences. The mathematical model for such a dynamical system is given as the system of ordinary linear differential equations, while modelling, exact estimation of quantity of toxic in discharge and fluid flow may not be possible so the considering the involved parameters represented by fuzzy numbers gives better realization.

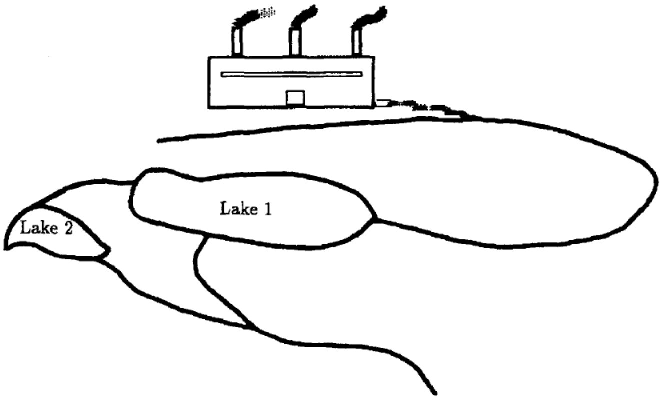


**Figure 1** Lake and factory model

This dynamical system is taken from [31], we consider two lakes as shown in Figure 1 above, with following assumptions:

The toxic material leaks into the river at constant rate for one day. We assume that the toxic material mixes quickly and thoroughly with lake waters as it enters each lake. The toxic discharge will also have an impact on the second lake. Let water of lake 1 barely trickles into lake 2, suggests contamination of lake 2 should be minimal. Final differential equations are formed as below,

$$\frac{d\tau_1}{dt} = r(t) - \frac{i_1}{v_1}\tau_1$$
$$\frac{d\tau_2}{dt} = \frac{i_2}{v_1}\tau_1 - \frac{o_2}{v_2}\tau_2$$

Putting numerical value,

$$\frac{d\tau_1}{dt} = \widetilde{100} - \widetilde{0.5}\tau_1$$
$$\frac{d\tau_2}{dt} = \widetilde{0.25}\,\tau_1 - \tilde{1}\tau_2$$

with initial conditions:

$$\tau_1^0(0) = \widetilde{50} = (40,50,60) = (40 + 10\alpha, 60 - 10\alpha),$$
$$\tau_2^0(0) = \widetilde{10} = (5,10,15) = (5 + 5\alpha, 15 - 5\alpha),$$

and $h = 0.1$,

$$\widetilde{100} = (90,100,110) = (90 + 10\alpha, 110 - 10\alpha),$$
$$\widetilde{0.5} = (0.4,0.5,0.6) = (0.4 + 0.1\alpha, 0.6 - 0.1\alpha)$$
$$\widetilde{0.25} = (0.20, .25,30) = (0.20 + .05\alpha, .30 - .05\alpha),$$
$$\tilde{1} = (0.5,1,1.5) = (0.5 + 0.5\alpha, 1.5 - .5\alpha)$$

Applying the proposed scheme and we get,

$$\tau_1^1(t) = [-0.1\,\alpha^2 + 12.2\,\alpha + 45.4, -0.1\,\alpha^2 - 11.8\,\alpha + 69.4]$$

$$\tau_2^1(t) = [-0.2\alpha^2 + 6.9\alpha + 3.55, -0.2\,\alpha^2 - 6.1\,\alpha + 16.55]$$

The other iteration values at different times for $\tau_1$ and $\tau_2$ are as shown in Figure 2 and Figure 3 respectively.

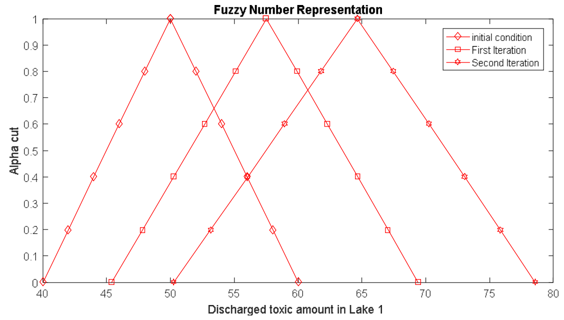


**Figure 2** Fuzzy number representation for discharged toxic in Lake 1

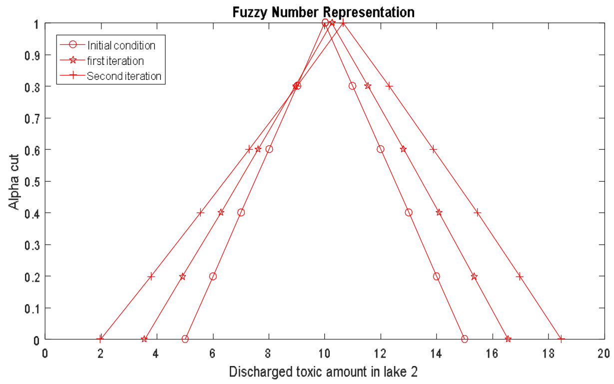


**Figure 3** Fuzzy number representation for discharged toxic in Lake 2

**Table 1**Number of iterations for amount of toxic discharge in lakes

| Time | $\tau_1$ (toxic discharge amount in lake1 as a fuzzy triplet) | Crisp solution for $\tau_1$ | $\tau_2$(toxic discharge amount in lake 2 as a fuzzy triplet) | Crisp solution for $\tau_2$ |
|---|---|---|---|---|
| t = 0 | (40, 50, 60) | 50 | (5, 10, 15) | 10 |
| t = 0.1 | (37.5, 57.5, 69.4) | 57.5 | (3.55, 10.25, 16.55) | 10.25 |
| t = 0.2 | (34.23, 64.625, 78.90) | 64.625 | (1.975, 10.6625, 18.4545) | 10.6625 |

We can see from the Table 1, core value of fuzzy solution matches with crisp solution by same numerical technique and support increases with time.

**Spread of Infectious Disease Model**

This application is taken from [32]. There is a total population of people is $N$ and certain contagious disease infecting the people. This population is divided into three parts, $x(t)$=those uninfected but may be infected, $y(t)$=those presently infected and may spread the disease, $z(t)$=already had the disease are dead, recovered and immune or cannot spread the disease.

$$\text{So,} N = x(t) + y(t) + z(t)$$

The rate of transfer from $x$ into $y$ is directly proportional to $xy$. So, $\dot{x}(t) = -kx(t)y(t)$

The rate of transfer into $y$ comes from $x$ and the rate of transfer out of $y$ goes to $z$ which is proportional to y, so, $\dot{y}(t) = kx(t)y(t) - cy(t)$ where $k$ and $c$ are positive constants and they are need to be estimated. There is no need of third differential equation because we may get $z$ by this relation,

$$N = x(t) + y(t) + z(t).$$
$$\dot{x} = -kxy$$
$$\dot{y} = kxy - cy$$
$$z(t) = N - (x + y)$$

with initial condition $x_0$,$y_0$ and $z_0 = 0, N = x_0 + y_0$

$k$ & $c$ depends on type of disease and season too so these parameters can be fuzzified.

Hence,

$$\tilde{k}^\alpha = (0.003,0.005,0.007) = (0.003 + 0.002\alpha, 0.007 - 0.002\alpha)$$

$$\tilde{c}^\alpha = (0.6,0.9,1.2) = (0.6 + 0.3\alpha, 1.2 - 0.3\alpha)$$

$$\tilde{x}^\alpha{}_0 = (920,950,980) = (920 + 30\alpha, 980 - 30\alpha)$$

$$\tilde{y}^\alpha{}_0 = (20,50,80) = (20 + 30\alpha, 80 - 30\alpha)$$ and $N = 1000$

Applying proposed numerical technique, we get first iteration which is given below,

$$[\underline{x_1}\overline{x_1}] = [0.18\alpha^3 - 6.99\alpha^2 + 67.94\alpha + 865.12, -0.18\alpha^3 - 5.91\alpha^2 - 42.14\alpha + 974.48]$$

$$[\underline{y_1}\overline{y_1}] = [0.18\alpha^3 + 5.01\alpha^2 + 48.14\alpha + 15.92, -0.18\alpha^3 + 6.90\alpha^2 - 70.34\alpha + 133.68]$$

For next iterations, Figures 4 & 5 are given below,

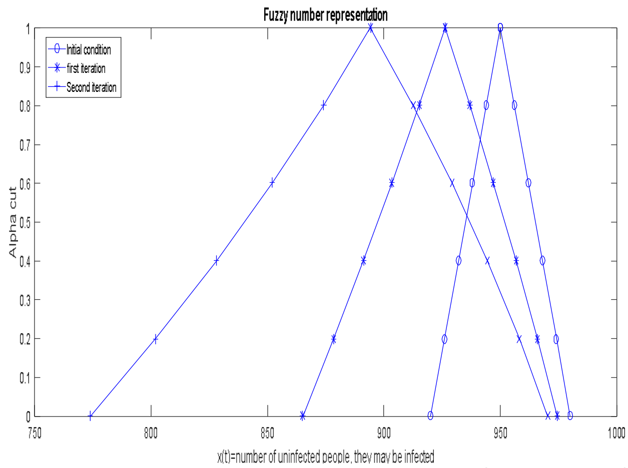


**Figure 4**Fuzzy number representation for number of uninfected people

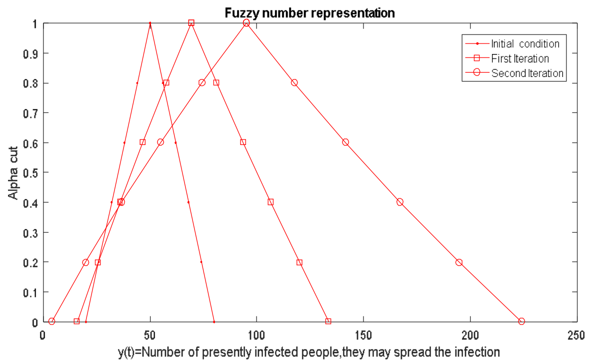


**Figure 5**Fuzzy number representation for number of infected people

**Table 2** Number of iterations for uninfected and infected people

| **Time** | ***x*** **(Uninfected people but may be infected as a fuzzy triplet)** | **Crisp solution for** ***x*** | ***y*** **(Infected people and may spread the disease as a fuzzy triplet)** | **Crisp solution for** ***y*** |
|---|---|---|---|---|
| t = 0 | (920, 950, 980) | 950 | (20, 50, 80) | 50 |
| t = 0.1 | (865.12, 926.25, 974.48) | 926.25 | (15.92, 69.25, 133.68) | 69.25 |
| t = 0.2 | (773.93, 894.178, 970.348) | 894.178 | (4.01, 95.08, 223.91) | 95.08 |

From table 2, core value of fuzzy solution matches with crisp solution and support increases as time increases.The value of $z(t)$ can be calculated in each iteration by this relation,

$$z(t) = 1000 - x(t) - y(t).$$

**Table 3**Number of iterations for already dead, recovered and immune people

| **Time** | ***z*** **(already had the disease are dead, recovered and immune or cannot spread the disease)** |
|---|---|
| t=0.1 | 4.5 |
| t=0.2 | 10.742 |

## 5.CONCLUSION

We have proposed numerical technique to solve fully fuzzy dynamical systems. Established convergence for the proposed scheme and applied it to two real world problems and compared the results with crisp solution. This technique can also be applied on fuzzy dynamical system under generalized differentiability, however, we may not get unique solution in some cases.

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