CHAPTER I

INTRODUCTION

A *compactification* of a topological space X is a compact space K together with an embedding $e: X \to K$ such that $e(X)$ is dense in K. We identify the space X with $e(X)$ and consider X as a subspace of K. A compactification of a completely regular Hausdorff space X in which X is embedded in such a way that every bounded continuous real valued function on X extends continuously to the compactification is called the *Stone-Čech compactification* of X and is denoted by βX. This compactification βX is a useful device to study relationships between topological characteristics of X and the algebraic structure of the set of all real valued continuous functions defined on X.

Once the term compactness was defined, it was a natural problem to try to "extend" a non-compact space to a compact one. The first general method was the one-point compactification due to Alexandroff (1924): For a locally compact Hausdorff space X, let $\omega X = X \cup \{\infty\}$, where ∞ is a point outside X. Describe topology on ωX as follows: Each open set of X is open in ωX and set of the form $\omega X - K$, where K is a compact subset of X is open in ωX. The resulting topological space ωX, called as one-point compactification of X, is a compact Hausdorff space and contains X as a dense subspace.

The year 1937 was an important year in the development of Topology and studying its relations with algebra. M. H. Stone **[28]** and E. Čech **[2]** published important papers, each providing independent proof of the existence of the compactification βX. Stone's paper deals with the relations of algebra and topology through applications of Boolean rings. The most important result in this theory is the representation of Boolean algebras utilizing totally disconnected compact Hausdorff spaces. On the other hand, Čech showed the existence of the compactification βX by extending Tychonoff's idea of embedding a completely regular Hausdorff space X in a cube and used it to investigate properties of X by embedding X into βX. Čech gave an additional characterization of βX which is important in the construction of βX via zero-sets as described in Gillman and Jerison **[9, Chapter 6]**. In 1938, Wallman gave a general method for constructing a T_1 compactification for any T_1 space X. If X is a normal space then this compactification coincides with βX.

Another interesting area in the theory of compactification is the study of remainder of a space: Let αX be a compactification of a Tychonoff space X. Then $\alpha X - X$ is called the *remainder* of X in αX. If X is a locally compact Hausdorff space then as observed earlier X has a one-point compactification ωX. Conversely, if X has a compactification αX with finite remainder then $\alpha X - X$ being finite, X is open in αX so that X is locally compact. Under what conditions a locally compact space X can have a compactification (besides ωX) with a finite or countable remainder? Magill answered these natural questions in **[14, 15, 16]**. In **[14]**, Magill gave the following characterization of spaces, having an N-point compactification.

Theorem 1.1. *For some compactification αX of X, $|\alpha X - X| = N$ if and only if X is locally compact and contains N non-empty, pairwise disjoint open sets $\{G_i | 1 \le i \le N\}$ such that $K = X - \bigcup_{i=1}^{N} G_i$ is compact, but for each i, $1 \le i \le N$, $K \cup G_i$ is not compact.*

From the above result it follows that if a space has an N-point compactification then it also has an M-point compactification for every positive integer $M < N$. Further, Magill observed that if a locally compact space X has the property that every compact subset of X is contained in a compact subset whose complement has at most N components then X has no M-point compactification for $M > N$. As a consequence of this result, we obtain that the space **R** of real numbers has no N-point compactification for $N > 2$. The only compactifications of **R** with finite remainder are 1-point and 2-point compactifications.

A compactification αX of a Hausdorff space X is called a *countable compactification* if $\alpha X - X$ is countable. Continuing the study of remainder of a space, Magill obtained the following characterization of those spaces which are locally compact and have countable compactifications **[15]**.

Theorem 1.2. *For a locally compact space X the following are equivalent:*

(i) *$\beta X - X$ has infinitely many components.*

(ii) *There is a compactification αX with $\alpha X - X$ infinite and totally disconnected.*

(iii) *For some compactification γX of X, $|\gamma X - X| = \aleph_0$.*

(iv) *For each $n \in \mathbf{N}$, X has a compactification with n points in its remainder.*

As a consequence of the above result one can obtain that no Euclidean n-space has a countable compactification. Further, given topological spaces X and K, can we always construct a compactification αX of X having K as a remainder? In **[16]**, Magill proved that if X is a locally compact, normal space containing an infinite, discrete, closed subset then for any Peano space K, there exists a compactification αX of X such that $\alpha X - X$ is homeomorphic to K.

Two compactifications αX and γX of a Tychonoff space X are said to be *equivalent* if there exists a homeomorphism from αX to γX leaving the points of X fixed. We identify equivalent compactifications of X, and denote by $K(X)$ the set of all such equivalence classes. The set $K(X)$ is partially ordered by the relation "$\leq$" defined by $\alpha X \leq \gamma X$ if there exists a continuous function from γX to αX which leaves the points of X fixed. In 1941, Lubben proved that for a Tychonoff space X, $(K(X), \leq)$ is a complete upper semi-lattice. Lubben further characterized that $K(X)$ is a complete lattice if and only if X is locally compact **[4]**.

Magill studied relation between $K(X)$ and $\beta X - X$. In **[17]**, Magill proved the following:

Theorem 1.3. *For locally compact non-compact Hausdorff spaces X and Y, the lattices $K(X)$ and $K(Y)$ are isomorphic if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic.*

In **[17]**, Magill determined the automorphism groups of the lattice $K(X)$. Magill proved that for a locally compact, non-compact space X, if $|\beta X - X| = 2$ then $A(K(X))$, the automorphism group of the lattice $K(X)$, is the group consisting of one element. Further if $|\beta X - X| \neq 2$, then $A(K(X))$ is isomorphic to the group (under composition) of all autohomeomorphism of $\beta X - X$. Magill further observed that every group can be regarded as the automorphism group of the lattice of all compactifications of a suitable locally compact space. In fact, given any group G, there exists a locally compact space X such that G is isomorphic to $A(K(X))$.

In **[12]**, V. Kannan and T. Thrivikraman have also explored the relation between lattice structure of $K(X)$ and the Stone- Čech remainder of a Tychonoff space X. In **[29]**, Thrivikraman has described a method to get back the space $\beta X - X$ from the $K(X)$ when X is locally compact, and the compact sets of $\beta X - X$ otherwise. We briefly describe Thrivikraman's work done in **[29]**. We recall the following definitions: For a Tychonoff space X, $\Im\alpha(X)$ denotes the family of partition classes of $\beta X - X$ corresponding to the compactification αX of X. In fact, if $q:\beta X \to \alpha X$ is the natural quotient map then $\Im\alpha(X)=\{q^{-1}(p) \mid p \in \alpha X - X\}$. Further, if K_1, K_2, ..., K_N are finitely many pairwise disjoint non-empty compact subsets of $\alpha X - X$ then a compactification $\alpha(X; K_1, K_2, ..., K_N)$ of X is obtained by collapsing K_1, K_2, ..., K_N to N distinct points.

We recall the following definitions.

Definition 1.4. An $\alpha X \in K(X)$ is a *dual atom* in $K(X)$ if there exist distinct points p and q in $\beta X - X$ such that $\alpha X = \alpha(X; \{p, q\})$. On the other hand $\alpha X \in K(X)$ is called an *atom* if $\alpha X = \alpha(X; K_1, K_2)$, where $\beta X - X = K_1 \cup K_2$.

For a completely regular Hausdorff space X, D denotes the set of all dual atoms of $K(X)$.

Definition 1.5. A compactification αX of a space X is called a *primary compactification* if $\Im\alpha(X)$ has precisely one non-singleton member.

Definition 1.6. Two distinct dual atoms of $K(X)$ are said to *overlap* if there are precisely three dual atoms above their lattice intersection and a dual d is said to be *hinged* with the overlapping duals d_1 and d_2 if d overlaps d_1 as well as d_2 and there are precisely six duals in D greater than the lattice intersection of d, d_1 and d_2.

Definition 1.7. Let d_1 and d_2 be two overlapping dual atoms of $K(X)$. Then the set of all dual atoms hinged with d_1 and d_2 is called the point $|d_1 d_2|$.

The set of all subsets of D of the form $|d_1 d_2|$ is denoted by F.

Definition 1.8. Let A be a subset of F with more than one element. Then

(i) a dual atom d is said to be *determined* by A if d occurs as the unique set intersection of two members of A.

(ii) A is said to be $F-$*compact* provided

(a) $\wedge_{d\in\lambda} d$ exists and

(b) $\lambda=\eta$, where η is the collection of all dual atoms $\geq \wedge_{d\in\lambda} d$ in $K(X)$,

where λ is the collection of all duals determined by A.

Theorem 1.9. *Let X be a Tychonoff space. Then there exists a bijection from F to $\beta X-X$ which carries $F-$compact sets to compact sets of $\beta X-X$ and vice versa. Further the complements of $F-$compact sets of F form a topology for F if and only if X is locally compact. In this case, F is homeomorphic to $\beta X-X$.*

Corollary 1.10. *Let X and Y be Tychonoff spaces. If $K(X)$ and $K(Y)$ are isomorphic then there exists a bijection h from $\beta X-X$ to $\beta Y-Y$ which preserves compact sets in both directions.*

Also, as a consequence of Theorem 1.9 Thrivikraman has obtained Magill's result [Theorem 1.3]. In **[29]**, Thrivikraman has also characterized some topological properties of $\beta X - X$ in terms of lattice theoretic properties of $K(X)$ as follows:

Theorem 1.11. *Let* X *be a locally compact space. Then,*

(i) $K(X)$ *is distributive if and only if* $|\beta X - X| < 3$.

(ii) $K(X)$ *is modular if and only if* $|\beta X - X| \leq 4$.

(iii) $K(X)$ *has a zero element but no atom if and only if* $\beta X - X$ *is compact and connected.*

(iv) $\beta X - X$ *is totally disconnected, if* $K(X)$ *is complemented.*

We recall the following definition:

Definition 1.12. Let B and C be categories. A *covariant functor* (*contravariant functor*) from B to C is a correspondence F which assigns to each object $X \in B$, a unique object in C, denoted as FX, and to each morphism f in B, a unique morphism in C, denoted by $F(f)$, satisfying following three properties:

(i)If $f: X \to Y$ is a morphism in B, then $F(f): FX \to FY$ (respectively $F(f): FY \to FX$) is a morphism in C.

(ii) For each object X of B, $F(I_X) = I_{FX}$.

(iii) For morphisms $f: X \to Y$ and $g: Y \to Z$ of B, $F(g \circ f) = F(g) \circ F(f)$ (respectively $F(g \circ f) = F(f) \circ F(g)$).

In **[30]**, Thrivikraman has obtained certain generalizations of Magill's result. Also, he has established that the association of $\beta X - X$ with $K(X)$ extends to a contravariant functor from the category of all compact Hausdorff spaces with morphisms as continuous surjections, into the category of all lattices with suitable morphisms and this functor is not injective but its restriction to the full subcategory of spaces with power different from 2 is injective.

In 1973 **[23]**, M. C. Rayburn has obtained generalizations of Magill's result [Theorem 1.3] by dropping the requirement that X and Y are locally compact spaces. For a Tychonoff space X, Rayburn considered the set $\mathrm{R}(X)$ of all points at which X is not locally compact. Note that for any compactification αX of X, $\mathrm{R}(X) = X \cap Cl_{\alpha X}(\alpha X - X)$. If $f : \beta X \to \alpha X$ the natural quotient map then $\Im(\alpha X) = \{f^{-1}(p) \mid p \in Cl_{\alpha X}(\alpha X - X)\}$ $\Im(\alpha X)$ is a partition of $Cl_{\beta X}(\beta X - X)$ into compact subsets. For αX, γX in $K(X)$, consider partial ordering '$\leq$' defined on $K(X)$ by $\alpha X \leq \gamma X$ if and only if $\Im(\gamma X)$ refines $\Im(\alpha X)$. In **[23]**, Rayburn proves the following results:

Theorem 1.13. *Let X and Y be completely regular Hausdorff spaces. If there exists a homeomorphism from $Cl_{\beta X}(\beta X - X)$ to $Cl_{\beta Y}(\beta Y - Y)$ which carries $\mathrm{R}(X)$ to $\mathrm{R}(Y)$ then $K(X)$ is isomorphic to $K(Y)$.*

Corollary 1.14. *Let X and Y be completely regular Hausdorff spaces with* $|\mathrm{R}(X)|=|\mathrm{R}(Y)|=1$. *If $\beta X - X$ is homeomorphic to $\beta Y - Y$, then $K(X)$ is isomorphic to $K(Y)$.*

A space X is called *compactly generated*, or a $k-$*space*, if every subset of X whose intersection with every compact subset of X is compact, is itself closed. To each space X one can associate a unique $k-$space kX with the same underlying set and the same compact sets by requiring that the closed sets are precisely those sets whose intersection with every compact set is compact. It follows that a space X is a $k-$space if and only if $X = kX$. A space X is called a $k-$*absolute space* if $\beta X - X$ is a $k-$space. In **[23]**, Rayburn proved following results:

Theorem 1.15. *Let X and Y be completely regular Hausdorff spaces. If $\psi : K(X) \to K(Y)$ is an isomorphism, then there exists a homeomorphism $f : k(\beta X - X) \to k(\beta Y - Y)$ such that for each αX in $K(X)$, $\Im(\psi(\alpha X)) \cap \beta Y - Y = \{f(H) \mid H \in \Im(\alpha X) \cap (\beta X - X)\}$. There are two such homeomorphisms if $|\beta X - X| = |\beta Y - Y| = 2$, otherwise homeomorphism is unique.*

Corollary 1.16. *Let X and Y be any two $k-$absolute spaces. If $K(X)$ is isomorphic to $K(Y)$ then $\beta X - X$ is homeomorphic to $\beta Y - Y$.*

In **[13]** Mack, Rayburn and Woods have obtained results from which some of Magill's result follow. For a completely regular Hausdorff space X, they have considered a topological property P which is

(a) closed-hereditary,

(b) productive under arbitrary products and

(c) such that if X is the union of a compact space and a space with P then X has P.

Recall that a topological space X is a *realcompact space* if every real maximal ideal in $C(X)$ is fixed. The '$P-$*reflection*' γX of a space X is defined by $\gamma X = \cap\{T \mid T$ has P and $X \subseteq T \subseteq \beta X\}$. It is known that if property P is compact, then $\gamma X = \beta X$ and if P is realcompact, then $\gamma X = \upsilon X$, where υX is the Hewitt Nachbin real compactification of X. They have studied the algebraic structure of the family of tight extensions of X i.e. those extensions which have property P and contain no proper extension with that property). When X has property P locally but not globally then relations between the complete lattice $P^*(X)$ of those tight extensions which are above the maximal one-point extension and the topology of the P-reflection, have been studied by them. They have obtained conditions under which $P^*(X)$ characterizes $\gamma X - X$. If P is taken to be compactness, then their work coincides with the Magill's work done in **[17]**.

It is interesting to know semilattices which are isomorphic to $K(X)$ for some X. Kannan and Thrivikraman in **[12]** have characterized dually atomic complete lattices in terms of lattices of closed equivalence relation of a T_1 space X and have used this result to obtain a characterization of $K(X)$, when X is a locally compact Hausdorff space.

Further the restriction to $\beta X - X$ of the Čech map of αX is perfect i.e. closed, continuous, onto and the pre-image of each point is compact. In **[22]**, Porter and Woods have studied perfect irreducible continuous surjection on a fixed domain X. A perfect irreducible continuous surjection is also termed as a *covering map*. For a Hausdorff space X, consider two covering maps f and g with range Rf and Rg, respectively. The covering maps f and g are said to be equivalent $(f \approx g)$ if there is a homeomorphism $h: Rf \to Rg$ such that $h \circ f = g$. We identify equivalent covering maps with domain X, and denote the equivalence classes of covering maps with domain X by $IP(X)$. A partial order '$\leq$' on $IP(X)$ is defined as follows: For covering maps f and g, $g \leq f$ if there exists a continuous map $h: Rf \to Rg$ such that $h \circ f = g$. For a Hausdorff space X without isolated points $(IP(X), \leq)$ is a complete upper semilattice. Porter and Woods have partially answered the question: When $IP(X)$ is a complete lattice? They have observed that if spaces X and Y are homeomorphic then the corresponding posets $IP(X)$ and $IP(Y)$ are order isomorphic. In **[22]** they have proved that the converse of the above statement is true when the underlying spaces are $k-$spaces.

Theorem 1.17. *Let X and Y be $k-$spaces without isolated points. Then $(IP(X), \leq)$ and $(IP(Y), \leq)$ are order-isomorphic if and only if X and Y are homeomorphic.*

For an open subset U of X, define $IP(X, U) = \{f \in IP(X) \mid |f^{-1}(f(x))| = 1, \text{ for each } x \in U\}$. Porter and Woods have obtained generalization of Theorem 1.17. They have proved the following result:

Theorem 1.18. *Let X and Y be a Hausdorff space and let U and V be open subsets of X and Y containing all isolated points of X and Y respectively. Suppose $\varphi : IP(X, U) \to IP(Y, V)$ is an order isomorphism. Then there is a bijection $F : X - U \to Y - V$ such that $\{F(A) \mid A$ is a compact nowhere dense subset of X and $A \subseteq X - U\}$ $= \{B \mid B$ is a compact nowhere dense subset of Y and $B \subseteq Y - V\}$ and if $f \in IP(X, U)$ then*

$$\wp(\varphi(f)) = \{\{x\} \mid x \in V\} \cup \{F(A) \mid A \in \wp(f) \text{ and } A \subseteq X - U\}.$$

Porter and Woods have used the above result to derive Theorem 1.3 due to Magill.

In **[18]**, Mendivil has proved using technique of function algebra that for locally compact spaces X and Y, $K(X)$ is order isomorphic to $K(Y)$ if and only if $C^*(X)|_{C_0(X)}$ is order isomorphic to $C^*(Y)|_{C_0(Y)}$, where

$C_0(X) = \{f \in C^*(X) \mid f^\beta|_{\beta X - X} = 0\}$, in which f^β denotes the extension of f to βX. Magill's [Theorem 1.3] result follows as a consequence of this result by using the fact that $C^*(X)|_{C0(X)}$ is isomorphic to $C^*(\beta X - X)$.

Another construction in general topology is the 'absolute' of a space. With each Hausdorff space X there is associated an extremally disconnected zero-dimensional Hausdorff space EX, called the *Iliadis absolute* of X and a perfect, irreducible, θ-continuous surjection from EX onto X **[11]**. The points of EX are certain ultrafilters on lattices associated with X. If the space X is regular then the space EX can be mapped onto X by a perfect continuous map k_X, which is also irreducible. If X is not regular, then also the extremally disconnected zero-dimensional Hausdorff space EX and the map k_X, which is a perfect surjection can be constructed but the map k_X is no longer continuous, it is θ-continuous. Uniqueness of the space EX follows from the following result:

Theorem 1.19. *Let X be a Hausdorff space and let (Y, f) be a pair consisting of an extremally disconnected zero-dimensional space Y and a perfect irreducible θ-continuous surjection f from Y to X. Then there exists a homeomorphism h from EX to Y such that $f \circ h = k_X$, where k_X is the natural map from EX to X.*

In **[6]**, Császár has constructed absolutes for arbitrary topological spaces.

The projective space EX together with the perfect irreducible mapping k_X is called the *projective cover* of X. Further, for a non-regular Hausdorff space X we can construct an extremally disconnected space PX, called the Banaschewski absolute of X and a perfect irreducible continuous surjection $\pi_X : PX \to X$. The space PX is extremally disconnected but not regular. For a regular Hausdorff space X, PX and EX coincide. That PX is not zero-dimensional follows by the following result **[21]**:

Theorem 1.20. *The following are equivalent for a Tychonoff space X:*

(i) X *is extremally disconnected,*

(ii) *each dense subspace of X is* C*-*embedded in X,*

(iii) *each open subspace of X is* C*-*embedded in X, and*

(iv) βX *is extremally disconnected.*

In **[32]**, R. C. Walker has discussed that the Stone-Čech compactification and the Hewitt-Nachbin realcompactification are both functorial i.e. they can be used to define functors. Using Gleason's result that every completely regular space has a projective cover which is unique up to homeomorphism and the uniqueness of EX, we have $E(\beta X) = \beta(EX)$. The extensions and absolutes, although conceptually very different, are constructed using similar tools. In **[19]**, H. Ohta has discussed topological extension properties and projective covers. Ohta has discussed under what conditions $\wp(EX) = E(\wp X)$, where $\wp$ is a topological property of a completely regular Hausdorff space. This question was raised by Woods and the case when $\wp$ is realcompactness was settled by Hardy and Woods **[10]**.

The *projective lift* of a map $f: X \to Y$ (if it exists) is the unique map $Ef: EX \to EY$ satisfying $k_Y \circ Ef = f \circ h_X$. In **[8]** Das has constructed projective lift of a dp-epimorphism, and hence proved that E is a covariant functor from the category C_d of regular Hausdorff spaces and continuous dp-epimorphisms to its coreflective subcategory consisting of projective objects of C_d.

We now briefly describe the work done in the present thesis. In **[8]**, Das has introduced the notion of a density preserving map which is defined as follows: A continuous map f from a topological space X into a topological space Y is called a *density preserving map* if $IntClf(A) \neq \varphi$, whenever $IntA \neq \varphi$, where A is a subset of X. We say two density preserving maps f and g defined on a topological space X with range Rf and Rg respectively are said to be equivalent if there exists a homeomorphism $h: Rf \to Rg$ satisfying $h \circ f = g$. We identify equivalent density preserving maps on a fixed domain X and denote by $DP(X)$, the set of all such equivalence classes of density preserving maps. We define an order relation $\leq$ on $DP(X)$ such that $(DP(X), \leq)$ becomes a partially ordered set (poset). Having observed that the poset $IP(X)$ is naturally contained in the poset $DP(X)$, we study the poset $DP(X)$ to obtain further results in the direction of

[22]. In Chapter 2, we show that for a compact Hausdorff space without isolated points $DP(X)$ is a complete lattice.

In Chapter 3 we obtain results similar to Theorem 1.17 for $DP(X)$. In fact we prove that if X and Y are countably compact T_3 spaces without isolated points then X and Y are homeomorphic if and only if $DP(X)$ is order isomorphic to $DP(Y)$.

In Chapter 4, for $A \subseteq X$ we define set $DP(X, A)$ of density preserving maps f on X satisfying $|f^{-1}(f(x))| = 1$ for each x in A and observe that for a dense subset U of a compact space X, $DP(X,U) = IP(X,U)$. In particular, for a locally compact space X we have $DP(\beta X,X) = IP(\beta X,X)$. As a consequence of this we obtain Magill's result [Theorem 1.3]. Using a result due to Porter and Woods **[22]**, we obtain that $DP(\beta X,X)$ is order isomorphic to $K(X)$. As a consequence of this we are able to use lattice theoretic properties of the complete lattice $DP(\beta X,X)$ to obtain topological properties of $\beta X - X$ when X is a locally compact Hausdorff space.

An $f \in DP(X)$ is said to be a *dual* if the only non-singleton fiber of f is a doubleton. In Chapter 5, we introduce and study the notion of overlapping duals and also the notion of duals hinged with overlapping duals. We topologize the collection $\Im$ of all subsets of the set of duals in $DP(X)$ which are hinged with overlapping duals in $DP(X)$, and study when the topological space $\Im$ is homeomorphic to X. We in fact prove that $\Im$ is homeomorphic to X when X is a countably compact T_3 space without isolated points. To prove this result we introduce the notion of $F-$closed sets and observe that for a locally compact Hausdorff space X, the notion of $F-$closed sets coincide with the notion of $F-$compact sets defined by Thrivikraman in **[29]**.

In Chapter 6, we attempt to construct a compactification rX for a non-Tychonoff space X by using the family $Rf(X)$ of all finite intersections of regular closed subsets of X. We observe that the resulting compactification rX is a non-Hausdorff T_1 space. A separation axiom stronger than T_1 but weaker than T_2 naturally exist on rX, which we term as nearly Hausdorffness. We introduce a topological property π for a space X and note that a space with the property π is a nearly Hausdorff space if and only if it is Urysohn. Further, we answer the natural question: When $rX = \beta X$? We observe that if $Rf(X)$ forms a Wallman base for a nearly Hausdorff space X, then $rX = \beta X$. As a consequence of this we have that if X is normal or zero-dimensional then $rX = \beta X$.

In Chapter 7, we describe the construction of projective cover (EX, h_X) for a compact nearly Hausdorff space X on the lines of Gleason's construction **[32]**. In this chapter we study projective lift and extension of a density preserving epimorphism $f : X \to Y$. We prove that both E and r are functorial and establish the commutativity of E and r.