CHAPTER V

HYPONORMAL OPERATORS AND THEIR ADJOINTS

In the year 1956, W. A. Beck and C. R. Putnam[1] proved the following theorem in a joint paper:

THEOREM 5.A. *Let N be a normal operator. If* $AN = N^*A$ *for an arbitrary invertible operator* $A = UR$, *for which U is cramped, then* $N = N^*$ *i.e. N is self-adjoint.*

The hypothesis in the above theorem does not only mean that the normal operator N is similar to its adjoint N^*, but imposes an additional condition on the operator A affecting this similarity in that U is cramped. As we known that a bilateral shift is unitary and is unitarily equivalent to its adjoint, without being self-adjoint, it follows that the restriction on A in the above theorem is optimal.

In the year 1962, S. K. Berberian[2] extended the above theorem of Beck and Putnam for any B^*-algebra as follows:

THEOREM 5.B. *Let Z be an element of a* B^**-algebra with unit I and U a unitary element of this algebra* (i.e. $UU^* = U^*U = I$) *with cramped spectrum. Then* $UZU^* = Z^*$ *implies that* $Z = Z^*$ i.e. *Z is self-adjoint.*

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1) W. A. Beck and C. R. Putnam [2]

2) S. K. Berberian [5]

The fact that this result of Berberian is an extension of that of Beck and Putnam follows from the following theorem due to Putnam[1]:

THEOREM 5.C. *Let N_1 and N_2 be two normal operators. If, for an invertible operator $A = UR$, $N_2 = AN_1 A^{-1}$, then* $N_2 = UN_1U^*$.

Recently, C. A. McCarthy[2] made a slight improvement in both these results in the following form:

THEOREM 5.D. *Let $\emptyset$ be a linear transformation on an algebra with involution such that*

(i) *if* X *is self-adjoint, then* $\emptyset(X)$ *is also self-adjoint and*

(ii) $-1 \notin p(\emptyset)$, *then*
$\emptyset(Z) = Z^*$ *implies* $Z = Z^*$.

In the present chapter, we extend to hyponormal operators the above result of Beck and Putnam on normal operators. Actually we prove the following theorem[3]:

THEOREM 5.1. *Let N be a hyponormal operator. If* $AN = N^*A$ *for an arbitrary operator* A, *for which* $0 \notin Cl(W(A))$, *then* $N = N^*$

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1) C. R. Putnam [19] 2) C. A. McCarthy [15] 3) I.H.Sheth [25]

Taking into consideration Berberian's result that an operator A is invertible with $A = UR$, where U is cramped provided that $0 \notin Cl(W(A))$, it follows that the condition $0 \notin Cl(W(A))$ in our result is stronger than the condition on A (via U is cramped) in Beck and Putnam's result.

We give below a list of lemmas which are needed in the proof of this result.

LEMMA 5.1. *Let T be a hyponormal operator and let* $\alpha_1, \alpha_2 \in a(T)$, $\alpha_1 \neq \alpha_2$. *If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of unit vectors of H such that* $\|(T - \alpha_1 I)x_n\| \to 0$ *and* $\|(T - \alpha_2 I)y_n\| \to 0$, *then* $(x_n, y_n) \to 0$.

PROOF: We have

$$(\alpha_1 - \alpha_2)(x_n, y_n) = (\alpha_1 x_n, y_n) - (x_n, \bar{\alpha}_2 y_n).$$

$$= ((\alpha_1 I - T)x_n, y_n) + (x_n, (T^* - \bar{\alpha}_2)y_n).$$

Hence

$$|(\alpha_1 - \alpha_2)(x_n, y_n)| \leq \|(T - \alpha_1 I)x_n\| \|y_n\| + \|(T^* - \bar{\alpha}_2 I)y_n\| \|x_n\|$$

$$\to 0 \text{ as } n \to \infty.$$

i.e. $(x_n, y_n) \to 0$.

LEMMA 5.2. *If T is hyponormal, then* $s(T^*) = a(T^*)$.

This result has been proved in chapter II (as theorem 2.6).

LEMMA 5.3. [C. R. Putnam [21]]. *If T is hyponormal such that $s(T)$ is a set of real numbers, then T is self-adjoint.*

LEMMA 5.4. *If an operator A is similar to an operator B, then A is bounded below if and only if B is bounded below. In other words, if A and B are similar, then $a(A) = a(B)$.*

PROOF: Let $A = T^{-1}B\,T$ for an invertible operator T. Now if B is bounded below, then $B^*B \geq \alpha I$ for some constant $\alpha > 0$. Since T is invertible, there exist constants $\beta > 0$ and $\gamma > 0$ such that $T^*\,T \geq \beta\, I$ and $(T\,T^*)^{-1} = T^{*-1}T^{-1} \geq \gamma\, I$. Now $A^*A = T^*B^*\,T^{*-1}\,T^{-1}\,B\,T = (BT)^*\,T^{*-1}\,T^{-1}\,B\,T \geq (BT)^*\gamma\, I\, B\, T$

$$= \gamma\, T^*\,B^*\,B\,T \geq \gamma\, T^*\,\alpha\, I\, T = \gamma\,\alpha\, T^*\,T \geq \alpha\,\beta\,\gamma\, I$$

i.e. A is bounded below. Since the above process is reversible, the stated result follows.

The relation $a(A) = a(B)$ follows from the following two observations:

(i) if A is similar to B, then $A - \alpha I$ is similar to $B - \alpha I$ for all complex number α ;

(ii) $\alpha \notin a(A)$ if and only if $A - \alpha I$ is bounded below.

PROOF OF THEOREM 5.1: Since $0 \notin Cl(W(A))$, A is invertible. Hence $N = A^{-1}\,N^*A$ and it follows from lemma 5.2 and 5.4 that $s(N) = s(N^*) = a(N^*) = a(N)$.

In order to complete the proof of the theorem, it is sufficient, by virtue of lemma 5.3, to prove that s(N) is real. Assume on the contrary, that there exists an $\alpha \in s(N)$ such that $\alpha \neq \bar{\alpha}$. Since $\alpha \in s(N) = a(N)$, there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(N^* - \bar{\alpha} I)x_n\| \leq \|(N - \alpha I)x_n\| \to 0.$$

Since $0 \notin Cl(W(A))$, the relation

$$\|(N^* - \bar{\alpha}I)x_n\| = \|(ANA^{-1} - \bar{\alpha}I)x_n\| = \|A(N - \bar{\alpha}I)A^{-1}x_n\| \to 0$$

implies that $\|(N - \bar{\alpha}I)A^{-1}x_n\| \to 0$. Hence

$$(x_n , A^{-1}x_n) = (AA^{-1}x_n , A^{-1}x_n) \to 0$$

by lemma 5.1 . Putting $y_n = A^{-1}x_n / \|A^{-1}x_n\|$, we have $\|y_n\| = 1$ and $(Ay_n , y_n) \to 0$ i.e. $0 \in Cl(W(A))$ which contradicts the hypothesis that $0 \notin Cl(W(A))$. This completes the proof of the theorem.

We deduce, as a corollary, the following result:

<u>COROLLARY 5.1</u>. <u>Let</u> N <u>be a semi-normal operator</u>. <u>If</u> $AN = N^*A$ <u>for an arbitrary operator</u> A, <u>for which</u> $0 \notin Cl(W(A))$, <u>then</u> N <u>is self-adjoint</u>.

<u>PROOF:</u> Let N^* be hyponormal. The proof of the main theorem shows that $0 \notin Cl(W(A))$ implies $0 \notin Cl(W(A^{-1}))$. Now the

relation $AN = N^*A$ implies that $A^{-1}N^* = N A^{-1}$ i.e. $BM = M^*B$ where $M = N^*$ is hyponormal and $0 \notin Cl(W(B)) = Cl(W(A^{-1}))$. Hence $M = M^*$ by the main theorem i.e. $N = N^*$.