CHAPTER IX

ON A PROBLEM OF T. NIEMINEN

T. Nieminen[1] posed the following question in his paper.

If T is a bounded operator on a Hilbert space H, the spectrum of which is a subset of the set $S = \{\alpha \,;\, |\alpha| = 1\}$ and $\|R_\alpha\| \leq \big|\, |\alpha| - 1 \,\big|^{-1}$ on the resolvent set, does it follow necessarily that T is unitary ?

William. F. Donoghue Jr.[2] answered this question affirmatively in the form of the following theorem:

THEOREM 9.A. The following three classes of bounded operators on a Hilbert space H are identical:

(I) The unitary operators ;

(II) The operators T for which

(a) T^{-1} exists and is everywhere defined and $\|Tx\| \geq \|x\|$ for every $x \in H$ and

(b) W(T) is a subset of the unit disc;

(III) The operators T for which

(a') 0 is in the resolvent set and $\|R_0\| \leq 1$ and

(b') for an unbounded sequence of numbers $f_n > 1$, $\|R_\alpha\| \leq (f_n - 1)^{-1}$ if $|\alpha| = f_n$

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1) T. Nieminen [17] 2) W. F. Donoghue Jr. [7]

Combining this theorem of W. F. Donoghue Jr, with S. K. Berberian's result that a unitary operator U is cramped if and only if $0 \notin Cl(W(U))$, we immediately get the following theorem for a subclass of unitary operators:

<u>THEOREM 9.1.</u> <u>The following two classes of bounded operators on a Hilbert space</u> H <u>are identical</u>:

(I') <u>The unitary operators with cramped spectrum</u> ;

(II') <u>The operators</u> T <u>for which</u>

(a) $\|Tx\| \geq \|x\|$ <u>for every</u> $x \in H$ <u>and</u>

(b) $0 \notin Cl(W(T))$ <u>and</u> W(T) <u>is a subset of the unit disc</u>.

We give a new and simple proof of this theorem in this chapter. The significance of the proof lies in the fact that it is not based on the 'Spectral Integral Theory' where as the proof of theorem 9.A leans heavily on the 'Spectral Integral Theory'.

To prove the theorem, we need the following lemma whose proof is obvious.

<u>LEMMA 9.1.</u> <u>Let</u> T <u>be an operator. Then a non-zero vector</u> x <u>is a proper element of</u> T <u>if and only if</u> $|(Tx , x)| = \|Tx\|.\|x\|$.

PROOF OF THEOREM 9.1: The fact that $(I') \Rightarrow (II')$ is obvious if we take Berberian's result into consideration.

To prove $(II') \Rightarrow (I')$

Since $0 \notin Cl(W(T))$, T^{-1} exists and is everywhere defined. The relation $s(T) \subseteq \Sigma(T) \subseteq Cl(W(T))$ and the hypothesis on $Cl(W(T))$ together imply that if $\alpha \in s(T)$, then $|\alpha| \leq 1$ i.e. if $\beta \in s(T^{-1})$, then $|\beta| \geq 1$. .....(9.1)

As $\|Tx\| \geq \|x\|$ for all $x \in H$, we have $\|T^{-1}x\| \leq \|x\|$ for all $x \in H$ i.e. $\|T^{-1}\| \leq 1$. This shows that if $\beta \in s(T^{-1})$, then $|\beta| \leq 1$ . ....(9.2)

It follows from (9.1) and (9.2) that $s(T^{-1})$ and consequently $s(T)$ is a subset of the set $S = \{\alpha \; ; \; |\alpha| = 1\}$ and $\|T^{-1}\| = \|T^{*-1}\| = 1$.

Now, if $T = UR$ is the polardecomposition of T, then $\|Rx\| = \|Tx\| \geq \|x\|$ for all $x \in H$. Since R is positive-definite, this implies that $(x , x) \leq (Rx , x)$ for all $x \in H$. Also, it follows from the relation $\|T^{*-1}\| = 1 = \|R^{-1}\|$ that $1 \in s(R^{-1})$ and hence $1 \in s(R) = a(R)$. As the property of unitarility or positivity of an operator T is preserved under *-isomorphism, we may assume that $1 \in p(R)$ [6].

Consider the two subsets L and M of H defined by the relations

$$L = \{ x \,;\, Rx = x \} \text{ and } M = \{ x \,;\, \|Tx\| = \|x\| \}.$$

Obviously, L is a non-empty closed linear subspace of H. Now if $x \in L$, then $\|x\| = \|Rx\| = \|Tx\|$ i.e. $x \in M$. Thus $L \subseteq M$. Conversely, if $x \in M$, then $\|x\| = \|Tx\| = \|Rx\|$. The relation $\|x\|^2 = (x , x) \leq (Rx , x) \leq \|Rx\|.\|x\| = \|x\|^2$ implies that $(Rx , x) = \|Rx\|.\|x\|$ i.e. $x \in L$ by lemma 9.1 i.e. $M \subseteq L$ and hence $M = L$. Thus, to complete the proof, it remains only to show that $L = M = H$ or in other words $L^{\perp} = \{ \theta \}$.

Assume, to the contrary that $L^{\perp} \neq \{\theta\}$. Then $L^{\perp}$ is invariant under T. Because, otherwise, we would have $(Ty , y) = 0$ for some unit vector $y \in L^{\perp}$, which contradicts the hypothesis that $0 \notin Cl(W(T))$. Hence, if T_1 be the restriction of T to $L^{\perp}$, then $\|T_1 x\| > \|x\|$ for all $x \in L^{\perp}$. Since T^{-1} exists, T_1^{-1} exists and $\|T_1^{-1}x\| < \|x\|$ for all $x \in L^{\perp}$ i.e. $\|T_1^{-1}\| < 1$. This implies that if $\alpha \in s(T_1^{-1})$, then $|\alpha| < 1$ and consequently, if $\beta \in s(T_1)$, then $|\beta| > 1$. This contradicts the relation $s(T_1) \subseteq s(T)$, hence $L^{\perp} = \{\theta\}$ i.e. $L = H$. As $0 \notin Cl(W(T))$, T has a cramped spectrum by Berberian's result. This completes the proof of theorem.