CHAPTER IV

CARTESIAN DECOMPOSITION OF A HYPONORMAL OPERATOR AND SPECTRAL RELATIONS

Let $T = A + iB$, where $A = \frac{T + T^*}{2}$, $B = \frac{T - T^*}{2i}$,
be the cartesian decomposition of a normal operator. With the help of the 'Spectral Integral Theory', it is very easy to obtain the following relations between the spectra of A , B and T :

$$s(A) = \{Re\alpha \; ; \; \alpha \in s(T)\}, \; s(B) = \{Im\alpha \; ; \; \alpha \in s(T)\} \qquad ....(4.1)$$

$$p(A) = \{Re\alpha \; ; \; \alpha \in p(T)\}, \; p(B) = \{Im\alpha \; ; \; \alpha \in p(T)\} \qquad ....(4.2)$$

It was, indeed, proved by S. L. Jamison[1)] that for $\alpha \in p(T)$, where $\alpha = s + it$, s and t being respectively the real and imaginary parts of α,
$s \in p(A)$, $t \in p(B)$ and $N_T(\alpha) = N_A(s) \cap N_B(t)$. ....(4.3)

Recently, C. R. Putnam[2)] extended the result (4.1) to hyponormal operators. He proved again these relations with the help of the 'Spectral Integral Theory'. In the present chapter, we prove these relations without taking recourse to the 'Spectral Integral Theory' and only by using

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1) S. L. Jamison [13] 2) C. R. Putnam [21]

elementary methods. Moreover, our proof is simpler and shorter than that of C. R. Putnam. Actually, we prove the following two theorems:

<u>THEOREM 4.1.</u> *If* $T = A + iB$ *is a hyponormal operator, then*

(i) $s(A) = \{\mathrm{Re}\,\alpha \; ; \; \alpha \in s(T)\}$;

$s(B) = \{\mathrm{Im}\,\alpha \; ; \; \alpha \in s(T)\}$

and in particular,

(ii) $p(A) = \{\mathrm{Re}\,\alpha \; ; \; \alpha \in p(T)\}$;

$p(B) = \{\mathrm{Im}\,\alpha \; ; \; \alpha \in p(T)\}$;

(iii) $N_A(s) = \mathrm{U} \oplus N_T(\alpha)$ *for all* $\alpha \in p(T)$ *and* $\mathrm{Re}\,\alpha = s$;

$N_B(t) = \mathrm{U} \oplus N_T(\alpha)$ *for all* $\alpha \in p(T)$ *and* $\mathrm{Im}\,\alpha = t$ *and*

(iv) *if* $\alpha = s + it \in p(T)$, *then* $s \in p(A)$, $t \in p(B)$ *and* $N_T(\alpha) = N_A(s) \cap N_B(t)$.

<u>THEOREM 4.2.</u> *If* $T = A + iB$ *is an operator such that* $\Sigma(T)$ *is a spectral set for* T, *then*

$s(A) = \{\mathrm{Re}\,\alpha \; ; \; \alpha \in s(T)\}$ *and*

$s(B) = \{\mathrm{Im}\,\alpha \; ; \; \alpha \in s(T)\}$.

It is sufficient to prove the results for A only as the proof of the corresponding results for B are similar in both of the theorems.

<u>PROOF OF THEOREM 4.1</u>: Let $\alpha \in p(T)$ and $N_T(\alpha)$ be the corresponding proper subspace. Then $\bar{\alpha} \in p(T^*)$ and $N_T(\alpha) \subseteq N_{T^*}(\bar{\alpha})$. Hence

$$Ax = \frac{T + T^*}{2} x = \frac{\alpha + \bar{\alpha}}{2} x = \mathrm{Re}\alpha . x \text{ for all } x \in N_T(\alpha).$$

Thus

$$\{\mathrm{Re}\alpha \; ; \; \alpha \in p(T)\} \subseteq p(A) \qquad \ldots\ldots\ldots\ldots(4.a)$$

and $U \oplus N_T(\alpha) \subseteq N_A(s)$ for all $\alpha \in p(T)$ and $\mathrm{Re}\alpha = s$ ...(4.b)

Conversely, let t(real) $\in p(A)$ and $N_A(t)$ be the corresponding proper subspace. Then

$$((T - tI) + (T - tI)^*)x = (C + C^*)x = \theta$$

for all $x \in N_A(t)$, where $C = T - tI$ is also hyponormal. Since $C C^* x = C^* Cx$ and $\|Cx\| = \|C^* x\|$ for every $x \in N_A(t)$, $N_A(t)$ reduces C and the restriction of C to $N_A(t)$ (i.e. $C/N_A(t)$) is normal. Hence there exists a complex number $\beta \in s(C/N_A(t)) \subseteq s(T - tI)$ such that $\|C/N_A(t)\| = |\beta|$. But $(C + C^*)x = \theta$ for all $x \in N_A(t)$ implies that $\mathrm{Re}\beta = 0$ and $\beta \in p(T - tI)$ i.e. $t + \beta \in p(T)$ and

$$N_A(t) \subseteq U \oplus N_T(\alpha) \text{ for } \alpha \in p(T) \text{ and } \mathrm{Re}\alpha = t \qquad \ldots\ldots\ldots\ldots(4.c)$$

Combining (4.c) with (4.a) and (4.b), we have

$$p(A) = \{\mathrm{Re}\alpha \; ; \; \alpha \in p(T)\} \quad \text{and}$$

$$N_A(s) = U \oplus N_T(\alpha) \text{ for } \alpha \in p(T) \text{ and } \mathrm{Re}\alpha = s.$$

Now, let $\alpha \in c(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(T^* - \bar{\alpha} I)x_n\| \leq \|(T - \alpha I)x_n\| \to 0.$$

Consequently,

$$\|(A - \text{Re}\alpha . I)x_n\| \to 0 \text{ i.e. } \text{Re}\alpha \in a(A) = s(A).$$

Thus, we have proved that $\{\text{Re}\alpha \; ; \; \alpha \in a(T)\} \subseteq s(A)$ ....(4.d)

If $\alpha \in r(T)$, then α is an interior point of $\Sigma(T)$. Let L be the line drawn through α parallel to the axis of y. If α_1, α_2 are the end-points of $\Sigma(T) \cap L$, then α_1, α_2 are the boundary points of $\Sigma(T)$. Hence α_1, $\alpha_2 \in a(T)$ and consequently $\text{Re}\alpha = \text{Re}\alpha_1 = \text{Re}\,\alpha_2 \in s(A)$ by (4.d). Since $s(T) = a(T) \cup r(T)$, we have $\{\text{Re}\alpha \; ; \; \alpha \in s(T)\} \subseteq s(A)$.

Now, if t(real) $\in s(A)$, then $t \in a(A)$. Since the hypomormality is preserved under *-isomorphism, we may assure that t is a proper value of A[6]. Then it follows from what we have already proved that there exists a complex number $\alpha \in s(T)$ such that $\text{Re}\alpha = t$. This completes the proofs of parts (i), (ii) and (iii).

Since the proper subspaces corresponding to distinct proper-values of a hyponormal operator are orthogonal, the proof of (iv) follows from (ii) and (iii).

The following interesting and important results can be derived as corollaries of this theorem:

COROLLARY 4.1. *If* T *is hyponormal, then* $\Sigma(T) = Cl(W(T))$.

PROOF: It is sufficient to prove that every extreme point of $Cl(W(T))$ belongs to $a(T)$. Let γ be an extreme point of $Cl(W(T))$. Since $\alpha T + \beta I$ is hyponormal, $Cl(W(\alpha T + \beta I)) = \alpha . Cl(W(T)) + \beta$ and $\alpha\gamma + \beta$ is an extreme point of $Cl(W(\alpha T + \beta I))$, there is no loss of generality in assuming that $\gamma = 0$ is an extreme point of $Cl(W(T))$ and $Re\ Cl(W(T)) \geq 0$. If $T = A + iB$ be the cartesian decomposition of T, then $A \geq 0$ by our assumption. The relation $(Tx , x) = (Ax , x) + i(Bx , x)$ for all $x \in H$ implies that 0 is an extreme point of $Cl(W(A))$ i.e. $0 \in Cl(W(A))$ and as $A \geq 0$, $0 \in a(A)$ by [A.E.Taylor [30] page 330]. Since 0 is an extreme point of $Cl(W(T))$, this implies that $0 \in a(T)$ by (i) of theorem 4.1 . In case $A = 0$, then $T = iB$ i.e. $\Sigma(T) = i\Sigma(B) = i\ Cl(W(B)) = Cl(W(T))$. This completes the proof of corollary 4.1 .

COROLLARY 4.2. *If* T *is hyponormal, then* $E(W(T)) \subseteq p(T)$.

PROOF: Since $p(\alpha T + \beta) = \alpha p(T) + \beta$, as in corollary 4.1 , there is no loss of generality in assuming that 0 is an extreme point of $W(T)$ with $Re\ W(T) \geq 0$. If $T = A + iB$, then again $A \geq 0$. Since $0 \in W(T)$, there exists a unit vector x such that

$$(Tx , x) = 0 = (Ax , x) + i(Bx , x)$$

i.e. $(Ax , x) = 0 = (Bx , x)$. This leads to the conclusion that $Ax = \theta$ by [A. C. Zaanen [32] page 216]. Hence if $A \neq 0$, then $0 \in p(A)$ and as 0 is an extreme point of $W(T)$, $0 \in p(T)$ by (ii) of theorem 4.1 . If $A = 0$, then $T = iB$ and 0 is an extreme point of $W(B)$ with $(Bx , x) = 0$. Hence $Bx = \theta$ i.e. $Tx = iBx = \theta$ or $0 \in p(T)$.

<u>COROLLARY 4.3.</u> <u>Let</u> T <u>be</u> <u>hyponormal,</u> <u>then</u> W(T) <u>is</u> <u>closed</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $E(\Sigma(T)) \subseteq p(T)$.

<u>PROOF:</u> Since $Cl(W(T)) = \Sigma(T)$ by corollary 4.1 and $p(T) \subset W(T)$, the 'if' part follows immediately from the definitions.

Now, if $W(T) = Cl(W(T)) = \Sigma(T)$, then $E(\Sigma(T)) = E(W(T))$. Hence, the 'only if' part follows from corollary 4.2 .

C. H. Meng[1)] proved the following theorem:

<u>THEOREM 4.A.</u> <u>If</u> N <u>is</u> <u>a</u> <u>normal</u> <u>operator,</u> <u>then</u> W(N) <u>is</u> <u>closed</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $E(\Sigma(N)) \subseteq p(N)$.

Also C. R. MacCLUER[2)] proved recently the following theorem:

<u>THEOREM 4.B.</u> <u>For</u> <u>a</u> <u>normal</u> <u>operator</u> N, $E(W(N)) \subseteq p(N)$.

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1) C. H. Meng [16] 2) C. R. MacCLUER [14]

It will be observed that our corollaries 4.2 and 4.3 proved above are extensions of theorems 4.B and 4.A respectively to hyponormal operators. It should be mentioned that J. G. Stampfli[1)] has also proved corollary 4.2 .

To prove theorem 4.2 , we need the following known result which we state below in the form of a lemma.

LEMMA 4.1. *For any operator* T, *if* $\alpha \in s(T)$ *and* $|\alpha| = \|T\|$, *then* $\alpha \in a(T)$ *and the approximate proper vectors of* T *belonging to* α *are those of* T^* *belonging to* $\bar{\alpha}$.

PROOF OF THEOREM 4.2: Let $\alpha \in s(T)$. Since $\Sigma(T)$ is a spectral set for T, we have

$$\|(T - \alpha I)\| \leq \sup \left\{ |t - \alpha| \ ; \ t \in \Sigma(T) \right\} \leq \|(T - \alpha I)\|$$

(Because $\Sigma(T) - \alpha = \Sigma(T - \alpha I)$) for the rational function $u(t) = t - \alpha$. Since $\Sigma(T)$ is the closed convex hull of $s(T)$, there exists an element $\beta \in s(T) \subseteq \Sigma(T)$ such that $\|T - \alpha I\| = |\beta - \alpha|$. Also $\beta - \alpha \in s(T - \alpha I)$.

Using the spectrality of $\Sigma(T)$ again, this implies that $\|(T - \beta I)\| = |\alpha - \beta|$ and $\alpha - \beta \in s(T - \beta I)$. Hence there exists a sequence $\{x_n\}$ of unit vectors such that

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1) J. G. Stampfli [28]

$\|((T - \beta I) - (\alpha - \beta)I)x_n\| = \|(T - \alpha I)x_n\| \to 0$ and

$\|((T^* - \bar{\beta}I) - (\bar{\alpha} - \bar{\beta})I)x_n\| = \|(T^* - \bar{\alpha}I)x_n\| \to 0$ by lemma 4.1.

Consequently, $\|(A - \mathrm{Re}\,\alpha\, I)x_n\| \to 0$ i.e. $\mathrm{Re}\,\alpha \in s(A)$. Thus

$$\{\mathrm{Re}\,\alpha \,;\, \alpha \in s(T)\} \subseteq s(A).$$

Conversely, let a(real) $\in s(A)$. Then $a \in \Sigma(A) = Cl(W(A))$. But the relation

$$(Tx , x) = (Ax , x) + i(Bx , x) \text{ for all } x \in H$$

implies that Re Cl(W(T)) = Cl(W(A)) i.e. $a \in$ Re Cl(W(T)). Since $\Sigma(T) = Cl(W(T))$ [24], we can choose a real number b such that a + ib is a boundary point of $\Sigma(T)$ i.e. $a + ib \in s(T)$. This completes the proof of the theorem.