CHAPTER VI

HYPONORMAL OPERATORS AND POINT SPECTRUM

It is known that a hyponormal operator T with a pure point spectrum is normal, whether the Hilbert space under consideration is separable or not. The question naturally arises as to whether the condition that T has a pure point spectrum can be replaced by a weaker condition such as $s(T) = p(T)$, in the above result. Thus we may pose the question:

If T is hyponormal with $s(T) = p(T)$, is T normal ?

The present chapter is devoted to the study of this question. In fact, our investigation show that for a separable Hilbert space, the answer to the above question is in the affirmative. If, however, the underlying Hilbert space is non-separable, then, in general, the answer to the above question is in the negative, as we show by means of an example. Indeed, we prove the following theorems:

THEOREM 6.1. *Let T be a hyponormal operator with $s(T) = p(T)$. Then T is normal if the Hilbert space under consideration is separable.*

THEOREM 6.2. *If H is non-separable, then there exists a non-normal hyponormal operator with* $s(T) = p(T)$.

We state below some known results in the form of lemmas which will be needed in the proof of theorem 6.1.

LEMMA 6.1. *Let* T *be hyponormal and suppose that* $s(T) = p(T)$, *then* $s(T + T^*) = p(T + T^*)$. *If also H is separable, then* $s(T + T^*)$ *is a countable set on the real-axis and consequently measure of* $s(T + T^*) = 0$.

This result has been proved in chapter IV (as theorem 4.1).

LEMMA 6.2. [C. R. Putnam [20]]. *If* $C = AB - BA$ *is a commutator of two operators A and B, then* 0 *belongs to the interior of* $Cl(W(C))$, *provided that A (or B) is self-adjoint with spectrum of measure* 0.

LEMMA 6.3. [C. R. Putnam [20]]. *If* $C = AA^* - A^*A$ *is semi-definite, then either* (i) $C = 0$ *or* (ii) $\int_Z dE(\alpha) < I$ *for every set Z of measure zero on the real-axis, where* $\int \alpha dE(\alpha)$ *is the spectral resolution of* $A + A^*$.

PROOF OF THEOREM 6.1:

1st PROOF: Let $C = T^*T - TT^* = (T + T^*)T - T(T + T^*) \geq 0$. By hypothesis, $s(T + T^*) = p(T + T^*)$ and measure of $s(T + T^*)$

is zero by lemma 6.1 . Hence 0 is an interior point of $Cl(W(C))$ by lemma 6.2. Since C is positive-definite, this implies that $Cl(W(C)) = \{0\}$ i.e. $C = 0$. In other words, T is normal.

<u>2nd PROOF:</u> The operator $C = T^*T - TT^*$ is positive-definite and $s(T + T^*) = p(T + T^*)$ by lemma 6.1. Now if $T + T^*$ has the spectral resolution $\int \alpha dE(\alpha)$, then $\int_{s(T+T^*)} dE(\alpha) = I$ and measure of $s(T + T^*)$ is zero. Hence $C = 0$ by lemma 6.3. i.e. T is normal.

The following corollaries can be derived from theorem 6.1:

<u>COROLLARY 6.1.</u> <u>If</u> H <u>is finite-dimensional, then every hyponormal operator is normal</u>.

<u>PROOF:</u> Since $s(T) = p(T)$ in this case, the corollary follows immediately from the theorem.

<u>COROLLARY 6.2.</u> <u>If</u> T <u>is quasi-normal</u> (<u>or sub-normal</u>) <u>with</u> $s(T) = p(T)$, <u>then</u> T <u>is normal provided</u> H <u>is separable</u>.

<u>PROOF:</u> The corollary follows from the inclusion relation between quasi-normal (or sub-normal) and hyponormal operators given in the introduction.

PROOF OF THEOREM 6.2: We prove this theorem by showing the existence, in a constructive way, of a non-normal hyponormal operator with a point spectrum.

We denote by $\mathcal{B}$, the space of all sequences $s = \{y_n\}$, where $y_n \in H$ and $\|y_n\|$ is bounded for $n = 1, 2, \dots$. We define a positive symmetric bilinear function on $\mathcal{B}$ as follows:

For the elements $s = \{y_n\}$, $t = \{z_n\}$ of $\mathcal{B}$, $\phi(s,t) = \text{glim}\,(y_n, z_n)$, where glim denotes the Banach limit of bounded sequence of complex numbers. Let $N = \{s\ ;\ \phi(s,s) = 0\}$ and let Hilbert space K be the completion of the quotient vector space $\mathcal{P} = \mathcal{B}/N$. Let T^o be the operator on K corresponding to the operator T on H as defined in [6] . The mapping $T \to T^o$ of $\mathcal{L}(H)$ into $\mathcal{L}(K)$ has the following properties:

$$(S + T)^o = S^o + T^o\ ;\ (ST)^o = S^oT^o,\ (T^*)^o = (T^o)^*,$$
$$I^o = I,\ \|T^o\| = \|T\| \text{ and } T \geq 0 \text{ if and only if } T^o \geq 0.$$

The main theorem of [6] is.

THEOREM 6.A. *For every operator T on H*, $a(T) = a(T^o) = p(T^o)$.

Let $\{x_n\}$ be an orthonormal basis of H. Let $S = \alpha\ ;\ |\alpha| \leq 1$ be the closed unit disc in the complex plane and $\alpha_n \in S$ be the sequence such that every $\alpha \in S$ is the limit of a subsequence of α_n . We define a mapping A of H into H as follows:

If $x = \Sigma\beta_n\, x_n \in H$, then $Ax = \Sigma\beta_n\, \alpha_n\, x_n$, in short, $Ax_n = \alpha_n\, x_n$ for $n = 1, 2, \ldots$. Then we have

(i) A is a normal operator ;

(ii) Every $\alpha \in S$ is an approximate proper value of A ;

(iii) $s(A) = a(A) = S$;

(iv) If A^o is the corresponding operator on K, then A^o is normal, and $s(A^o) = a(A^o) = a(A) = S = p(A^o)$ by theorem 6.A.

Let B be the one-sided shift operator on H i.e. $Bx_n = x_{n+1}$ for $n = 1, 2, \ldots$. Then we have

(i') B is hyponormal ;

(ii') $s(B) = S$ and $a(B) = S - \{0\}$;

(iii') If B^o is the corresponding operator on K, then B^o is hyponormal, $s(B^o) = S$ and $a(B^o) = p(B^o) = a(B) = S - \{0\}$.

Now we consider the tensor product $T = A^o \oplus B^o$ of A^o and B^o on the product space $K \oplus K$.

Since

$$T^*T - T\,T^* = (A^\circ \oplus B^\circ)^*(A^\circ \oplus B^\circ) - (A^\circ \oplus B^\circ)\,(A^\circ \oplus B^\circ)^*,$$

$$= ((A^*)^\circ \oplus (B^*)^\circ)\,(A^\circ \oplus B^\circ) - (A^\circ \oplus B^\circ)((A^*)^\circ \oplus (B^*)^\circ),$$

$$= ((A^*)^\circ A^\circ - A^\circ(A^*)^\circ) \oplus (B^*)^\circ B^\circ +$$

$$A^\circ(A^*)^\circ \oplus ((B^*)^\circ B^\circ - B^\circ(B^*)^\circ),$$

$$\geq 0.$$

i.e. T is hyponormal.

Also, we have $s(T) = p(T) = S$. Because, if $\alpha = re^{i\theta}$, $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$, then $r \in p(A^\circ)$ and $e^{i\theta} \in p(B^\circ)$ by (iv) and (iii'). Hence there exist unit vectors x and y of K such that $A^\circ x = rx$ and $B^\circ y = e^{i\theta}y$. Consequently,

$$(A^\circ \oplus B^\circ)\,(x \oplus y) = (A^\circ x) \oplus (B^\circ y),$$

$$= rx \oplus e^{i\theta}y,$$

$$= (re^{i\theta})\,(x \oplus y)$$

i.e. $re^{i\theta} \in p(T)$.

Also, if $\alpha = 0$, then $\alpha = 0.\beta$, where $\beta \neq 0 \in S$. Now $0 \in p(A^\circ)$ and $\beta \in p(B^\circ)$. Hence by an argument similar to the one given above, $\alpha = 0.\beta \in p(T)$. In other words, $S \subseteq p(T) \subseteq s(T)$. Since T is hyponormal and $\|T\| \leq \|A^\circ\|.\|B^\circ\|=1.1=1$, we have $s(T) = p(T) = S$.

Thus T is a non-normal hyponormal operator with a point spectrum.