CHAPTER I

INTRODUCTION

1.1. The present thesis is devoted to the study of certain problems relating to linear operators and particularly hyponormal operators in a Hilbert space. As the title suggests, this chapter is of introductory character and seeks to give a brief survey of problems dealt with in the thesis. It will be convenient to begin by defining certain notions which we shall constantly use.

1.2. DEFINITIONS AND NOTATIONS

Let X be a complex vector space. X is called an INNER PRODUCT SPACE if there is defined on $X \times X$ a complex-valued function (x , y) (called the inner product of vectors x and y) having the following properties:

1. $(x + y , z) = (x , z) + (y , z), \quad z \in X$
2. $(x , y) = \overline{(y , x)}$ (the bar denotes the complex conjugate)
3. $(\alpha x , y) = \alpha(x , y)$, α scalar
4. $(x , x) > 0$, when $x \neq \theta$.

If X is an inner-product space, then the non-negative real number $\sqrt{(x , x)}$ has the properties of a norm and X becomes a normed vector space with the norm $\|x\|$ of a vector x defined as $\|x\| = \sqrt{(x, x)}$.

An inner product space X is called a HILBERT SPACE if it is complete in the metric topology associated with the norm. If X is a finite-dimensional inner-product space, then it is called a EUCLIDEAN SPACE (or UNITARY SPACE). Through out the thesis, H will denote a Hilbert space. The Roman letters x, y, z etc. with or without suffixes will be used to denote the elements of H. The Greek letters α, β, γ etc. will denote complex numbers. A vector $x \in H$ will be called a UNIT VECTOR if $\|x\| = 1$.

A mapping $T : H \to H$ is said to be linear in case

$$T(\alpha x + \beta y) = \alpha(Tx) + \beta(Ty)$$

for all vectors x, y and complex numbers α, β. T is said to be CONTINUOUS at a point $x \in H$ if for every convergent sequence $\{x_n\}$ with limit x, $\lim\limits_{n\to\infty} Tx_n = Tx$.

T is said to be continuous on H if it is continuous at every point of H. A mapping T of H into H is said to be bounded if there exists a constant $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in H$, $x \neq \theta$. The smallest number

M satisfying the above inequality is called the norm of T and is denoted by $\|T\|$. Thus

$$\|T\| = \underset{\substack{x \in H \\ x \neq \theta}}{\text{l.u.b}} \quad \frac{\|Tx\|}{\|x\|} \quad .$$

It is easy to prove that a linear mapping T is continuous if and only if T is bounded. A continuous (or bounded) linear mapping of H into H is called an OPERATOR.

If T is an operator on H, then there exists a unique operator T^*, called the ADJOINT of T, on H such that $(Tx , y) = (x , T^*y)$ for all $x , y \in H$.

It is not difficult to see that any two operators T_1 and T_2 defined on a Hilbert space H can be added and multiplied in an obvious way. That is

$$(T_1 + T_2)x = T_1x + T_2x \; , \quad x \in H$$

$$(T_1T_2)x = T_1(T_2x), \qquad x \in H.$$

In fact, the set of all operators on a Hilbert space forms an algebra.

The subspace of H on which T is defined is called the DOMAIN of T and is denoted by D(T). The RANGE R(T) and the NULL-SPACE N(T) of an operator T are the sets defined by the relations:

$$R(T) = \{ y \; ; \; Tx = y \quad \text{for some} \quad x \in D(T) \}$$

and $$N(T) = \{ x \; ; \; Tx = \theta \}.$$

The smallest positive integer n, for which $N(T^n) = N(T^{n+1})$ is called the ASCENT of T. T is said to be ONTO if $R(T) = H$. T is said to be DENSE in H if $Cl(R(T))$ [i.e. the closure of $R(T)$] $= H$ and it is ONE-TO-ONE if $Tx = \theta$ implies that $x = \theta$. T is said to be REVERSIBLE if it is one-to-one and it is said to be INVERTIBLE if it is both one-to-one and onto.

1.3. THE SPECTRUM OF AN OPERATOR T.

The operator which maps every element x into x is called the IDENTITY OPERATOR and is denoted by I. Thus $Ix = x$ for every $x \in H$. The POINT SPECTRUM $p(T)$, the CONTINUOUS SPECTRUM $c(T)$, the RESIDUAL SPECTRUM $r(T)$ and the APPROXIMATE POINT SPECTRUM $a(T)$ of an operator T are defined as follows:

$$p(T) = \{ \alpha \; ; \; T - \alpha I \ \text{ is not one-to-one} \},$$

$$c(T) = \{ \alpha \; ; \; T - \alpha I \ \text{ is one-to-one and } R(T - \alpha I) \text{ is dense in } H, \text{ but not equal to } H \},$$

$$r(T) = \{ \alpha \; ; \; T - \alpha I \ \text{ is one-to-one and } Cl(R(T - \alpha I)) \neq H \},$$

$$a(T) = \{\alpha \ ; \text{ there exists a sequence } \{x_n\} \text{ of unit vectors such that } \|(T - \alpha I)x_n\| \to 0\}.$$

The sets $p(T)$, $c(T)$ and $r(T)$ are disjoint and their union is called the <u>SPECTRUM</u> of T and is denoted by $s(T)$. Thus

$$s(T) = p(T) \cup c(T) \cup r(T).$$

Equivalently,

$$s(T) = \{\alpha \ ; \ T - \alpha I \text{ is not invertible}\}.$$

If T is a bounded linear operator, then it is known that $s(T)$ is a non-empty closed bounded set. The complement of the set $s(T)$ is called the <u>RESOLVENT SET</u> of T and is denoted by $\rho(T)$. The convex hull of the set $s(T)$ is denoted by $\Sigma(T)$ i.e. $\Sigma(T)$ is the intersection of all convex sets which contain $s(T)$. If $\alpha \notin s(T)$, then $(T - \alpha I)^{-1}$ exists and is denoted by R_α. It is not difficult to see that $p(T) \subseteq a(T) \subseteq s(T)$. Every point α of the set $p(T)$ is called a <u>PROPER-VALUE</u> of T. For a proper-value α of T, we defin the α-th <u>PROPER SUBSPACE</u> $N_T(\alpha)$ of T by the relation:

$$N_T(\alpha) = \{x \ ; \ Tx = \alpha x\}.$$

If $x \in N_T(\alpha)$, then x is called a <u>PROPER-ELEMENT</u> of T corresponding to the proper-value α of T.

A family of closed linear subspaces is said to be TOTAL if the null-vector is the only vector orthogonal to every subspace belonging to the family.

An operator T is said to have a PURE POINT SPECTRUM if the proper subspaces of T constitute a total family.

1.4. THE NUMERICAL RANGE OF AN OPERATOR T

The NUMERICAL RANGE W(T) of an operator T is the set of all complex numbers (Tx , x) such that $\|x\| = 1$. It is a convex set in the complex plane and its closure Cl(W(T)) contains the set $\Sigma(T)$, the convex hull of s(T) as a subset i.e. $\Sigma(T) \subseteq Cl(W(T))$.

A point α of a non-empty convex set S is said to be an EXTREME POINT of S, if no line segment joining any two points of S, each different from α, contains α. We denote by E(S), the set of all extreme points of S. Clearly, $E(\Sigma(T)) \subseteq s(T)$ and the convex closure of $E(\Sigma(T))$ is exactly $\Sigma(T)$.

Following C. R. Putnam[1], we shall say that a complex number α will belong to the INTERIOR of Cl(W(T)), if α is in Cl(W(T)) and one of the following three

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1) C. R. Putnam [20]

Note: Numbers in square brackets [ ] in the body of the thesis refer to the corresponding item listed in the bibliography at the end of the thesis.

conditions holds:

(i) if $Cl(W(T))$ is two-dimensional, then α does not lie in the boundary of $Cl(W(T))$;

(ii) if $Cl(W(T))$ is a line-segment, then α is not an end-point ;

(iii) $Cl(W(T))$ consists of α alone.

We shall call with T. Yoshino[1], a point α of a closed bounded set S a SEMI-BARE POINT of S, if there exists a circle C through α such that $S \cap C = \{\alpha\}$. The symbol $B_s(S)$ will denote the set of all semi-bare points of S.

J. Von-Neumann[2] introduced the notion of a SPECTRAL SET. According to him, a closed proper subset S of the complex plane is a spectral set for an operator T if

$$\|u(T)\| \leq \sup\left\{ |u(\alpha)| \ ; \ \alpha \in S \right\}$$

for every rational function u of α having no poles on S.

The distance between a point α and a set S is denoted by $d(\alpha, S)$.

1.5. DIFFERENT TYPES OF OPERATORS

An operator T is said to be POSITIVE-DEFINITE if $(Tx, x) \geq 0$ for all $x \in H$. The positive-definiteness

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1) T. Yoshino [31] 2) J.Von-Neumann [22 , pp 440.]

of T is also expressed in symbol by $T \geq 0$. T is said to be SELF-ADJOINT if $T = T^*$, UNITARY if $T\,T^* = T^*T = I$, NORMAL if $T\,T^* = T^*T$, HYPONORMAL if $T^*T - T\,T^* \geq 0$. Equivalently, T is hyponormal if $\|T^*x\| \leq \|Tx\|$ for all $x \in H$. T is said to be QUASI-NORMAL if $(T^*T)T = T(T^*T)$, SEMI-NORMAL if either T or T^* is hyponormal. We say that T is an OPERATOR OF CLASS (N) if $\|Tx\|^2 \leq \|T^2x\|$ for all unit vectors $x \in H$. Given any bounded sequence of vectors $\{x_n\}$ if the sequence $\{Tx_n\}$ has a convergent subsequence, then T is said to be a COMPACT OPERATOR. If there exists a Hilbert space H', containing H as a subspace and a normal operator B on H' such that $Tx = Bx$ for $x \in H$, then T is said to be a SUBNORMAL OPERATOR. We have the following proper inclusion relation for classes of operators:

Normal $\subseteq$ Quasi-normal $\subseteq$ Sub-normal $\subseteq$ Hyponormal $\subseteq$ Operator of class (N).

An operator T is said to be ISOMETRIC if $\|Tx\| = \|x\|$ for every $x \in H$. It is known that T is isometric if and only if $T^*T = I$. T is said to satisfy the condition G_1, if the resolvent of T has exactly first order rate of growth with respect to the spectrum of T i.e.

$$\|R_\alpha\| \leq d(\alpha, s(T))^{-1} \quad \text{for } \alpha \in \rho(T).$$

A unitary operator U is said to be CRAMPED if its spectrum is contained in an open semi-circle $\{e^{i\theta}\ ;\ \theta_0 < \theta < \theta_0 + \pi\}$ of the unit circle.

For any operator T, there exists two self-adjoint operators A and B such that $T = A + iB$. The self-adjoint operators A and B are given by $A = \frac{T + T^*}{2}$, $B = \frac{T - T^*}{2i}$. The representation of T in the form $T = A + iB$ is called the CARTESIAN DECOMPOSITION of T. If T is positive-definite, then there exists a positive-definite operator R such that $R^2 = T$. The operator R is called the POSITIVE SYMMETRIC SQUARE ROOT of T. Given an operator T, there exists an isometric operator U and a positive-definite operator R such that $T = UR$. Indeed, R is the positive symmetric square root of the positive-definite operator T^*T. The representation of an operator T in the form $T = UR$, where U is isometric and R is positive-definite, is called the POLARDECOMPOSITION of T. If T is invertible, then U is unitary and R is invertible.

Finally, a closed linear subspace L is said to be INVARIANT under an operator T if $Tx \in L$ for all $x \in L$. In this case, we denote the restriction of T to L by T/L. Thus T/L is a mapping from L to L such that $(T/L)x = Tx$

for $x \in L$. If a closed linear subspace L and its orthogonal complement $L^{\perp}$ are both invariant under T, then L is said to reduce T.

1.6. The basic nature of hyponormal operators was noticed for the first time in the year 1950 by P. R. Halmos[1]. He observed that if an operator T has a normal extension, then $T^{*}T \geq T\,T^{*}$. Taking his clue from this observation, he called an operator T subnormal if $T^{*}T \geq T\,T^{*}$. What Halmos called a subnormal operator came to be known as a hyponormal operator later on. Using the concept of trace, Halmos also proved that the notion of hyponormality coincides with that of normality on a finite dimensional space. However, hyponormal operators exist in large numbers to justify a study of their properties. The study of hyponormal operators can be divided broadly in three parts namely

(i) study of those properties which are similar to the properties of normal operators;

(ii) investigation of sufficient conditions under which a hyponormal operator is normal and

(iii) study of their general properties.

It is known that for an invertible normal operator N,

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1) P. R. Halmos [10]

its inverse N^{-1} is also normal. We obtain, in chapter II, a similar result pertaining to hyponormal operators. In fact, the analogous result for hyponormal operators is deduced as a corollary to the following theorem proved in chapter II :

(1.1) THEOREM. *An operator* T *is hyponormal if and only if there exists an operator* V *with* $D(V) = R(T)$ *such that* $T^* = VT$ *and* $\|V\| = 1$.

We also prove in chapter II the following theorem:

(1.2) THEOREM. *If* T *is hyponormal, then the ascent of* T *is 0 or 1.*

It will be observed that this theorem extends to hyponormal operators a property possessed by normal operators.

Using the notion of semi-bare points of a closed bounded set, Takashi Yoshino[1)] proved that for a hyponormal operator T, $B_s(s(T)) \cap r(T) = \emptyset$. We extend this result to any operator satisfying the condition G_1 in the form of the following:

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1) Takashi Yoshino [31]

(1.3) <u>THEOREM</u>. *If T is an operator whose resolvent has first order rate of growth* (i.e. T *satisfies the condition* G_1), *then* $B_s(s(T)) \cap r(T) = \emptyset$.

In view of the fact established in chapter III, that a hyponormal operator satisfies the condition G_1, it follows that the result of T. Yoshino is a special case of our theorem. A second corollary of our theorem is the following:

If T *satisfies the condition* G_1 *and* α *is an isolated point of* $s(T)$, *then* $\alpha \in a(T)$.

It may be noted here that if α is an isolated point of $s(T)$, when T is hyponormal, then, in accordance with a result proved by J. G. Stampfli[1)], $\alpha \in p(T)$.

It is known that the product $T_1 T_2$ of two hyponormal operators T_1 and T_2 need not be hyponormal even if they commute. In view of this situation, it seems to be worthwhile to investigate conditions under which the product of two hyponormal operators is also hyponormal. We prove in chapter II the following theorem in this direction:

(1.4) <u>THEOREM</u>. *If* T_1 *and* T_2 *are two hyponormal operators such that* $T_1 T_2^* = T_2^* T_1$, *then* $T_1 T_2$ *is hyponormal.*

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1) J. G. Stampfli [26]

T. Andô[1], S. K. Berberian[2] and J.G.Stampfli[3] proved independently of each other that a compact hyponormal operator is necessarily normal. We generalise this result in the following form:

(1.5) THEOREM. <u>If</u> T <u>is a hyponormal operator such that</u> T^p <u>is compact, where</u> p <u>is an integer</u> ≥ 1, <u>then</u> T <u>is normal and hence compact</u>.

We also give in chapter II a different and simple proof of the following theorem, which was proved by S. K. Berberian[2].

(1.6) THEOREM. <u>If</u> T <u>is hyponormal, then</u> $r(T^*) = \emptyset$.

The relation $\Sigma(T) \subseteq Cl(W(T))$ is true for every bounded operator T. M. H. Stone[4] was first to prove that if T is normal, then $\Sigma(T) = Cl(W(T))$. S. K. Berberian[5] conjectured that this result is also true for hyponormal operators. This conjecture of Berberian was proved to be true only recently by C. R. Putnam[6], J. G. Stampfli[7], T. Yoshino and T. Saito[8] independently of each other. We give in chapter III another proof of the same result namely

(1.7) <u>for a hyponormal operator</u> T, $\Sigma(T) = Cl(W(T))$.

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1) T. Andô [1] 2) S. K. Berberian [3] 3) J.G.Stampfli [26

4) M. H. Stone [29] 5) S. K. Berberian [4] 6) C.R.Putnam [21]

7) J. G. Stampfli [27] 8) T.Yoshino and T. Saito [23]

In the course of the proof of this result, we are able to give a new proof of the following result:

(1.8) <u>THEOREM</u>. <u>If</u> T <u>is hyponormal and</u> s(T) <u>lies on the unit circle</u>, <u>then</u> T <u>is unitary</u>.

Recently, C. R. Putnam[1)] proved the spectral relations between a semi-normal operator T and its cartesian decompositions A and B. His results are:

$$s(A) = \{\mathrm{Re}\alpha \ ; \ \alpha \in s(T)\}$$
$$s(B) = \{\mathrm{Im}\alpha \ ; \ \alpha \in s(T)\} .$$

Actually, C. R. Putnam made use of the 'Spectral Integral Theory' in proving these results. In chapter IV, we prove these relations by using only elementary methods without resorting to the 'Spectral Integral Theory'. In addition to this, we also prove in chapter IV the following theorem:

(1.9) <u>THEOREM</u>. If $T = A + iB$ <u>is hyponormal</u>, <u>then</u>

(i) $p(A) = \{\mathrm{Re}\alpha \ ; \ \alpha \in p(T)\}$, $p(B) = \{\mathrm{Im}\alpha \ ; \ \alpha \in p(T)\}$,

(ii) $N_A(s) = \bigcup \oplus N_T(\alpha)$ for all $\alpha \in p(T)$ and $\mathrm{Re}\alpha = s$,
$N_B(t) = \bigcup \oplus N_T(\alpha)$ for all $\alpha \in p(T)$ and $\mathrm{Im}\alpha = \beta$.

(iii) If $\alpha = s + it$ (s and t being real) $\in p(T)$, then $N_T(\alpha) = N_A(s) \cap N_B(t)$.

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1) C. R. Putnam [21]

The result that $\Sigma(T) = \check{C}l(W(T))$ for a hyponormal operator T can also be derived as a corollary of the theorem (1.9). As a second corollary of this result we show that for a hyponormal operator T, $E(W(T)) \subseteq p(T)$, which extends to hyponormal operators a result proved earlier by C. R. MacCLUER[1)] (and consequently that of C. H. Meng[2)]) regarding extreme points of the numerical range of a normal operator.

W. A. Beck and C. R. Putnam[3)] have proved the following theorem:

(1.10) <u>THEOREM.</u> <u>Let</u> N <u>be a normal operator</u>. <u>If</u> $AN = N^*A$ <u>for an arbitrary invertible operator</u> $A = UR$, <u>for which</u> U <u>is cramped, then</u> $N = N^*$ i.e. N <u>is self-adjoint</u>.

S. K. Berberian[4)] has given an abstract proof of this theorem for any B^*-algebra and C. A. McCarthy[5)] has given another proof of a slightly improved version of this result.

In chapter V, we extend this result to hyponormal operators under slightly more restrictive conditions on A. In fact, we prove the following:[6)]

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1) C. R. MacCLUER [14] 2) C. H. Meng [16] 3) W. A. Beck and C. R. Putnam [2]
4) S. K. Berberian [5] 5) C. A. McCarthy [15] 6) I.H.Sheth [25]

(1.11) <u>THEOREM. Let</u> N <u>be a hyponormal operator. If</u> $AN = N^*A$ <u>for an arbitrary operator</u> A, <u>for which</u> $0 \notin Cl(W(A))$, <u>then</u> N <u>is self-adjoint</u>.

Now if $0 \notin Cl(W(A))$, then A is invertible. Also S. K. Berberian[1)] has proved that for such an operator A, with polardecomposition $A = UR$, U is cramped. Hence it follows that the condition $0 \notin Cl(W(A))$ is stronger than the condition that U is cramped.

It is known that for any Hilbert space (separable or nonseparable) a hyponormal operator with a pure point spectrum is normal. In chapter VI, we examine the question whether the hyponormality of T together with $s(T) = p(T)$ imply the normality of T and we prove the following two theorems in this respect:

(1.12) <u>THEOREM. Let</u> T <u>be a hyponormal operator with</u> $s(T) = p(T)$. <u>Then</u> T <u>is normal if the Hilbert space under consideration is separable</u>.

(1.13) <u>THEOREM. If</u> H <u>is nonseparable then there exists a non-normal hyponormal operator with</u> $s(T) = p(T)$.

In his paper [4], S. K. Berberian raised the following question.

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1) S. K. Berberian [4]

If $T = UR$ is an invertible operator such that U is cramped, does it follow that $0 \notin Cl(W(T))$?

He proved that if, in addition, T is normal, then the answer to the above question is in the affirmative. We construct an example in chapter VII to show that there exist invertible operators T with polardecomposition $T = UR$, where U is cramped, such that $0 \in Cl(W(T))$. We also show that the answer to Berberian's question is still in the affirmative for certain special types of operators and more particularly for hyponormal operators. In fact, we prove the following three theorems in chapter VII:

(1.14) THEOREM. *If $T = UR$ is an invertible hyponormal operator such that U is cramped, then $0 \notin Cl(W(T))$.*

(1.15) THEOREM. *If $T = UR$ is an invertible operator such that U is cramped, then $0 \notin Cl(W(T))$ provided that $\Sigma(T)$ is a spectral set for T.*

(1.16) THEOREM. *If H is finite dimensional, then there exists an invertible operator $T = UR$ such that U is cramped and $0 \in W(T)$.*

P. R. Halmos[1)] showed by means of an example that the hyponormality of T does not necessarily imply that

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1) P. R. Halmos [10]

T^2 is hyponormal. But our investigation shows that the situation with regard to the operators of class (N) is different, in that for every operator T of class (N), T^n is also of class (N) for every positive integer n. In fact, we prove the following three theorems pertaining the operators of class (N) in chapter VIII:

(1.17) THEOREM. If T is of class (N), then T^n is also of class (N) for every positive integer n.

(1.18) THEOREM. If T is of class (N), then the ascent of T is either 0 or 1.

(1.19) THEOREM. If T is of class (N) and s(T) lies on the unit circle, then T is unitary.

In answering the question raised by T. Nieminen[1], W. F. Donoghue Jr.[2] proved the following theorem:

(1.20) THEOREM. The following three classes of bounded operators on a Hilbert space H are identical:

(I) The unitary operators;

(II) The operators T for which

(a) T^{-1} exists and is everywhere defined and $\|Tx\| \geq \|x\|$ for every $x \in H$ and

(b) W(T) is a subset of the unit disc;

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1) T. Nieminen [17]   2) W. F. Donoghue Jr [7]

(III) The operators T for which

(a') 0 is in the resolvent set and $\|R_0\| \leq 1$ and

(b') for an unbounded sequence of numbers $f_n > 1$,

$\|R_\alpha\| < (f_n - 1)^{-1}$ if $|\alpha| = f_n$.

Combining this theorem of W. F. Donoghue Jr., with the fact that a unitary operator U is cramped if and only if $0 \notin Cl(W(U))$, which was proved by S. K. Berberian[1)], we immediately get the following theorem:

(1.21) THEOREM. The following two classes of bounded operators on a Hilbert space H are identical:

(I') The unitary operators with cramped spectrum and

(II') The operators T for which

(a) $\|Tx\| \geqq \|x\|$ for every $x \in H$ and

(b) $0 \notin Cl(W(T))$ and $W(T)$ is a subset of the unit disc.

We give a new and simple proof of this theorem in chapter IX. Our proof seems to have same interest in view of the fact that it is not based on the 'Spectral Integral Theory', where as the proof of theorem (1.20) leans on the 'Spectral Integral Theory'.

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1) $^+$S. K. Berberian [4]