CHAPTER II

GENERAL PROPERTIES OF HYPONORMAL OPERATORS

The following theorem gives a necessary and sufficient condition for an operator T to be normal. The significance of the condition lies in the fact that it involves the adjoint operator:

THEOREM. A. *An operator T is normal if and only if there exists a unitary operator U such that* $T^* = UT$.

We begin by proving a result regarding hyponormal operators, which is analogous to the above mentioned theorem.

THEOREM. 2.1. *An operator T is hyponormal if and only if there exists an operator V with* $D(V) = R(T)$ *such that* $T^* = VT$ *and* $\|V\| = 1$.

PROOF: Let $T^* = VT$ with $\|V\| = 1$ and $D(V) = R(T)$. Then $\|T^*x\| = \|VTx\| \leq \|V\|.\|Tx\| = \|Tx\|$ for all $x \in H$ i.e. T is hyponormal.

To prove that the conditions are necessary, let T be hyponormal. If $Tx = z$ and $T^*x = y$ for an arbitrary $x \in H$, we define a mapping V on R(T) by the relation $y = Vz$. Obviously, V is linear and $T^*x = VTx$ for every $x \in H$

i.e. $T^* = VT$. The relation $\|V\| = 1$ follows from the facts that $T^* = VT$ and $\|T^*\| = \|T\|$.

From this theorem, we derive the following:

COROLLARY 2.1. *If T is an invertible hyponormal operator, then T^{-1} is also hyponormal.*

PROOF: It follows from the definition of V in theorem 2.1 that $D(V) = H$ and V^{-1} exists when T is an invertible hyponormal operator. First, taking adjoint and then inverse of the relation $T^* = VT$, we get $T^{-1} = (V^*)^{-1}(T^{-1})^*$ i.e. $(T^{-1})^* = V^* T^{-1}$, where $\|V\| = \|V^*\| = 1$. Hence T^{-1} is hyponormal by theorem 2.1.

We next prove a theorem pertaining to the ascent of a hyponormal operator T. This theorem is an extension to hyponormal operators of known result concerning normal operators.

THEOREM 2.2. *If T is hyponormal, then the ascent of T is 0 or 1.*

PROOF: If $x \in N(T^2)$, then $T^2x = \theta$. T being hyponormal, the relation $\|T^*Tx\| \leq \|T^2x\|$ implies that $\|T^*Tx\| = 0$ i.e. $x \in N(T^*T) = N(T)$ [A. E. Taylor [30] page 250]. Hence $N(T^2) \subseteq N(T)$. Since the reverse inclusion is true for any operator, we have $N(T) = N(T^2)$. This completes the proof of the theorem.

The notion of semi-bare points, which is more general than that of bare points, leads to some interesting results. This notion was introduced by T. Yoshino[1)] who proved the following theorem:

THEOREM 2.B. *If* T *is hyponormal, then* $B_s(s(T)) \cap r(T) = \emptyset$.

We give the following generalisation of the theorem 2.B.

THEOREM 2.3. *If* T *is an operator whose resolvent has first order rate of growth* (i.e. T *satisfies the condition* G_1), *then* $B_s(s(T)) \cap r(T) = \emptyset$.

To prove this theorem, we need the following known results:

LEMMA 2.1. *If* T *is an operator for which* $\|T\| = |\alpha|$ *for* $\alpha \in s(T)$, *then* $\alpha \in a(T)$.

LEMMA 2.2. *If* T *is an invertible operator, then* $\alpha \in a(T)$ *if and only if* $\alpha^{-1} \in a(T^{-1})$.

PROOF OF THEOREM 2.3: Assume, to the contrary, that $\alpha \in B_s(s(T)) \cap r(T)$. Then there exists a complex number $\beta \notin s(T)$ such that $|\alpha - \beta| = d(\beta , s(T))$. Since $\beta \notin s(T)$, $(T - \beta I)^{-1} = R_\beta$ exists and $\|R_\beta\| \leq |\alpha - \beta|^{-1}$ by hypothesis.

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1) T. Yoshino [31]

But $|\alpha - \beta|^{-1} \leq \|R_\beta\|$ [Dunford-Schwartz [8] page 566] i.e. $\|R_\beta\| = |\alpha - \beta|^{-1} = |(\alpha - \beta)^{-1}|$. Now $\alpha - \beta \in s(T - \beta I)$ and hence $(\alpha - \beta)^{-1} \in s((T - \beta I)^{-1})$. Using lemmas 2.1 and 2.2, we have $(\alpha - \beta)^{-1} \in a((T - \beta I)^{-1})$ and $\alpha - \beta \in a((T - \beta I))$ which in turn implies that $\alpha \in a(T)$. This contradiction proves the theorem.

In the next chapter, we will prove that the hyponormal operators satisfy the condition G_1 and hence theorem 2.B becomes a particular case of theorem 2.3. We derive the following corollary from this theorem:

COROLLARY 2.2. *If* T *satisfies the condition* G_1 *and* α *is an isolated point of* $s(T)$, *then* $\alpha \in a(T)$.

PROOF: Since every isolated point of $s(T)$ is a semi-bare point of $s(T)$, the corollary follows immediately from the main theorem.

Regarding the isolated points of $s(T)$, when T is hyponormal, J. G. Stampfli[1)] proved the following theorem:

THEOREM 2.C. *If* T *is hyponormal and* α *is an isolated point of* $s(T)$, *then* $\alpha \in p(T)$.

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1) J. G. Stampfli [26]

For a hyponormal operator T, T^2 need not be hyponormal as was shown by P. R. Halmos[1] by means of an example. This implies that if T_1 and T_2 are two hyponormal operators which commute, then their product $T_1 T_2$ is not necessarily hyponormal. As mere commutativity is not enough to ensure the hyponormality of the product, it seems desirable to find conditions under which the product of two hyponormal operators is also hyponormal. Our next theorem is an attempt in that direction.

<u>THEOREM 2.4.</u> <u>If</u> T_1 <u>and</u> T_2 <u>are two hyponormal operators such that</u> $T_1 T_2^* = T_2^* T_1$, <u>then</u> $T_1 T_2$ <u>is hyponormal</u>.

<u>PROOF:</u> $(T_1T_2)^* (T_1T_2) = T_2^* T_1^* T_1 T_2 \geq T_2^* T_1 T_1^* T_2 =$

$T_1 T_2^* T_2 T_1^* \geq T_1 T_2 T_2^* T_1^* = (T_1 T_2) (T_1 T_2)^*$

i.e. $T_1 T_2$ is hyponormal.

T. Andô[2], S. K. Berberian[3] and J.G.Stampfli[4] proved independently of each other that if T is a compact hyponormal operator, then T is normal. We generalise this result in the form of the following theorem:

<u>THEOREM 2.5.</u> <u>If</u> T <u>is hyponormal such that</u> T^p <u>is compact, where</u> p <u>is an integer</u> ≥ 1, <u>then</u> T <u>is normal and consequently compact</u>.

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1) P. R. Halmos [10] 2) T. Andô [1] 3) S. K. Berberian [3]
4) J. G. Stampfli [26]

To prove this theorem, we need certain results which we state below in the form of lemmas.

LEMMA 2.3: [T. Andô [1]] If T is hyponormal, then s(T) contains a complex number α such that $\|T\| = |\alpha|$.

LEMMA 2.4. [A. C. Zaanen [32] page 317] If T is a normal operator such that T^p (p is an integer ≥ 1) is compact, then T is itself compact.

LEMMA 2.5. Let T be a hyponormal operator. If the proper subspaces of T are a total family, then T is normal.

PROOF OF LEMMA 2.5: If $N_T(\alpha)$ and $N_T(\beta)$ are the proper subspaces of T corresponding to two distinct proper values α and β, then we have

(i) $N_T(\alpha) \subseteq N_{T^*}(\bar{\alpha})$;

(ii) $N_T(\alpha)$ reduces T and $T/N_T(\alpha)$ is normal ;

(iii) $N_T(\alpha) \perp N_T(\beta)$.

Now the lemma follows from the given hypothesis with the help of properties listed above.

PROOF OF THEOREM 2.5: For the case p = 1, see S. K. Berberian [3] .

Consider the case p > 1. T being hyponormal, we have $\|T\| = |\alpha|$ for some $\alpha \in s(T)$ by lemma 2.3. Since

T^p is compact, $\alpha^p \in p(T^p)$. Now the relation $p(T^p) = \{\alpha^p ; \alpha \in p(T)\}$ implies that $\alpha \in p(T)$ i.e. $p(T) \neq \emptyset$.

Denote by H_1, the smallest linear subspace of H which contains every proper subspaces of T. Then $H_1 \neq \{\theta\}$. Let $H_2 = H_1^{\perp}$. The subspace H_2 reduces T and the restrictions T/H_2 and T^p/H_2 are respectively hyponormal and compact. If $s(T/H_2) \neq \{0\}$, it would contain a non-zero number β such that $\|T/H_2\| = |\beta|$. Again, as above, β is a proper value of T/H_2. This is contrary to the definition of H_1. Hence $s(T/H_2) = \{\theta\}$ i.e. $H_2 = \{\theta\}$. In other words, the proper subspaces of T are total. Hence T is normal by lemma 2.5 and compact by lemma 2.4 .

S. K. Berberian[1)] proved that if T is hyponormal, then $s(T^*) = a(T^*)$. Here we give an easy and direct proof of this result.

<u>THEOREM 2.6:</u> <u>If</u> T <u>is</u> <u>hyponormal</u>, <u>then</u> $r(T^*) = \emptyset$.

<u>PROOF:</u> If possible, let $\alpha \in r(T^*)$. Then $\bar{\alpha} \in p(T)$. Since T is hyponormal, this leads to the conclusion that $\alpha \in p(T^*)$, a contradiction. Hence $r(T^*) = \emptyset$.

This elementary result plays an important part in the proof of the main theorem of chapter V of the present thesis.

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1) S. K. Berberian [3]