Mathematical Modelling, Analysis and Applications of Fuzzy Systems

Thesis Submitted to

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Executive Summary of Thesis

(Title: Mathematical Modelling, Analysis and Applications of Fuzzy Systems)

In this research work, we began our work, with a system of fuzzy differential equations involving fuzzy initial conditions and then extended it to a fully fuzzy system in which both the parameters and the initial condition are considered fuzzy. Such a system in general form is given as follows,

$$\dot{\tilde{X}} = \tilde{f}(t, \tilde{X});\ \tilde{X}(0) = \tilde{X}_0$$

We have focused on the techniques for the solution of the system of linear and nonlinear fuzzy differential equations. The summary of work consists following sections, introduction, analytical technique, numerical techniques, transformation technique, semi-analytical techniques and lastly an application of solar energy under fuzzy environment.

1. Introduction

Modelling of various physical phenomenon under the influence of physical laws are represented in form of differential equations. In these models, various parameters involved are obtained from experiments or observations. These values such obtained may have certain measurement errors or imprecision. These kinds of error in turn leads to the imprecision in the modelling of physical problem. To model such phenomenon with imprecise parameters, using fuzzy sets give more realistic realizations. Such general dynamical models involving fuzzy parameters are given as,

$$\dot{\tilde{X}} = \tilde{f}\,(t, \tilde{X});\ \tilde{X}(0) = \tilde{X}_0$$

The above equation explains real world problem with imprecision in more realistic way than crisp counterpart.

The concept of fuzzy set was first discussed in seminal paper by Zadeh [1] in 1965. The concept of fuzzy derivative was first introduced by Chang and Zadeh [2] in 1972. Dubois and Prade [3] defined derivative based on extension principle in 1982. Puri and Ralescue [4] introduced H-derivative of fuzzy valued function based on Hukuhara difference [5]. Kaleva [6] and Seikkala [7], first simultaneously solved the fuzzy initial value problem with fuzzy initial condition.

Kandel and Byatt [8] used this theory for the applications of fuzzy dynamical system. The basic and most popular approach to solve fuzzy differential equation (FDE) is Hukuhara differentiability which is based on H-difference. The drawback in using Hukuhara derivative is that solution does not remain fuzzy as time increases. To overcome this situation, Generalized Hukuhara derivative [9] is proposed, which is very popular among other fuzzy derivatives. When generalized Hukuhara derivative is used to solve the fuzzy differential equation, the solution is obtained as possible set of solutions from which one needs to choose the solution which best satisfies the problem. This is one of the disadvantages of generalized Hukuhara derivative. To overcome this, we have proposed Modified Hukuhra derivative which gives unique fuzzy solution.

Initially, most of the authors have worked on scalar differential equation with fuzzy initial condition using different techniques like analytical, numerical, transformation and semi-analytical. The numerical method for solving FDE is introduced by M. Ma, Friedman and Kandel refer [10]. They used classical Euler method which is followed by complete error analysis. Javed Shokri in [11] solved FDE by modified Euler method. S. Abbasbandy and T. Allahviranloo refer [12] solved FDE by Taylor's method of order p and proved the order of convergence is $O(h^p)$ which is better than the order of convergence of Euler's method. FDE is solved by some other numerical methods like Runge-Kutta and Corrector-Predictor method as in [13], [14] respectively. T. Jayakumar, D. Mahesh Kumar and Rangrajan [15] presented solution of FDE by 5th order of Runge Kutta method. Numerical solution of second order FDE is given by N. Parandin in [16] by using Runge Kutta method. Solution of FDE under generalized differentiability by Improved Euler method under strongly generalized differentiability is given by K. Kangrajan and R. Suresh refer [17], here only the fuzzy initial condition is considered.

Allahviranloo and Ahmadi [18] proposed fuzzy Laplace transforms (FLT) for solving first order differential equations under generalized H-differentiability without giving the existence condition. S. Salahsour and T. Allahviranloo [19] described the existence condition for Laplace transform and its inverse. Many other authors in [20], [21] used FLT to solve FDE. They solve second order fuzzy differential equation and n^{th} order fuzzy differential equation with fuzzy initial condition under Generalized Hukuhara derivative.

Some authors have used semi-analytical techniques to solve nonlinear differential equation like Homotopy perturbation method (HPM), Adomian Decomposition method (ADM). Variational Iteration method (VIM) etc. as in [22], [23], [24], [25], [26], [27]. These techniques are also used in solving fuzzy nonlinear differential equations. Other methods for solving FDE and important results pertaining to continuity, existence of solution and various applications are given in [28-58].

Our work comprises of solution of system of linear fuzzy differential equation with fuzzy parameters as well as fuzzy initial condition that is fully fuzzy linear dynamical system, followed by some work on nonlinear fully fuzzy differential equation. To solve fully fuzzy dynamical systems, we have used different techniques like analytical, numerical, transformation and semi-analytical techniques. We have also proposed and proved various existence and uniqueness results.

In Next section, we solved Prey-Predator model with fuzzy initial conditions using analytical technique.

2. Analytical Technique (Fuzzy Prey-Predator Model)

We have considered following Prey-Predator model,

$$\dot{x} = ax - bxy \qquad (1)$$
$$\dot{y} = -cy + dxy$$

with $x(0) = \tilde{x}_0$ and $y(0) = \tilde{y}_0$.

The matrix form of equation (1.1) is,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -bxy \\ dxy \end{bmatrix}; \begin{bmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{bmatrix}$$

To solve the equation (1), we linearize it around the equilibrium points and we have obtained closed form solution by using eigen value and eigen vector method.

Now, the linearized form of equation (1) around the equilibrium point is,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{-ac}{d} \\ \frac{-ac}{b} \end{bmatrix}; \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad (2)$$

i.e. $\dot{X}(t) = PX + Q;\ \tilde{X}_0$

where, $P = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}, Q = \begin{bmatrix} \frac{-ac}{d} \\ \frac{-ac}{b} \end{bmatrix}$ and $\tilde{X}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

For equation (2), fundamental matrix $\psi(t)$ is found to obtain state transition matrix $\phi(t, 0)$. Then the solution of (2) exists and is given as,

$$\tilde{X}(t) = \phi(t, 0)\tilde{X}_0 + \int_0^t \phi(t, \tau)Qdt$$

After this work, we solved system of differential equations involving fuzzy parameters as well as fuzzy initial condition using numerical techniques.

3. Numerical Techniques

Using two numerical techniques, we solved the fully fuzzy system with fuzzy parameters and fuzzy initial conditions. First one numerical technique is based on discretization of Hukuhara derivative and the other one is Improved Euler Method.

We have considered following system of differential equations,

$$\dot{\tilde{X}} = \tilde{f}(t, \tilde{X}) \tag{3}$$

with initial condition $\tilde{X}(0) = \tilde{X}_0$.

where, $\tilde{f} : I \times E^n \to E^n$ and

$$\tilde{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_n \end{bmatrix}, \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}, \tilde{X}_0 = \begin{bmatrix} \tilde{x}_{10} \\ \tilde{x}_{20} \\ \vdots \\ \tilde{x}_{n0} \end{bmatrix}$$

Here each $\tilde{f}_i$ is Hukuhara differentiable, $\forall\, i = 1,2,3 \ldots ..$

$\tilde{f}$ can be linear or nonlinear. When we consider $\tilde{f}$ or

$$\tilde{f}(t, \tilde{X}) = \tilde{A} \otimes \tilde{X} \oplus \tilde{B};\ \tilde{X}_0. \tag{4}$$

where, $\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \ldots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \ldots & \tilde{a}_{nn} \end{bmatrix}$ is $n \times n$ fuzzy matrix and $\tilde{B} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$ is $n \times 1$ column vector.

The parametric numerical scheme for equation (3), with $\tilde{f}$ as in equation (4) is given by,

$$\begin{bmatrix} \underline{X}_{k+1} \\ \overline{X}_{k+1} \end{bmatrix} = \begin{bmatrix} \underline{A} & 0 \\ 0 & \overline{A} \end{bmatrix} \begin{bmatrix} \underline{X}_k \\ \overline{X}_k \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \overline{B} \end{bmatrix}; \tag{5}$$

where, $k = 0, 1, 2, \ldots \ldots \ldots$

$$\underline{X}(0) = \underline{X}_0 \text{ and } \overline{X}(0) = \overline{X}_0$$

For equation (3), results for existence of solution and convergence analysis are proposed and proved.

Further for a system such as equation (3), a numerical scheme on the line of Improved Euler Method is proposed and proved along with convergence analysis. We have solved infectious disease model under fuzzy environment using these numerical techniques.

In next section, we have solved fully fuzzy dynamical system by Transformation technique i.e., Fuzzy Laplace Transform.

4. Transformation Technique

We used Fuzzy Laplace Transformation (FLT) technique, to solve linear homogeneous as well as non-homogeneous system. For solution of both the systems, we used diagonalization concept then applied fuzzy Laplace transform.

If the system is homogeneous, i.e., $\tilde{f}\left(t,\tilde{X}\right) = \tilde{A} \otimes \tilde{X}; \tilde{X}_0$.

The solution of above system is given as,

$$\underline{X} = \underline{X}_0 e^{\underline{D}t}, \overline{X} = \overline{X}_0 e^{\overline{D}t}.$$

where, $\underline{D}$ and $\overline{D}$ both are diagonal matrices, obtained from $\underline{A}$ and $\overline{A}$.

For equivalent non homogeneous linear systems i.e., $\tilde{f}\left(t,\tilde{X}\right) = \tilde{A} \otimes \tilde{X} \oplus \tilde{B}\ ; \tilde{X}_0$.

The solution of above system is given as,

$$\underline{X} = \underline{P}\, L^{-1}\{\left[sI - \underline{D}\,\right]^{-1}[\underline{P}^{-1}\, \underline{X}(0) + \underline{P}^{-1}L\left[\underline{B}\right]]\}$$

$$\overline{X} = \overline{P}L^{-1}\{\left[sI - \overline{D}\right]^{-1}[\overline{P}^{\,-1}\, \overline{X}(0) + \overline{P}^{\,-1}L[\overline{B}\,]]\}$$

where, $\underline{P}$ and $\overline{P}$ are orthogonal matrices. The result for the existence of solution using fuzzy Laplace Transform also proved. We have solved arm race model in fuzzy environment using proposed method.

For the rigorous development of solution, to fully fuzzy dynamical system using Laplace Transform technique, we afresh defined Modified Generalized Hukuhara derivative (mgh-derivative for the first time. Our proposed derivative has advantage that it gives unique solution of fuzzy dynamical system automatically without the author to select the suitable one as in case of Generalized Hukuhara derivative shown in [59]. The proposed Modified Generalized Hukuhare derivative is defined as follows.

4.1 Modified Generalized Hukuhare derivative (mgh-derivative)

Let $f: I \to E^n$ is said to be modified generalized Hukuhara differentiable at t_0 if there exist an element, $\dot{\tilde{f}}(t_0) \in E^n$, such that for all $h > 0$ sufficiently small, $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ should exist and the limits,

$$\lim_{h\to 0+} \frac{\tilde{f}(t_0+h)\ominus \tilde{f}(t_0)}{h} = \lim_{h\to 0-} \frac{\tilde{f}(t_0)\ominus \tilde{f}(t_0-h)}{h} = \dot{\tilde{f}}(t_0)$$

Its equivalent parametric form is given below,

$$\lim_{h\to 0+} \frac{{}^{\alpha}\tilde{f}(t_0+h) - {}^{\alpha}\tilde{f}(t_0)}{h}$$

$$= \left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h) - \underline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0+h) - \overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h) - \overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h) - \underline{f}(t_0)\right)}{h}\right\},\right.$$

$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h) - \underline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0+h) - \overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h) - \overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h) - \underline{f}(t_0)\right)}{h}\right\}\right]$$

And,

$$\lim_{h\to 0-} \frac{{}^{\alpha}\tilde{f}(t_0) - {}^{\alpha}\tilde{f}(t_0-h)}{h} =$$

$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0) - \underline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0) - \overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0) - \overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0) - \underline{f}(t_0-h)\right)}{h}\right\},\right.$$

$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0) - \underline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0) - \overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0) - \overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0) - \underline{f}(t_0-h)\right)}{h}\right\}\right]$$

Along with the proposed mgh-derivative in [65], various results pertaining to the existence of fuzzy Laplace Transform for the function, derivative for function and fuzzy convolution theorem are given as listed below.

4.2 Fuzzy Laplace Transform under mgh-derivative

Consider fuzzy valued function ${}^{\alpha}\tilde{f}(t) = [\underline{f}(t), \overline{f}(t)]$in parametric form, is bounded and piecewise continuous on the interval $[0, \infty)$ and suppose that $\tilde{f}(t) \otimes e^{-st}$ is improper fuzzy Riemann integrable [48], then fuzzy Laplace Transform of $\tilde{f}(t)$ defined as,

$$\tilde{F}(s) = \mathcal{L}\left(\tilde{f}(t)\right) = \int_0^{\infty} e^{-st} \otimes \tilde{f}(t)dt$$

$$\tilde{F}(s) = \mathcal{L}\left(\tilde{f}(t)\right) = \lim_{t\to\infty}\int_0^t e^{-st} \otimes \tilde{f}(t)dt$$

Taking alpha cut on both sides,

$$\mathcal{L}\left({}^{\alpha}\tilde{f}(t)\right) = \lim_{t\to\infty}\int_0^t e^{-st} \otimes \left[\underline{f}(t), \overline{f}(t)\right] dt$$

$$\mathcal{L}\left(\left[\underline{f}(t), \overline{f}(t)\right]\right) = \lim_{t\to\infty}\left[\int_0^t e^{-s}\, \underline{f}(t)\, dt, \int_0^t e^{-st}\overline{f}(t)dt\right]$$

4.3 Fuzzy Laplace of Derivative

If ${}^{\alpha}\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ be continuous fuzzy valued function, $\lim_{t\to\infty} e^{-st}\underline{f}(t) \to 0$ and $\lim_{t\to\infty} e^{-st}\overline{f}(t) \to 0$ for large value of s and $\dot{\tilde{f}}(t)$ is piecewise continuous then $\mathcal{L}\left(\dot{\tilde{f}}(t)\right)$ exist, and is given by,

$$\mathcal{L}\left(\dot{\tilde{f}}(t)\right) = s\mathcal{L}\left(\tilde{f}(t)\right) \ominus \tilde{f}_0$$

4.4 Fuzzy convolution Theorem

Let $\tilde{f}(s)$ and $\tilde{g}(s)$ denote the fuzzy inverse Laplace transforms of $\tilde{f}(t)$ and $\tilde{g}(t)$ respectively. Then the product given by $\tilde{f}(s)\tilde{g}(s)$ is the fuzzy inverse Laplace transform of the convolution of $\tilde{f}$ and $\tilde{g}$, is given by,

$$\mathcal{L}\left(\tilde{f}(t) * \tilde{g}(t)\right) = \tilde{f}(s) * \tilde{g}(s)$$

After that, we have solved fully fuzzy dynamical system by semi-analytical technique which is given below.

5. Semi-Analytical Technique

Following semi-analytical techniques are used.

- Fuzzy Adomian Decomposition Method in parametric form (FADMP)
- Fuzzy Adomian Decomposition Method (FADM)

5.1 Fuzzy Adomian Decomposition Method in Parametric form (FADMP)

This topic investigates solution of fully fuzzy system by FADM in parametric form.

In this method, we decompose the nonlinear part of fully fuzzy system into the series of fuzzy Adomian polynomials and obtain the solution of system in series form. We have also discussed the convergence of FADM in parametric form.

Consider a nonlinear fuzzy differential equation,

$$\mathbb{L}\tilde{u} \oplus \mathbb{N}\tilde{u} \oplus \mathbb{R}\tilde{u} = \tilde{g}$$

where, $\mathbb{L}$ is linear (modified generalized Hukuhara) differentiable operator, $\mathbb{N}$ is nonlinear operator, $\mathbb{R}$ is the operator of less order than that of $\mathbb{L}$ and $\tilde{g}$ is source term. Then applying, $\mathbb{L}^{-1}$ i.e., fuzzy integration operator on both sides we get,

$$\tilde{u} = \mathbb{L}^{-1}\tilde{g} \ominus \mathbb{L}^{-1}(N\tilde{u}) \oplus \mathbb{L}^{-1}(R\tilde{u})$$

Parametric form of above equation is as follows,

$$\begin{aligned}[\underline{u}\ \overline{u}] = \mathbb{L}^{-1}\left[\min\left(\underline{g}, \overline{g}\right), \max\left(\underline{g}, \overline{g}\right)\right] \ominus L^{-1}[\min(\mathbb{N}\underline{u}, \mathbb{N}\overline{u}), \max(\mathbb{N}\underline{u}, \mathbb{N}\overline{u})] \\ \oplus L^{-1}[\min(\mathbb{R}\underline{u}, \mathbb{R}\overline{u}), \max(\mathbb{R}\underline{u}, \mathbb{R}\overline{u})]\end{aligned} \tag{6}$$

Now, we decompose the nonlinear term, $[\min(\mathbb{N}\underline{u}, \mathbb{N}\overline{u}), \max(\mathbb{N}\underline{u}, \mathbb{N}\overline{u})]$ into the series of Adomian polynomials $\underline{A}_1, \underline{A}_2, \underline{A}_3, \ldots$ and $\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots$ with the general polynomial as,

$$\underline{A}_n = \min\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\mathbb{N}\Big(\sum_{k=0}^{\infty}\underline{u}_k\lambda^k\Big), \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathbb{N}\Big(\sum_{k=0}^{\infty}\overline{u}_k\lambda^k\Big)\right]$$

$$\overline{A}_n = \max\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\mathbb{N}\Big(\sum_{k=0}^{\infty}\underline{u}_k\lambda^k\Big), \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathbb{N}\Big(\sum_{k=0}^{\infty}\overline{u}_k\lambda^k\Big)\right]$$

So, substituting these Adomian polynomials in equation (1.8), the desirable solution obtained as,

$$[\underline{u}, \overline{u}] = [\underline{u}_0, \overline{u}_0] + [\underline{u}_1, \overline{u}_1] + [\underline{u}_2, \overline{u}_2] + [\underline{u}_3, \overline{u}_3] + \cdots$$

Till now the solution technique always required the fuzzy systems to be converted to the crisp by considering its equivalent parametric form. This motivated us to develop the complete fuzzy technique so that we can solve fuzzy systems directly.

5.2 Fuzzy Adomian Decomposition Method (FADM)

To propose FADM in complete fuzzy environment, we have redefined the fuzzy function and extend the Modified Generalized Hukuhara derivative. We have developed few results required in fuzzy calculus like fuzzy power series, its convergence and Taylor's theorem under Modified generalized Hukuhara derivative.

The above-mentioned results to develop FADM in complete fuzzy environment, are as follows.

5.3 Fuzzy Function

Consider a fuzzy valued scalar function with fuzzy argument $\tilde{f}: E \rightarrow E$. Its parametric form can be defined as follows,

$$ {}^{\alpha}\tilde{f}(\tilde{x}) = [\underline{f}(\tilde{x}), \overline{f}(\tilde{x})], \text{ where, } \underline{f}(\tilde{x}) = \min \tilde{f}(\tilde{x}) \text{ and } \overline{f}(\tilde{x}) = \max \tilde{f}(\tilde{x}) $$

Further, $ {}^{\alpha}\tilde{f}(\tilde{x}) = \left[\underline{\underline{f}}(\underline{x}, \overline{x}), \overline{\overline{f}}(\underline{x}, \overline{x})\right]$

where, $\underline{\underline{f}}(\underline{x}, \overline{x}) = \min\left(\underline{f}(\underline{x}, \overline{x}), \overline{f}(\underline{x}, \overline{x})\right)$, $\overline{\overline{f}}(\underline{x}, \overline{x}) = \max\left(\underline{f}(\underline{x}, \overline{x}), \overline{f}(\underline{x}, \overline{x})\right)$

This definition can be extended to n dimensional fuzzy valued function, by now considering $\tilde{f}$ as, $\tilde{f}: E^n \rightarrow E^n$ where, $\tilde{f} = \{\tilde{f}_1(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n), \tilde{f}_2(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n), \dots \tilde{f}_n(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n)\}$.

On the basis of this definition of fuzzy function, we redefined Modified Generalized Hukuhara derivative in new manner as follows.

5.4 Definition

A function $\tilde{f}: E \rightarrow E$ is said to be Modified Generalized Hukuhara differentiable for an element $\dot{\tilde{f}}(\tilde{x}_0) \in E$, such that for small $h > 0$, $\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)$, $\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)$ should exist and

$$ \lim_{h \to 0+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)}{h} = \lim_{h \to 0-} \frac{\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)}{h} = \dot{\tilde{f}}(\tilde{x}_0) $$

The equivalent parametric form for the first limit is given as,

$$ \lim_{h \to 0+} \frac{{}^{\alpha}\tilde{f}(\tilde{x}_0 + h) - {}^{\alpha}\tilde{f}(\tilde{x}_0)}{h} $$

$$=\lim_{h\to 0}\frac{1}{h}\left[\min\left(\underline{f}(\tilde{x}_0+h)-\underline{f}(\tilde{x}_0),\overline{f}(\tilde{x}_0+h)-\overline{f}(\tilde{x}_0)\right),\max\left(\underline{f}(\tilde{x}_0+h)-\underline{f}(\tilde{x}_0),\overline{f}(\tilde{x}_0+h)-\overline{f}(\tilde{x}_0)\right)\right]$$

Further,

$$=\lim_{h\to 0}\frac{1}{h}\left[\min\left(\underline{\underline{f}}\left(\underline{x}_0+h,\overline{x}_0+h\right)-\underline{\underline{f}}\left(\underline{x}_0,\overline{x}_0\right),\overline{\overline{f}}\left(\underline{x}_0+h,\overline{x}_0+h\right)-\overline{\overline{f}}\left(\underline{x}_0,\overline{x}_0\right)\right),\max\left(\underline{\underline{f}}\left(\underline{x}_0+h,\overline{x}_0+h\right)-\underline{\underline{f}}\left(\underline{x}_0,\overline{x}_0\right),\overline{\overline{f}}\left(\underline{x}_0+h,\overline{x}_0+h\right)-\overline{\overline{f}}\left(\underline{x}_0,\overline{x}_0\right)\right)\right]$$

Similarly, the second limit in equation (5.12) can be given as,

$$\lim_{h\to 0+}\frac{{}^{\alpha}\tilde{f}(\tilde{x}_0)-{}^{\alpha}\tilde{f}(\tilde{x}_0-\mathrm{h})}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\min\left(\underline{f}(\tilde{x}_0)-\underline{f}(\tilde{x}_0-h),\overline{f}(\tilde{x}_0)-\overline{f}(\tilde{x}_0-h)\right),\max\left(\underline{f}(\tilde{x}_0)-\underline{f}(\tilde{x}_0-h),\overline{f}(\tilde{x}_0)-\overline{f}(\tilde{x}_0-h)\right)\right]$$

Further,

$$=\lim_{h\to 0}\frac{1}{h}\left[\min\left(\underline{\underline{f}}\left(\underline{x}_0,\overline{x}_0\right)-\underline{\underline{f}}\left(\underline{x}_0-h,\overline{x}_0-h\right),\overline{\overline{f}}\left(\underline{x}_0,\overline{x}_0\right)-\overline{\overline{f}}\left(\underline{x}_0-h,\overline{x}_0-h\right)\right),\max\left(\underline{\underline{f}}\left(\underline{x}_0,\overline{x}_0\right)-\underline{\underline{f}}\left(\underline{x}_0-h,\overline{x}_0-h\right),\overline{\overline{f}}\left(\underline{x}_0,\overline{x}_0\right)-\overline{\overline{f}}\left(\underline{x}_0-h,\overline{x}_0-h\right)\right)\right]$$

5.5 Fuzzy Taylor's Theorem:

If a fuzzy valued function, $\tilde{f}(x): E^n \to E^n$ is, n times modified generalized Hukuhara differentiable, then fuzzy Taylor's expansion is given as,

$$\tilde{f}(\tilde{x})=\tilde{f}(\tilde{x}_0)\oplus\dot{\tilde{f}}(\tilde{x}_0)\otimes(\tilde{x}\ominus\tilde{x}_0)\oplus\frac{\ddot{\tilde{f}}(\tilde{x}_0)}{2!}\otimes(\tilde{x}\ominus\tilde{x}_0)^2\oplus\frac{\dddot{\tilde{f}}(\tilde{x}_0)}{3!}\otimes(\tilde{x}\ominus\tilde{x}_0)^3\oplus\frac{\tilde{f}^4(\tilde{x}_0)}{4!}\otimes(\tilde{x}\ominus\tilde{x}_0)^4\oplus\frac{\tilde{f}^5(\tilde{x}_0)}{5!}\otimes(\tilde{x}\ominus\tilde{x}_0)^5\oplus\ldots\ldots\oplus\frac{\tilde{f}^n(\tilde{x}_0)}{n!}\otimes(\tilde{x}\ominus\tilde{x}_0)^n+\cdots$$

5.6 Fuzzy Power series and its radius of convergence:

A power series of fuzzy valued function around the point $\tilde{x}_0$ can be given as, $\sum\tilde{a}_n\otimes(\tilde{x}\ominus\tilde{x}_0)^n=\tilde{a}_0\oplus\tilde{a}_1\otimes(\tilde{x}\ominus\tilde{x}_0)\oplus\tilde{a}_2\otimes(\tilde{x}\ominus\tilde{x}_0)^2\oplus\ldots$ where, $\tilde{a}_n$ are any fuzzy coefficients and n is positive integer. The radius of convergence for this power series is given by,

$$\tilde{R} = \lim_{n\to\infty} \left|\frac{\tilde{a}_n}{\tilde{a}_{n+1}}\right|$$

The above-mentioned results are used for FADM technique as follows.

5.7 FADM in Fuzzy Environment

Consider a nonlinear fuzzy differential equation,

$$\mathbb{L}\tilde{u} \oplus \mathbb{N}\tilde{u} \oplus \mathbb{R}\tilde{u} = \tilde{g}$$

where, $\mathbb{L}$ is linear (modified generalized Hukuhara) differentiable operator, $\mathbb{N}$ is nonlinear operator, $\mathbb{R}$ is the operator of less order than that of $\mathbb{L}$ and $\tilde{g}$ is source term.

$$\tilde{u} = L^{-1}\tilde{g} \ominus L^{-1}(\mathbb{N}\tilde{u} \oplus \mathbb{R}\tilde{u})$$

This method gives that unknown fuzzy function can be expressed as series of, $\tilde{u} = \sum_{n=0}^{\infty} \tilde{u}_n$ The method defines nonlinear fuzzy term by Adomian polynomials, and it is defined as,

$$\mathbb{N}\tilde{u} = \sum_{n=0}^{\infty} \tilde{A}_n\,(\tilde{u}_0, \tilde{u}_1, \dots \tilde{u}_n)$$

where, $$\tilde{A}_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\mathbb{N}\sum_{k=0}^{\infty} \tilde{u}_k \lambda^k\right)$$

We have also proved convergence of fuzzy Adomian Decomposition method.

The advantage of this technique is that we solve a problem completely in fuzzy environment instead of bringing it to the crisp for solving and going back in fuzzy as done in conventionally.

We have solved one application based on solar energy which is given as follows.

6. Application: Mathematical Modelling of Air Heating Solar Collectors with Fuzzy Parameters

Problem of solar collector to convert solar radiation into energy, which can be used for electricity generation, blow drying process, heating of water for many industrial applications, is considered. Various physical factors play vital role in mathematical formulation [68]. The ambient temperature depends on weather and also varies from morning to evening, so taking range of such value is more realistic than the specific value of the same. Similarly, rate of mass of air flow also varies. Due to this reason, we have proposed the mathematical model of solar air collector involving fuzzy initial temperature and fuzzy rate of mass of air flow. Thus, the fuzzy model [67] is proposed,

$$\left(\left(\frac{\dot{\tilde{m}}}{W}\right)\otimes a\otimes\left(\frac{d\tilde{T}}{dx}\right)\right)\oplus\left(\left(\frac{\dot{\tilde{m}}}{W}\right)\otimes b\otimes\left(\frac{d\tilde{T}}{dx}\right)\otimes\tilde{T}(x)\right)\oplus\left(\tilde{F}\otimes U_l\otimes\tilde{T}(x)\right)$$
$$=\tilde{F}\otimes(S+T_aU_l)\ ;\quad \tilde{T}(0)$$

Above model is solved by FADMP techniques.

In next section, the research methodology of the research work is given as follows.

7. Research Methodology

We have used following methods to solve fuzzy dynamical system in this research work.

- We have used Analytical technique, i.e., Eigen value and Eigen vector method. We solved fuzzy nonlinear differential equations using this method.
- We have solved fuzzy dynamical system using two numerical schemes, one is based on discretization of Hukuhara derivative and other in Improved Euler method.
- We have used Fuzzy Laplace Transform technique to solve fuzzy dynamical system.
- We have used semi-analytical technique Fuzzy Adomian Decomposition Method (FADM) in parametric and FADM in complete fuzzy environment.

In next section, the key findings of the research work are given as follows.

8. Key findings

- We have solved fully fuzzy dynamical system i.e., initial condition and parameters both are taken as fuzzy number by different techniques like Analytical, Numerical, Transformation and Semi-Analytical technique.

- We have proposed and proved convergence results for Numerical, Transformation and Semi-Analytical technique.

- We have developed new fuzzy derivative, Modified Generalized Hukuhara derivative. Under this derivative, as time increases, support remains bounded and unique.

- We have redefined Fuzzy Laplace transform (FLT) along with existence result, FLT of derivative, convolution theorem under mgh derivative.

- We have developed Fuzzy Calculus under this new derivative. We have proposed and proved Fuzzy Power series with its convergence and Fuzzy Taylor's theorem.

- We have given Fuzzy Adomian Decomposition Method in complete fuzzy environment. The advantage of this technique is that we solve a problem completely in fuzzy environment instead of bringing it to the crisp for solving and going back in fuzzy as done in conventionally.

- We solved an application of mathematical model of air heating solar collector with fuzzy parameters.

The following sections contain some suggestions and conclusion.

9. Suggestions

- The research scholars and academicians can take reference of this research work to learn complete solution techniques for solving fuzzy dynamical system.
- The research scholars and academicians can use our proposed fuzzy derivative for research.
- They can use our proposed Fuzzy Adomian Decomposition Method for solving nonlinear ODE and PDE in complete fuzzy environment.

10. Conclusion

In this research work, we have applied different techniques on fully fuzzy linear as well as nonlinear dynamical system and compared result at core. We have proposed new fuzzy derivative and used it for fuzzy Laplace transform. Further we have proposed a method and established results for solving nonlinear dynamical system namely Fuzzy Adomian Decomposition method. Advantage of this technique is that any fuzzy dynamical system can be solved completely in fuzzy environment. We have used it to solve real-life application i.e., air heating solar collectors with fuzzy parameters.

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