CHAPTER - VI

ELASTICITY OF SUBSTITUTION

1. Nature of the CES Production Function :

The constant elasticity of substitution (CES) or SMAC production function[1] is a definite improvement over the Douglas type of production function examined in the preceding chapter. This function allows the elasticity of substitution to take any (constant) value. Since the function is no longer bound to unit elasticity, it becomes suggestive of the direction of change in relative shares of the factors of production. The assumption of constant elasticity of substitution in the function, although, is a restriction on the form of production possibilities, we have to have the precise nature of the elasticity of substitution to arrive at certain estimates, particularly when the general formula of the elasticity of substitution does not allow for its direct calculations. The obvious advantage in using this function lies in the fact that, to explain the substitutability between different inputs, the economists need not rely on the extreme production

1. K.J.Arrow ; H.B.Chenery, B.Minhas and R.M.Solow : "Capital Labor Substitution and Economic Efficiency", Review of Economics and Statistics, August, 1961. For the generalisation of the production function, see V. Mukerji, "A Generalised SMAC Function with constant ratios of Elasticity of Substitution", Review of Economic Studies, October, 1963.

functions like Leontief and Douglas types where the values of elasticity of substitution are tied up either with zero or unity. The CES production function in fact includes both these functions as special cases.

The authors of the CES production function found different degrees of substitutability in different types of production. Because of numerous technological alternatives available in different sectors, they argue that the uniform substitutability in different industries is most unlikely. Their study is primarily based on the empirical observations that the value added per worker within a given industry varies across countries with the wage rate. "A regression of the labour productivity on the wage rate shows a highly significant correlation in all industries and also a considerable variation in the regression coefficients".[2]

SMAC tested two relations :

$$Y = c + dw \qquad \ldots(i)$$

$$\text{and} \quad Y = Aw^{\sigma} \text{ or } \log Y = \log A + \sigma \log w \qquad \ldots(ii)$$

Where Y = product per worker, w = wage rate, and σ is a constant equal to the elasticity of substitution, (for proof see mathematical note given in Appendix VI-2).

The authors fitted these relations to international data relating to 24 three-digit industries of 19 countries.

2. SMAC, Ibid, p.225.

Both the relations showed good fits. However, the logarithmic (second) relation, according to the authors, is found better. The logarithmic relation is also tested in the case of the U.S. manufacturing industries by Solow.[3]

The CES production function (under constant returns to scale assumption) is defined as :

$$V = \gamma \left\{ \delta b^{-\rho} + (1 - \delta) a^{-\rho} \right\}^{-1/\rho}$$

where V = output; a and b are labour and capital inputs respectively; and δ, ρ and γ are distribution, substitution and efficiency parameters respectively. The production function defined in this way gives the value of elasticity of substitution as :

$\sigma = \frac{1}{1+\rho}$ which also happens to be the value of the exponent of the wage rate in the relation

$Y = A\ w^{\sigma}$ (see Appendix VI-2)

2. Scope of the Present Analysis :

The authors of the CES production function, as noted above, have tested the relation to different industries spread over 19 countries. But, as Solow[4] points out, the technological conditions are bound to differ from country to country due to differences in industrial

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3. R.M.Solow: "Capital, Labor and Income in Manufacturing" in the Behaviour of Income Shares, Studies in Income and Wealth, NBER,New York), Vol.27,1964.

4. R.M.Solow: Ibid, p.

strategies followed by different countries. The technological conditions between the ragions within a country, however, are definitely more homogeneous than between countries for a given industry. This section, thus, attempts to estimate the relation on the basis of cross-section data of two-digit ASI manufacturing industries for the year 1962 which has been found to be relatively normal year. Incidentally, it falls exactly in the middle of the period 1959-1965 for which the Annual Survey of Industries data are available. The following regions constitute the observations of the study: Andhra Pradesh, Assam, Bihar, Gujarat, Kerala, Madhya Pradesh, Madras (Tamil Nadu), Maharashtra, Mysore, Orissa, Punjab, Rajasthan, Uttar Pradesh, West Bengal, Delhi and other regions.

By fitting the logarithmic relation to the above data, the elasticities of substitution and other parameters for different industries are estimated. The hypothesis that value added per worker and wages per worker are uncorrelated is being tested in section 3. The relative share of labour in relation to whether the industry is elastic ($\sigma > 1$) or inelastic ($0 < \sigma < 1$) i.e. the distributive aspect of the SMAC production function, is examined in Section 4.

Value added is used as a measure of output.[5] Labour input refers to the number of workers employed[6]. Wages to workers include all payments made in cash as compensation for work done during the year.

3. Estimates of Elasticity of Substitution :

Table VI-1 shows the results of the regressions of value added per worker on wage rate taken in the logarithmic forms. Column 4 of the table shows the estimates of the elasticity of substitution between labour and capital in different industries. To check the goodness of the fit the standard errors of the coefficients are presented in column 5. It can be seen that there are eleven industries namely, food (including beverage), tobacco, textiles, furniture and fixtures, paper and paper products, printing and publishing, rubber products, chemicals, basic metal

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5. For calculations that follow we have used output net of depreciation. As long as depreciation remains a stable proportion of output over the observations for a given industry, the estimate of σ obtained from logarithmic relationship will not be affected even if we take output gross of depreciation (as done in the preceding chapter).

6. The number of workers is computed by taking the total attendance of workers in all the shifts on all working days and dividing it by the number of days worked. While the number of man-hours worked during the year is calculated by multiplying the number of workers employed in each shift by the number of hours in the shift and aggregating the products for all shifts on all the working days in the year. Thus, so long as a working day (or shift) has more or less a uniform number of hours, it makes no difference whether we consider man-hours worked or total number of workers.

Table VI-1

Regressions of Value Added Per Worker on Wages per Worker; and Rank Correlation Between Wage Share and Wage Rate.

I. No.	Industry	Degrees of freedom	Log a	b	S.e. of b	Rank correlation coeff. between wageshare and wage rate
	1	2	3			6
20-21	Food(including Beverage)	14	0.2537	1.1136	0.1145	- .0705
22	Tobacco	5	0.9946	0.8027	0.1355	- .1785
23	Textiles	11	0.8478	1.9705	0.1635	+ .2308
24-29	Wearing Apparel*	4	1.2084	0.7060	0.3082	+ .3715
25	Wood and Cork	6	1.4579	0.6450	0.2892	+ .8334
26	Furniture and fixtures	8	-0.3620	1.1809	0.1233	- .3696
27	Paper	6	-0.3713	1.2845	0.1463	- .1904
28	Printing & Publishing	13	-1.4323	1.5392	0.0865	- .4071
30	Rubber products	6	-0.6251	1.3685	0.0905	- .5952
31	Chemicals	9	1.7286	0.6456	0.1760	+ .3546
32-33	Petroleum,Coal & non-metallic mineral	12	3.3862	0.0328	0.2728	- .4417
34	Basic metal industries	9	-0.9428	1.4209	0.0770	- .5727
35	Metal products	8	1.6259	0.6005	0.2718	+ .1031
36	Machinery	11	-0.1227	1.1509	0.1220	- .1373

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Table VI-1 (concluded)

Regressions of Value Added Per Worker on Wages Per Worker; and Rank Correlation Between Wage Share and Wage Rate. (continued).

I. No.	Industry	Degrees of freedom	Log a	b	S.e. of b	Rank correlation coefft. between wageshare and wage rate
	1	2	3	4	5	6
37	Electrical machinery	8	2.3700	0.3884	0.2884	+ .3697
38	Transport-equipment	14	-3.5215	2.1874	0.0610	- .5735
39	Miscellaneous	6	2.7956	0.1767	0.5123	- .3333

* Wearing Apparel includes foot-wear, made up textile goods, leather and Fur products.

Source: Calculated on the basis of data given in Appendix VI-1.

industries, machinery (excluding electrical machinery), and transport equipment which show excellent fit. Wearing apparel, wood & cork, and metal products show the coefficient to be significant at 10 per cent level. This leaves complete failure in the case of petroleum, coal & non-metallic mineral, electrical machinery and miscellaneous industries. On the whole, therefore, the results are quite satisfactory - the SMAC relation fits well to the Indian manufacturing industries. For the poor results in some of the industries it is enough to remember that "The hardy econometrician must learn to take his lickings in the conscious realisation that you can't win 'em all".[7]

Out of the 17 two-digit industries examined, nine industries namely, food (including beverage), textiles, furniture and fixtures, papers and paper products, printing and publishing, rubber products, basic metal, machinery, and transport-equipment show the value of elasticity of substitution greater than one. There are six industries which give the estimates of elasticity of substitution to be statistically different from one. These industries are textiles, printing and publishing, rubber products, petroleum, coal and non-metallic mineral, basic metal, and transport equipment. The values of the elasticity of substitution has ranged from as low as 0.033 in Petroleum,

7. R.M.Solow : Ibid, p.114.

coal and non-metallic mineral to as high as 2.187 in transport equipment. The simple average of the values of the elasticity, however, has turned out to be 1.013.

4. <u>Distributive Aspect of the Function</u> :

The estimates of the elasticity of substitution between labour and capital enable us to relate the observed labour's share with capital labour ratio and/or relative prices of factors of production, and hence to test the distributive aspect of the SMAC relation. Assuming no technical change, the increase in the price of labour relative to the cost of capital would mean an increase in marginal rate of substitution between labour and capital - the shift in inputs would lead to an increase in marginal product of labour relative to marginal product of capital. But the increase in marginal rate of substitution would make the capital/labour ratio to rise more than proportionately if the elasticity of substitution between labour and capital is greater than one. This implies that the relative share of labour would decline with a relative increase in the wage rate. Similarly, the relative share of labour will increase with the increase in wage rate if the elasticity of substitution is less than unity.[8]

8. c.f. C.E.Ferguson: "Cross-section Production Functions and the Elasticity of Substitution in American Manufacturing Industries", <u>Review of Economics and Statistics</u>, August, 1963.

Assuming the cost of capital to be uniform over the regions, thus, we should find the wage rate and the relative labour share to move in the opposite directions in those industries where elasticity of substitution is greater than one, and to move in the same direction where the elasticity of substitution is less than one.

Column 6 of Table VI-1 gives the estimates of coefficients of rank correlation between wage rate and the corresponding share of labour in value added in different industries. It can be easily seen that out of nine elastic industries (industries with elasticity of substitution greater than unity) eight industries show the negative signs before the coefficients. In the case of remaining (eight) inelastic industries, five show positive signs before the coefficients. Thus, out of the total of seventeen industries, thirteen industries confirm the relations; the SMAC production function fits well to the Indian manufacturing industries so far as the distributive aspect is concerned.

5. Conclusion :

The index of substitutability of the factors of production is the elasticity of substitution between factors of production. The formula of the elasticity of substitution, however, does not allow for its direct calculation. It is the SMAC production function that comes to our rescue. The function, although restricted to the constant value of elasticity of substitution, does provide a method of its estimation in different industries. From the estimates of elasticity of substitution, we are in a position to test the distributive aspect of the function.

The estimated values of the elasticity of substitution in different industries has ranged from as low as 0.03 to as high as 2.19. There are nine industries which show the values of elasticity of substitution more than unity. The unweighted arithmetic mean of the values of the elasticity turns out to be 1.013, slightly more than unity. There is, thus, an indication of wage share being inversely related with the variables like capital/labour ratio and wage rate - the hypothesis to be tested in the following Chapter.

APPENDIX VI-1

Value Added Per Worker, Wages Per Worker and Wages Share, By Industry and Region, 1962.

Industry No.		Andhra-Pradesh	Assam	Bihar	Gujarat	Kerala	Madhya Pradesh	Madras	Maharashtra	Mysore	Orissa	Punjab	Rajasthan	Uttar Pradesh	West Bengal	Delhi	others
1		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
20& 21	A	3641	4434	3099	3133	906	2040	8492	7169	2595	2146	3984	2715	3085	3529	5331	6439
	B	921	731	1084	1098	397	759	1063	1562	663	473	1173	652	1355	826	1191	1222
	C	.2529	.1649	.3499	.3505	.4378	.3722	.1251	.2172	.2556	.2202	.2944	.2400	.4394	.2276	.2214	.1897
22	A	817	-	-	-	-	1100	3206	2112	682	-	-	-	-	1385	-	8216
	B	453	-	-	-	-	655	795	863	159	-	-	-	-	723	-	2010
	C	.5549	-	-	-	-	.5951	.2479	.4087	.2339	-	-	-	-	.5228	-	.2445
23	A	1523	-	-	3032	2110	2384	3537	3428	2232	-	3120	1876	2494	2579	3977	2369
	B	1063	-	-	1786	1209	1504	1723	2085	1077	-	1152	1028	1397	1255	2107	1091
	C	.6981	-	-	.5892	.5729	.6309	.4873	.6052	.4822	-	.3693	.5478	.5602	.4866	.5297	.4606
24& 29	A	-	-	-	-	1739	-	3367	3276	-	-	-	-	1806	2014	-	2615
	B	-	-	-	-	928	-	1135	1386	-	-	-	-	1183	1043	-	1070
	C	-	-	-	-	.5337	-	.3371	.5864	-	-	-	-	.6550	.5178	-	.4092
25	A	-	3844	-	-	2100	-	1253	2079	1901	-	2214	-	-	2269	-	2506
	B	-	866	-	-	757	-	414	1117	704	-	1011	-	-	993	-	956
	C	-	.2253	-	-	.3606	-	.3293	.5372	.3702	-	.4573	-	-	.4376	-	.3813
26	A	94	-	768	2014	-	-	717	5055	-	-	913	-	770	2659	3480	887
	B	189	-	1067	970	-	-	205	2771	-	-	761	-	668	1148	1804	754
	C	2.0103	-	1.3850	.4812	-	-	.2853	.4691	-	-	.8320	-	.8658	.4314	.5186	.8502
27	A	-	-	-	2602	-	-	-	5608	5500	4916	7132	-	4009	5517	-	4372
	B	-	-	-	1065	-	-	-	1524	1386	1204	1570	-	1551	1752	-	1458
	C	-	-	-	.4064	-	-	-	.2717	.2520	.2449	.2200	-	.3868	.3176	-	.3335
28	A	1708	-	3717	3373	3214	1770	4283	3842	3711	1322	2600	2025	2211	3808	6046	2973
	B	1229	-	1757	1536	1465	1379	1759	1597	1110	1140	1632	1614	1315	1733	2219	1398
	C	.7191	-	.4727	.4553	.4557	.7791	.4106	.4417	.2990	.8622	.6275	.7971	.5946	.4551	.3670	.4702
29	A	-	-	-	3470	4474	-	9108	5123	-	-	9911	-	-	9275	2739	11388
	B	-	-	-	1380	1123	-	2107	1553	-	-	2155	-	-	2643	951	2341
	C	-	-	-	.3978	.2509	-	.2313	.3031	-	-	.2174	-	-	.2849	.3467	.2055

APPENDIX VI-1 (concluded)

Value Added Per Worker, Wages Per Worker and Wages Share, By Industry and Region, 1962 (concluded).

Industry No.		Andhra-Pradesh	Assam	Bihar	Gujarat	Kerala	Madhya-Pradesh	Madras	Maharashtra	Mysore	Orissa	Punjab	Rajasthan	Uttar Pradesh	West Bengal	Delhi	Others
1		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
31	A	6763	-	4340	10175	9581	-	4345	14514	8916	-	2204	-	6122	7203	-	9268
	B	1528	-	2021	1740	2619	-	1098	2517	829	-	1146	-	1005	1678	-	1892
	C	.2259	-	.4655	.1710	.2733	-	.2527	.1734	.0929	-	.5201	-	.1614	.2330	-	.2041
32& 33	A	5014	-	3364	3266	1730	4679	6048	9191	1879	-	1236	4426	1349	2008	1913	5186
	B	1533	-	1229	950	850	1306	1378	1435	898	-	1057	734	886	903	938	1439
	C	.3057	-	.3652	.2908	.4913	.2791	.2277	.1561	.4780	-	.8547	.1656	.6567	.4499	.4900	.2773
34	A	-	-	9879	2130	-	-	2971	4798	2730	1093	2504	-	1860	4920	2431	6032
	B	-	-	3213	1156	-	-	1091	2044	1162	763	1245	-	1007	1788	982	1476
	C	-	-	.3252	.5428	-	-	.3671	.4259	.4258	.6983	.4971	-	.5417	.3633	.4039	.2446
35	A	-	-	3066	1820	4516	-	3916	4615	-	-	3041	-	2443	4525	1990	3043
	B	-	-	620	977	1950	-	1481	1647	-	-	1741	-	1200	1780	1044	1139
	C	-	-	.2021	.5370	.4321	-	.3782	.4002	-	-	.5727	-	.4910	.3934	.5247	.3743
36	A	-	5143	3069	3137	3169	-	3469	4915	7344	1687	2799	-	1774	5046	2150	4538
	B	-	1819	1693	1095	988	-	1497	1902	1725	1417	1378	-	1092	1821	1597	1855
	C	-	.3538	.5517	.3492	.3119	-	.4316	.3865	.2348	.8402	.4923	-	.6156	.3609	.7429	.4087
37	A	3090	2176	-	-	4256	-	3759	6285	-	-	4211	-	3361	6016	3254	4401
	B	795	1826	-	-	1597	-	1459	1851	-	-	1077	-	1312	1797	1003	1773
	C	.2569	.3391	-	-	.3752	-	.3881	.2944	-	-	.2558	-	.3903	.2966	.3030	.4028
38	A	2093	2921	1762	1839	1693	1288	5473	5217	10475	1987	2217	2081	2204	4438	2569	3938
	B	1539	1372	1160	1408	1424	1215	1939	2212	1569	1418	1379	1475	1353	1931	1467	1937
	C	.7351	.4703	.6583	.7655	.8414	.9433	.3543	.4240	.1497	.7121	.6216	.7088	.6136	.4350	.5712	.4919
39	A	-	-	-	1248	799	-	1791	3552	-	-	4675	-	1417	3929	-	3666
	B	-	-	-	427	535	-	1009	1345	-	-	1499	-	884	1473	-	1576
	C	-	-	-	.3422	.6693	-	.5632	.3785	-	-	.3209	-	.6234	.3749	-	.4298

A = Value added per worker; B = Wages per worker; C = Wages share in value added.

Source: Calculated on the basis of data provided in Annual Survey of Industries, 1962.

APPENDIX VI-2

Derivation and Properties of the CES Production Function :

Let the production function be :

$$V = F\ (a,b)$$

where V = output, a and b inputs.

Assuming constant returns to scale,

$V = a\ f(b/a)$

$f_a = f\ (b/a) - \frac{b}{a} f'\ (b/a) = f(x) - x\ f'(x)$ (see also mathematical note of Chapter V).

$f_b = f'(b/a) = f'(x)$ (putting $b/a = x$)

$f_{ab} = -\ f''(b/a)\ b/a^2$

The elasticity of substitution is defined as :

$\sigma = \frac{f_a f_b}{f_{ab} V}$ (see R.G.D. Allen, Mathematical Analysis for Economists, 1960, p.343)

$$= \frac{\{f(x) - x\ f'(x)\}\ f'(x)}{-f''(x).\frac{x}{a}.V}$$

$$= \frac{f'(x)\ \{f(x) - x\ f'(x)\}}{-f''(x)\ x.y}\ ,\ (Y = \frac{V}{a})$$

(i) Under competitive market :

$f_a = w =$ wage rate (a= labour input)

i.e. $f_a = w = f(x) - x\ f'(x)$

$$\therefore \quad \frac{dw}{dw} = f'(x)\,\frac{dx}{dw} - x\,f''(x)\,\frac{dx}{dw} - f'(x)\,\frac{dx}{dw}$$

$$\therefore \quad 1 = -\,x\,f''(x)\,\frac{dx}{dw}$$

Now, $\frac{dx}{dw} = \frac{dx}{dy} \cdot \frac{dy}{dw}$

$$\therefore \quad 1 = -\,x\,f''(x)\,\frac{dx}{dy} \cdot \frac{dy}{dw}$$

$$= -\,\frac{x\,f''(x)}{f'(x)}\,\frac{dy}{dw}$$

Since $y = f(x)$

$\frac{dy}{dx} = f'(x)$

$$\therefore \quad \frac{dy}{dw} = -\,\frac{f'(x)}{x\,f''(x)}$$

$$\therefore \quad \frac{dy}{dw} \cdot \frac{w}{y} = -\,\frac{f'(x)\,w}{f''(x)xy}$$

$$= \frac{f'(x)\,\{f(x) - x\,f'(x)\}}{-f''(x)xy} = \sigma$$

i.e. $\sigma = \frac{dy}{y} \Big/ \frac{dw}{w} = \frac{d \log y}{d \log w}$

or $d \log y = \sigma\, d \log w$

$$\therefore \quad \log y = \sigma \log w + \log A$$

$\therefore \quad y = AW^{\sigma}$ where y = output per worker

w = wages per worker.

(ii) If $\sigma = 1$ (Cobb-Douglas)

Then $\log y = \log A + \log w$

$$\therefore \quad -\log A = \log w - \log y$$

$$\therefore \quad \log A^{-1} = \log \left(\frac{w}{y} \right)$$

$\therefore A^{-1} = \frac{w}{y} = \frac{aw}{V}$ (= labour's share.)

i.e. $\frac{1}{A} = \alpha$ (of Cobb-Douglas)

and property share $= 1 - \frac{1}{A} = \beta$

and $\frac{\alpha}{\beta} = \frac{1}{A-1}$ (in CES)

(iii) If $\sigma \neq 1$ (e.g. CES)

Then $\log y = \log A + \sigma \log w$

$$= \log A + \sigma \log \{f(x) - x f'(x)\}$$

$$\therefore y = A \{f(x) - x f'(x)\}^{\sigma}$$

$$= A \left\{y - x \frac{dy}{dx}\right\}^{\sigma}$$

$$\therefore y^{1/\sigma} = \left\{A\left(y - x \frac{dy}{dx}\right)^{\sigma}\right\}^{1/\sigma}$$

$$= A^{1/\sigma}\left(y - x \frac{dy}{dx}\right)$$

$$\therefore y^{1/\sigma} A^{-1/\sigma} = y - x \frac{dy}{dx}$$

$$\therefore x \frac{dy}{dx} = y - y^{1/\sigma} A^{-1/\sigma}$$

$$\therefore \frac{dy}{dx} = \frac{y - y^{1/\sigma} A^{-1/\sigma}}{x} = \frac{y(1 - y^{1/\sigma - 1} A^{-1/\sigma})}{x}$$

If we put $A^{-1/\sigma} = \alpha$ and $\frac{1}{\sigma} - 1 = \rho$

Then $\frac{dy}{dx} = \frac{y(1 - \alpha y^{\rho})}{x} = \frac{y - y^{\rho+1}\alpha}{x}$

Put $u = \frac{1}{y^{\rho}} = y^{-\rho}$ i.e. $y = u^{-1/\rho}$

$$\therefore \frac{dy}{du} = -\frac{1}{\rho} u^{-1/\rho - 1} = -\frac{1}{\rho} u^{-\left(\frac{\rho+1}{\rho}\right)}$$

$$\therefore \frac{dy}{dx} = \frac{u^{-1/\rho} - u^{-1/\rho(\rho+1)}}{x} \cdot \alpha = \frac{u^{-1/\rho} - u^{-\left(\frac{\rho+1}{\rho}\right)}}{x} \cdot \alpha$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

i.e. $$\frac{u^{-1/\rho} - u^{-\left(\frac{\rho+1}{\rho}\right)} \cdot \alpha}{x} = -\frac{1}{\rho} u^{-\left(\frac{\rho+1}{\rho}\right)} \cdot \frac{du}{dx}$$

$$\therefore \frac{du}{dx} = \frac{u^{-1/\rho} - u^{-\left(\frac{\rho+1}{\rho}\right)} \cdot \alpha}{-x \frac{1}{\rho} u^{-\left(\frac{\rho+1}{\rho}\right)}}$$

$$= \frac{u^{-1/\rho} \cdot \rho \cdot u^{\left(\frac{\rho+1}{\rho}\right)} - \alpha u^{-\left(\frac{\rho+1}{\rho}\right)} \cdot \rho \cdot u^{\left(\frac{\rho+1}{\rho}\right)}}{-x}$$

$$= \frac{\rho u - \alpha \rho}{-x} = \frac{-\rho(u-\alpha)}{x}$$

$$\therefore \frac{du}{u-\alpha} = \frac{-\rho\, dx}{x}$$

$$\therefore \int \frac{du}{u-\alpha} = \int \frac{-\rho\, dx}{x}$$

$$\therefore \log(u-\alpha) = -\rho \log x + \log \beta$$

$$\therefore u - \alpha = \beta x^{-\rho}$$

$$\therefore u = \beta x^{-\rho} + \alpha$$

$$\therefore y^{-\rho} = \beta x^{-\rho} + \alpha$$

$$\therefore y = (\beta x^{-\rho} + \alpha)^{-1/\rho}$$

$$\therefore \frac{V}{a} = \left\{\beta (b/a)^{-\rho} + \alpha\right\}^{-1/\rho}$$

$$V = a\left\{\beta (b/a)^{-\rho} + \alpha\right\}^{-1/\rho}$$

$$\therefore V^{-\rho} = a^{-\rho}\left\{\beta (b/a)^{-\rho} + \alpha\right\}$$

$$= a^{-\rho} \beta b^{-\rho} a^{\rho} + \alpha a^{-\rho}$$

$$= \beta b^{-\rho} + \alpha a^{-\rho}$$

$\therefore V = (\beta b^{-\rho} + \alpha a^{-\rho})^{-1/\rho}$ (CES prod. $f^{\underline{n}}$. in its simple form)

(iv) Putting $\alpha+\beta = \gamma^{-\rho}$, $\beta\gamma^{\rho} = \delta$

$$\text{i.e. } \beta = \delta\gamma^{-\rho}$$

$$\text{and } \alpha = \gamma^{-\rho} - \beta = \gamma^{-\rho} - \delta\gamma^{-\rho} = \gamma^{-\rho}(1-\delta)$$

Then $V = \left\{\delta\gamma^{-\rho}b^{-\rho} + \gamma^{-\rho}(1-\delta)\,a^{-\rho}\right\}^{-1/\rho}$

$$= \left\{\gamma^{-\rho}(\delta b^{-\rho} + (1-\delta)a^{-\rho})\right\}^{-1/\rho}$$

$$= (\gamma^{-\rho})^{-1/\rho}\left\{\delta b^{-\rho} + (1-\delta)\,a^{-\rho}\right\}^{-1/\rho}$$

$$= \gamma\left\{\delta\, b^{-\rho} + (1-\delta)\,a^{-\rho}\right\}^{-1/\rho}$$

where γ = efficiency parameter

δ = distribution parameter

and ρ = substitution parameter.

(v) To introduce the degree of scale :

$V = \gamma\left\{\lambda^{-\rho}(\delta b^{-\rho} + (1-\delta)a^{-\rho})\right\}^{-1/\rho}$ $\left\{\gamma(\lambda^{-\rho})^{-\upsilon/\rho} = \gamma\lambda^{\upsilon}\right.$ where υ = degree of scale.

Thus, $V = \gamma\left\{\delta b^{-\rho} + (1-\delta)\,a^{-\rho}\right\}^{-\upsilon/\rho}$

$$\lambda V = v(\lambda a, \lambda b) = \gamma\left\{\delta(\lambda b)^{-\rho} + (1-\delta)(\lambda a)^{-\rho}\right\}^{-\upsilon/\rho}$$

$$= \gamma\left\{\delta\lambda^{-\rho}b^{-\rho} + (1-\delta)\lambda^{-\rho}a^{-\rho}\right\}^{-\upsilon/\rho}$$

$$= \gamma\left[\lambda^{-\rho}\left\{\delta b^{-\rho} + (1-\delta)\,a^{-\rho}\right\}\right]^{-\upsilon/\rho}$$

$$= \gamma(\lambda^{-\rho})^{-\upsilon/\rho}\left\{\delta b^{-\rho} + (1-\delta)\,a^{-\rho}\right\}^{-\upsilon/\rho}$$

$$= \lambda^{\upsilon}\gamma\left\{\delta b^{-\rho} + (1-\delta)\,a^{-\rho}\right\}^{-\upsilon/\rho}$$

It is homogeneous of order υ: $\upsilon=1$ indicates constant returns to scale; $\upsilon>1$, increasing returns to scale; and $\upsilon<1$, decreasing returns to scale.

(vi) Derivation of σ given the CES production function :

$$V = \gamma\left\{\delta b^{-\rho} + (1-\delta)\, a^{-\rho}\right\}^{-\upsilon/\rho}$$

$$= \gamma\left(\delta_1 b^{-\rho} + \delta_2 a^{-\rho}\right)^{-\upsilon/\rho} \qquad \text{(putting } \delta = \delta_1 \text{ and } 1-\delta = \delta_2\text{)}$$

$$\therefore \log V = \log \gamma - \frac{\upsilon}{\rho} \log\left(\delta_1 b^{-\rho} + \delta_2 a^{-\rho}\right)$$

$$\therefore \frac{1}{V}\frac{\partial V}{\partial b} = -\frac{\upsilon}{\rho} \cdot \frac{-\rho\delta_1 b^{-\rho-1}}{\delta_1 b^{-\rho} + \delta_2 a^{-\rho}}$$

$$= \frac{\upsilon\delta_1 b^{-(\rho+1)}}{\left(\frac{V}{\gamma}\right)^{-\rho/\upsilon}}$$

$$= \upsilon\delta_1 b^{-(\rho+1)} \cdot \left(\frac{V}{\gamma}\right)^{\rho/\upsilon}$$

$$= \upsilon\delta_1 b^{-(\rho+1)} \cdot V^{\rho/\upsilon} \cdot \gamma^{-\rho/\upsilon}$$

$$\left\{\begin{array}{l} V = \gamma\left(\delta_1 b^{-\rho} + \delta_2 a^{-\rho}\right)^{-\upsilon/\rho} \\ \frac{V}{\gamma} = \left(\delta_1 b^{-\rho} + \delta_2 a^{-\rho}\right)^{-\upsilon/\rho} \\ \left(\frac{V}{\gamma}\right)^{-\rho/\upsilon} = \delta_1 b^{-\rho} + \delta_2 a^{-\rho} \end{array}\right.$$

$$\therefore f_b = \frac{\partial V}{\partial b} = \upsilon\delta_1 \gamma^{-\rho/\upsilon} \cdot b^{-(\rho+1)} \cdot V^{\rho/\upsilon} \cdot V$$

$$= \upsilon\delta_1\gamma^{-\rho/\upsilon} \cdot b^{-(\rho+1)} \cdot V^{(1+\rho/\upsilon)} > 0$$

Similarly $f_a = \upsilon\delta_2\gamma^{-\rho/\upsilon}\, a^{-(\rho+1)} \cdot V^{(1+\rho/\upsilon)} > 0$

and $R = \dfrac{f_a}{f_b} = \dfrac{\delta_2}{\delta_1}\,(b/a)^{\rho+1}$ $\quad\{$ R = marginal rate of substitution.

Now $\sigma = \dfrac{a/b \;\; d(b/a)}{\frac{1}{R}\, dR} = \dfrac{d(b/a)}{dR} \cdot \dfrac{R}{b/a}$

$$= \frac{d(b/a)}{\frac{\delta_2}{\delta_1}(\rho+1)(b/a)^{\rho}\cdot d(b/a)} \cdot \frac{\frac{\delta_2}{\delta_1}(b/a)^{\rho+1}}{b/a} \qquad \left\{ \begin{array}{l} R = \frac{\delta_2}{\delta_1}(b/a)^{\rho+1} \\ \therefore dR = \frac{\delta_2}{\delta_1}(\rho+1) \\ \qquad \cdot (b/a)^{\rho}\cdot d(b/a) \end{array} \right.$$

$$= \frac{1}{\rho+1}$$