CHAPTER VIII

THE CLASS (N) OF OPERATORS

A new class of operators, called the class (N), was introduced by Vasily Istratescu[1] in a recent paper.

DEFINITION: An operator T is said to be of class (N) if $\|Tx\|^2 \leq \|T^2x\|$ for all unit vectors $x \in H$.

It is not difficult to see that a hyponormal operator is of class (N). For, if T is hyponormal, then $\|T^*x\| \leq \|Tx\|$ for all $x \in H$. Hence if $\|x\| = 1$, then $\|Tx\|^2 = (Tx , Tx) = (T^*Tx , x) \leq \|T^*Tx\| \leq \|T^2x\|$ which shows that T is of class (N). As consequence of a theorem to be proved presently, it will be seen that an operator may belong to the class (N) without being hyponormal.

The present chapter is devoted to the study of some properties of the operators of class (N). We begin by proving the following theorem:

THEOREM 8.1. *If T is of class (N), then T^{n+1} is also of class (N) for $n = 1, 2, \ldots$.*

To prove this theorem, we need the following result which we prove in the form of a lemma:

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1) Vasily Istratescu [11]

<u>LEMMA 8.1.</u> <u>If</u> T <u>is an operator of class</u> (N), <u>then for a unit vector</u> $x \in H$,

$$\|T^{2n+1}x\| \geq \|T^{2n}x\|.\|Tx\| \ , \ n = 1,2, \ldots \ .$$

Both the theorem and the lemma will be proved by induction.

<u>PROOF OF LEMMA 8.1:</u> Let x be a unit vector of H. Then we have

$$\|T^3x\| = \|T^2Tx\| = \|Tx\|.\|T^2 \frac{Tx}{\|Tx\|}\|$$

$$\geq \|Tx\|.\|T \frac{Tx}{\|Tx\|}\|^2$$

$$= \frac{\|Tx\|.\|T^2x\|^2}{\|Tx\|^2}$$

$$= \frac{\|T^2x\|^2}{\|Tx\|}$$

$$\geq \|T^2x\|.\|Tx\|$$

Thus the lemma is true for n = 1. Let us suppose that the lemma is true for n = k i.e. we have

$$\|T^{2k+1}x\| \geq \|T^{2k}x\|.\|Tx\| \qquad \ldots\ldots(8.1)$$

Now $\|T^{2(k+1)+1}x\| = \|T^{2k+3}x\| = \|T^2 \ T^{2k+1}x\| \geq \dfrac{\|T^{2k+2}x\|^2}{\|T^{2k+1}x\|}$

Again $\|T^{2k+2}x\| = \|T^2 T^{2k}x\| \geq \dfrac{\|T^{2k+1}x\|^2}{\|T^{2k}x\|}$.

Hence $\|T^{2(k+1)+1}x\| \geq \dfrac{\|T^{2k+2}x\| . \|T^{2k+1}x\|^2}{\|T^{2k+1}x\| . \|T^{2k}x\|}$

$$= \frac{\|T^{2k+2}x\| . \|T^{2k+1}x\|}{\|T^{2k}x\|}$$

$$\geq \|T^{2k+2}x\| . \|Tx\|$$

by (8.1) i.e. the inequality is true for $n = k+1$. Hence the lemma follows from induction.

<u>PROOF OF THEOREM 8.1</u>: For a unit vector $x \in H$ we have

$$\|T^4x\| = \|T^2T^2x\| \geq \frac{\|T^3x\|^2}{\|Tx\|} .$$

Again, $\|T^3x\| = \|T^2Tx\| \geq \dfrac{\|T^2x\|^2}{\|Tx\|}$.

Thus $\|T^4x\| \geq \dfrac{\|T^2x\|^4}{\|T^2x\| . \|Tx\|^2} = \dfrac{\|T^2x\|^3}{\|Tx\|^2}$

$$\geq \|T^2x\|^2$$

i.e. the theorem is true for $n = 1$. Equivalently, T^2 is also of class (N).

Let us suppose that the theorem is true for $n = k - 1$ i.e. we suppose that T^k is of class (N). Hence

$\|T^{2k}x\| \geq \|T^kx\|^2$ for every unit vector $x \in H$.

Now, $$\|T^{2(k+1)}x\| = \|T^2.T^{2k}x\| \geq \frac{\|T^{2k+1}x\|^2}{\|T^{2k}x\|} .$$

Also $$\|T^{2k+1}x\| = \|T^{2k}Tx\| \geq \frac{\|T^{k+1}x\|^2}{\|Tx\|} .$$

Hence $$\|T^{2(k+1)}x\| \geq \frac{\|T^{2k+1}x\|.\|T^{k+1}x\|^2}{\|T^{2k}x\|.\|Tx\|}$$

$$= \|T^{k+1}x\|^2.\frac{\|T^{2k+1}x\|}{\|T^{2k}x\|.\|Tx\|}$$

With the help of lemma 8.1, this reduces to

$$\|T^{2(k+1)}x\| \geq \|T^{k+1}x\|^2 .$$

Thus the theorem is true for $n = k$ i.e. T^{k+1} is also of class (N) and this completes the proof of the theorem.

Remark: P. R. Halmos[1)] showed by means of an example that if T is hyponormal, then T^2 is not necessarily hyponormal. Thus, the class of hyponormal operators is narrower than the class of operators of class (N).

Next, we prove a theorem concerning the ascent of an operator of class (N).

THEOREM 8.2. *If T is of class (N), then the ascent of T is either 0 or 1.*

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1) P. R. Halmos [10]

PROOF : We know that $N(T^2)$ is a closed subspace of H and $N(T) \subseteq N(T^2)$. Hence to prove the theorem, it remains only to show the above inclusion relation in the opposite way. Let $\{x_n\}$ be the orthonormal basis of $N(T^2)$. Then $\|Tx_n\|^2 \leq \|T^2x_n\| = 0$ implies that $Tx_n = \theta$ i.e. $x_n \in N(T)$ or $N(T^2) \subseteq N(T)$. This is what we wanted to prove.

An interesting result concerning hyponormal operators, which was proved in chapter III, corollary 3.1, can be extended to the operators of class (N) as follows:

THEOREM 8.3. If T is an operator of class (N), then

$$T_1\|x\| \leq \|Tx\| \leq T_L\|x\| \text{ for all } x \in H.$$

We need the following result of V. Istratescu, T. Saito and T. Yoshino[1] to prove our theorem.

THEOREM 8.A. If T is an operator of class (N), then

(i) T is normaloid and

(ii) T^{-1} is also of class (N) provided that T is invertible.

PROOF OF THEOREM 8.3: Since T is normaloid by (i) of theorem 8.A, s(T) contains a complex number α such that $\|T\|=|\alpha|$.

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1) V. Istratescu, T. Saito and T. Yoshino [12]

Also, by definition, $T_L = |\alpha|$. Hence $\|Tx\| \leq T_L\|x\|$ for every $x \in H$. If T is non-invertible, then $0 \in s(T)$ and $T_1 = 0$ i.e. $T_1\|x\| \leq \|Tx\|$ for all $x \in H$. If T is invertible, then T^{-1} is normaloid by (ii) of theorem 8.A and again, as proved above, $\|T^{-1}x\| \leq T_L^{-1}\|x\|$ for all $x \in H$ i.e. equivalently, $\|x\| \leq T_L^{-1}\|Tx\|$ for all $x \in H$. Since a complex number $\alpha \in s(T)$ if and only if $\alpha^{-1} \in s(T^{-1})$, we have $T_L^{-1} = \frac{1}{T_1}$. Thus $T_1\|x\| \leq \|Tx\|$ for all $x \in H$. This completes the proof of theorem.

The following corollary can be derived from theorem 8.3:

<u>COROLLARY 8.1.</u> <u>If</u> T <u>is of class</u> (N) <u>and</u> s(T) <u>lies on the unit circle, then</u> T <u>is unitary</u>.

<u>PROOF:</u> Here we have $T_1 = T_L = 1$. Hence $\|Tx\| = \|x\|$ for all $x \in H$. Since, in addition, T is invertible, T is unitary.

It should be mentioned that V. Istratescu, T. Saito and T. Yoshino[1)] have also proved the result given in corollary 8.1.

It has been brought to the notice of the author in a personal communication that V. Istratescu has further

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1) V. Istratescu, T. Saito and T. Yoshino [12]

generalised the notion of operators of class (N) by introducing the following definition:

<u>DEFINITION</u>: An operator T is of class (N) and order k if

$$\|Tx\|^k \leq \|T^k x\| \text{ for all unit vectors } x \in H.$$

The class of operators of class (N) and order k is denoted by $\mathscr{C}(N.k)$. Thus $\mathscr{C}(N,2)$ is identical with the class (N) of operators.

V. Istratescu proved the following interesting result regarding operators of class (N) and order k.

<u>THEOREM 8.B</u>. <u>If</u> T $\in \mathscr{C}(N,k)$, <u>then</u> T <u>is</u> <u>normaloid</u>.

We give below a simple proof of this theorem.

<u>PROOF</u>: Since $\|T\| = \sup_{x \neq \theta} \frac{\|Tx\|}{\|x\|}$, there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\| \to \|T\| = 1$ (say).

Now, by definition, we have

$$\|Tx_n\|^k \leq \|T^k x_n\| \leq \|T^{k-1}\| . \|Tx_n\| \leq \|T\|^{k-1}\|Tx_n\| \leq \|Tx_n\|$$

Taking the limits,

$$1 \leq \lim \|T^k x_n\| \leq 1$$

i.e. $\lim \|T^k x_n\| \to 1$ or $\|T^k\| = 1$.

Also, for k > 2,

$$\|T^k\| = \|T^2.T^{k-2}\| \leq \|T^2\|.\|T^{k-2}\|$$

i.e. $$1 \leq \|T^2\|.\|T^{k-2}\| \leq \|T\|^2.\|T\|^{k-2} \leq 1$$

or $$\|T^2\| = 1 = \|T\|^2 .$$

Now it is proved that an operator T is normaloid provided that $\|T^2\| = \|T\|^2$. Thus T is normaloid and the proof is complete.