CHAPTER III

ORDER STRUCTURE OF POSET $DP(X)$

Since last several years topologists have been studying the order structure of an associated family of extensions of a space and have obtained results like Theorem 1.3. obtained by Magill in 1968. The situation in which the topology of a space is determined by the order structure of an associated family of mappings is illustrated by Theorem 1.17.

In case of the poset $DP(X)$, the order structure of $DP(X)$ is always determined by the topology of the space X, i.e. if topological spaces X and Y are homeomorphic then $DP(X)$ and $DP(Y)$ are order isomorphic. In this chapter we deal with the converse problem, that is, if $DP(X)$ and $DP(Y)$ are order isomorphic then can we say that X is homeomorphic to Y? In section 1 of this chapter we define and study primary and dual member in $DP(X)$. In section 2, we introduce the notion of cln-bijection and using cln-bijections we prove in the last section that if X and Y are countably compact T_3 spaces without isolated points then $DP(X)$ and $DP(Y)$ are order isomorphic implies X is homeomorphic to Y.

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1. Primary and dual members in $DP(X)$.

In this section we define primary and dual members in $DP(X)$ and characterize them. We also derive the formula for the greatest lower bound of two primary members in $DP(X)$.

Definition 3.1.1. Let X be a topological space. Then an f in $DP(X)$ is said to be

(i) *primary* if the corresponding dp-partition $\wp(f)$ generated by f has at most one non-singleton member.

(ii) *dual* if f is primary and the corresponding dp-partition $\wp(f)$ generated by f contains exactly one doubleton.

Notation. If $f \in DP(X)$ is such that $\wp(f)$ contains n non-singleton members, say $K_1, K_2,...., K_n$, then f is denoted by $(f; K_1, K_2,...., K_n)$. In particular, if K is a non-singleton closed nowhere dense set in X, then $(f; K)$ denotes the natural density preserving map defined on X obtained by collapsing K to a point.

Examples 3.1.2.(a) Consider the usual space **R** of real numbers and take closed nowhere dense set as the set **Z** of all integers. The natural quotient map $q: \mathbf{R} \to \mathbf{R}|_{\mathbf{Z}}$ obtained by identifying **Z**, is a density preserving map and the only non-singleton member in $\wp(f)$ is **Z**. Therefore (q; **Z**) is a primary member in DP(**R**) but is not a dual member.

3.1.2(b) The natural quotient map f on the usual space **R** of real numbers obtained by identifying two distinct points x and y in **R** is a density preserving map. Observe that the only non-singleton member of $\wp(f)$ is the set $\{x, y\}$. Hence this primary member f in DP(**R**) is also a dual.

Theorem 3.1.3. *Let X be a topological space. Then an f in $DP(X)$, $f \neq I_X$, is primary if and only if there do not exist duals $g, h \in DP(X)$ such that $f \wedge g = f \wedge h \neq f$ and the only dual points greater than $g \wedge h$ are g and h.*

Proof. If f in $DP(X)$ represents the family of constant members then for any two duals g, h in $DP(X)$, we get $f \wedge g = f \wedge h = f$. Therefore f is primary. Suppose f is a non constant member in $DP(X)$ such that $f \neq I_X$ and there do not exist duals g and h in $DP(X)$ such that $f \wedge g = f \wedge h \neq f$ and the only duals greater than $g \wedge h$ are g and h. Then we need to show that f is primary or equivalently dp-partition $\wp(f)$ generated by f contains exactly one non-singleton member. Suppose $\wp(f)$ contains more than one non-singleton member say K and H. Then choose distinct points a, $b \in H$ and c, $d \in K$ and consider the natural dual members $(g;\{a,c\})$ and $(h;\{b,d\})$ in $DP(X)$. Since $H \cap K = \varphi$, the dp-partition $\wp(f \wedge (g;\{a,c\}))$ generated by $f \wedge (g;\{a,c\})$ consists of $[\wp(f) - \{K, H\}] \cup \{K \cup H\}$. Observing similarly for $f \wedge (h;\{b,d\})$, we obtain

$$f \wedge (g;\{a,c\}) = f \wedge (h;\{b,d\}) \neq f .$$

Also, besides g and h, there are other duals in $DP(X)$ greater than $g \wedge h$. In fact $g \wedge h$ is the natural quotient map ($g \wedge h; \{a,b.c,d\}$) and $(k;\{a,b\})$ is a dual member in $DP(X)$ different from g and h satisfying $(g \wedge h) \leq k$. This contradicts our hypothesis. Hence such an f is primary.

Conversely, suppose $f \neq I_X$ is primary. In case f is constant we are through. Let f be non-constant and let K be the non-singleton member of $\wp(f)$. Let $(g;\{a,b\})$ and $(h;\{c,d\})$ be two distinct dual members in $DP(X)$. Then $g \neq h$ implies

$$\{a,b\} \neq \{c,d\}$$

$$\Rightarrow \quad \{a,b\} \cap \{c,d\} = \varphi \text{ or } \{a,b\} \cap \{c,d\} \text{ is a singleton.}$$

Now suppose $\{a,b\} \cap \{c,d\} = \varphi$. In case $\{a, b\} \cap K = \varphi$ and $\{c, d\} \cap K = \varphi$, then

$$f \wedge g \approx (f \wedge g;\ K,\ \{a,\ b\}) \qquad (1)$$

and

$$f \wedge h \approx (f \wedge h;\ K,\ \{c,\ d\}). \qquad (2)$$

On the other hand consider the case when $\{a, b\} \cap K \neq \varphi$ and $\{c, d\} \cap K \neq \varphi$. Let $b \in K$ and $d \in K$. Then

$$f \wedge g \approx (f \wedge g;\ K \cup \{a\}) \qquad (3)$$

and

$$f \wedge h \approx (f \wedge h;\ K \cup \{c\}). \qquad (4)$$

Relations (1) to (4) imply that if $\{a,b\} \cap \{c,d\} = \varphi$, then $f \wedge g \neq f \wedge h$ and we are through.

Next, suppose $\{a,b\} \cap \{c,d\}$ is a singleton. In case $f \wedge g \neq f \wedge h$, we are through. Suppose $f \wedge g \approx f \wedge h \neq f$. In this case we obtain a dual member η different from g and h which is greater than $h \wedge g$. In fact, if $\{a,b\} \cap \{c,d\} = \{a\}$, where $a = c$, then

$$(h; \{a, b\}) \wedge (g; \{c, d\}) \approx (h \wedge g; \{a, b, d\}).$$

and the dual member $(\eta; \{b, d\})$ is different from g and h and is greater than $h \wedge g$.

Therefore if $\wp(f)$ contains exactly one non-singleton member then there do not exist dual points g and h such that $f \wedge g \approx f \wedge h \neq f$ and the only duals greater than $h \wedge g$ are g and h.

Theorem 3.1.4. *Let X be a topological space. Then an f in $DP(X)$ is a dual if and only if there does not exist g in $DP(X)$ such that $f < g < I_X$.*

Proof. Suppose $f \in DP(X)$ is a dual. Then there exists precisely one non-singleton member in $\wp(f)$ which is a doubleton. If possible, suppose there exists $g \in DP(X)$ such that $f < g < I_X$. Then

$$f < g \Rightarrow \wp(g) \subseteq \wp(f)$$

$\Rightarrow$ $g^{-1}(z)$ is a singleton for each z in Rg

$\Rightarrow$ $g \approx I_X$ - a contradiction.

Conversely, if possible, suppose f is not a dual. Then the dp-partition $\wp(f)$ contains one non-singleton member, say K, containing more than two elements. Choose distinct points a,b,c in K. Then $(h; \{a, b\})$ is a dual member in $DP(X)$ satisfying $f < h < I_X$.

Theorem 3.1.5. *Let X be a topological space. Then for any two closed nowhere dense subsets K_1 and K_2 of X,*

$$(f; K_1) \wedge (g; K_2) = \begin{cases} (h;\ K_1,\ K_2), & if\ K_1 \cap K_2 = \phi \\ (h;\ K_1 \cup K_2), & if\ K_1 \cap K_2 \neq \phi. \end{cases}$$

Proof. Suppose $K_1 \cap K_2 = \varphi$. Then by Lemma 2.3.3,

$$\wp(f) \subseteq \wp(h) \text{ and } \wp(g) \subseteq \wp(h)$$

$$\Rightarrow \quad (f; K_1) \geq (h;\ K_1,\ K_2) \text{ and } (g; K_2) \geq (h;\ K_1,\ K_2)$$

If $\sigma \in DP(X)$ is another member such that $f \geq \sigma$ and $g \geq \sigma$, then by Lemma 2.3.3,

$$f \geq \sigma \text{ and } g \geq \sigma$$

$$\Rightarrow \quad \wp(f) \subseteq \wp(\sigma) \text{ and } \wp(g) \subseteq \wp(\sigma),$$

which implies K_1 is a subset of some member in $\wp(\sigma)$ and K_2 is also a subset of some member in $\wp(\sigma)$. But this gives $\sigma \leq (h; K_1, K_2)$ which proves $(f; K_1) \wedge (g; K_2) = (h; K_1, K_2)$.

Next, suppose $K_1 \cap K_2 \neq \phi$, then $(f; K_1) \geq (h; K_1 \cup K_2)$ and $(g; K_2) \geq (h; K_1 \cup K_2)$. Suppose $\sigma \in DP(X)$ is such that $f \geq \sigma$ and $g \geq \sigma$. Then by Lemma 2.3.3,

$$f \geq \sigma \text{ and } g \geq \sigma$$

$$\Rightarrow \quad \wp(f) \subseteq \wp(\sigma) \text{ and } \wp(g) \subseteq \wp(\sigma),$$

which implies that K_1 is a subset of some member, say H_1 in $\wp(\sigma)$ and K_2 is a subset of some member, say H_2 in $\wp(\sigma)$. Since $K_1 \cap K_2 \neq \phi$, we have $H_1 = H_2$. Thus $(h; K_1 \cup K_2) \geq \sigma$. This proves that $(f; K_1) \wedge (g; K_2) = (h; K_1 \cup K_2)$, if $K_1 \cap K_2 \neq \phi$.

Theorem 3.1.6. *Let X and Y be topological spaces. Then an order isomorphism $\varphi : DP(X) \to DP(Y)$ maps duals to duals.*

Proof. Suppose $f \in DP(X)$ is a dual. If $\varphi(f)$ is not a dual member then by Theorem 3.1.4, there exists $g \in DP(Y)$ such that $\varphi(f) < g < I_Y$. Since φ is an order isomorphism, we have $f < \varphi^{-1}(g) < I_X$ which contradicts that $f \in DP(X)$ is a dual. Hence an order isomorphism $\varphi : DP(X) \to DP(Y)$ maps duals to duals.

2. cln-bijection.

In this section we define cln-bijection and prove that for Hausdorff spaces X and Y without isolated points, an order isomorphism $\varphi : DP(X) \to DP(Y)$ induces a cln-bijection from X to Y.

Definition 3.2.1. A bijection $f : X \to Y$ from a topological space X to a topological space Y is called a *cln-bijection* if the family $\{ f(A) \mid A$ is a closed nowhere dense subset of $X\}$ is precisely the family of all closed nowhere dense subsets of Y.

We recall that a point p in the Stone-Čech compactification βX of a Tychonoff space X is called a *remote point* of X if $p \in \beta X - X$ but $p \notin Cl_{\beta X} A$, for every nowhere dense subset A of X.

Example 3.2.2. Consider the usual space Q of rational numbers. Suppose p and q are remote points of Q such that Stone extension of none of the self-homeomorphism of Q maps p to q. Consider the subspaces $Q \cup \{p\}$ and $Q \cup \{q\}$ of the Stone-Čech compactification βQ and the map $f : Q \cup \{p\} \to Q \cup \{q\}$ defined by $f(x) = x$ if $x \in Q$ and $f(p) = q$. Then f is a cln-bijection.

Lemma 3.2.3. *Let X and Y be Hausdorff spaces without isolated points and let φ: $DP(X) \to DP(Y)$ be an order isomorphism. Then there exists a cln-bijection $F : X \to Y$ such that for each $f \in DP(X)$ we have $\wp(\varphi(f)) = \{F(A) \mid A \in \wp(f)\}$.*

Proof. Let $p \in X$. Choose distinct points $q,\ r \in X - \{p\}$. By Theorem 3.1.6, $\varphi(f, \{p, q\})$, $\varphi(g, \{p, r\})$ are dual points of $DP(Y)$ say $(\bar{f}, \{a, b\})$ and $(\bar{g}, \{c, d\})$ respectively. Clearly

$$(\bar{f}; \{a, b\}) \wedge (\bar{g}; \{c, d\}) = \varphi(f \wedge g; \{p, q, r\}).$$

If $\{a, b\} \cap \{c, d\} = \phi$, then

$$(\bar{f}; \{a, b\}) \wedge (\bar{g}; \{c, d\}) = (\bar{f} \wedge \bar{g}; \{a, b\}, \{c, d\}),$$

Thus in this case $(f; \{p, q\})$, $(g; \{p, r\})$, $(h; \{q, r\})$ are three dual points in $DP(X)$ greater than $(f \wedge g; \{p, q, r\})$ where as $(\bar{f}; \{a, b\})$, $(\bar{g}; \{c, d\})$ are the only two dual points in $DP(Y)$ greater than $(\bar{f} \wedge \bar{g}; \{a, b\}, \{c, d\})$ which is not possible. Therefore $\{a, b\} \cap \{c, d\} \neq \phi$, in fact it is a singleton, say $\{a\}$. Define $F : X \to Y$ by $F(p) = a$. We now show that the choice of a does not depends upon the choices of r and q. Let $s \in X - \{p,q,r\}$. Then there exist points y and z in Y such that $\varphi(k;\{p,s\}) = (\bar{k};\{y,z\})$. We have

$\varphi(f;\{p,q\}) = (\overline{f};\{a,b\})$. Assume $\varphi(g;\{p,r\}) = (\overline{g};\{a,c\})$. Using similar arguments we conclude that $\{y, z\}$ intersects both $\{a,b\}$ and $\{a,c\}$ in exactly one point. If $a \notin \{y,z\}$ then $\{y,z\} = \{b,c\}$. Therefore by Theorem 3.1.5,

$$\varphi(f \wedge g \wedge k; \{p, q, r, s\})$$

$$= (\overline{f};\{a, b\}) \wedge (\overline{g}; \{a, c\}) \wedge (\overline{k};\{b, c\})$$

$$= (\overline{f} \wedge \overline{g} \wedge \overline{k}; \{a, b, c\}).$$

This implies there are six dual points greater than $(f \wedge g \wedge k;\{p,q,r,s\})$ while there are only three duals greater than $(\overline{f} \wedge \overline{g} \wedge \overline{k};\{a,b,c\})$, which is impossible as φ is an order isomorphism. Thus our assumption that $a \notin \{y,z\}$ is not possible. This proves that for any $s \in X - \{p\}$, if $\varphi(k;\{p,s\}) = (\overline{k};\{y,z\})$ then $a \in \{y,z\}$. Also if s' is any other point in $X - \{p,q\}$ and if $\varphi(\sigma;\{p,s'\}) = (\overline{\sigma};\{y',z'\})$, then $\{y',z'\} \cap \{y,z\} = \{a\}$. Thus we have defined map F.

We now show that F maps closed nowhere dense sets to closed nowhere dense sets. Let H be a closed nowhere dense set in X. Consider $f \in DP(X)$ of the form $(f; H)$. If $\varphi(f; H) = \overline{f}$ then $\overline{f} = (\overline{f}; K)$ for some closed nowhere dense subset K of Y. Further, if $p, q \in H$, $p \neq q$, then by Lemma 2.3.3 $(g; \{p, q\}) \geq (f; H)$ which implies $(\overline{g}; \{a, b\}) \geq (\overline{f}; K)$. Hence $\{a,b\} \subseteq K$. This proves that $F(\{p, q\}) = \{a, b\} \subseteq K$. Since $p,q \in H$ are arbitrary, it follows that $F(H) \subseteq K$.

Similarly, using φ^{-1}, we can define $\overline{F}: Y \to X$ as follows: Let $a \in Y$. Choose distinct points b and c in $Y - \{a\}$. Then, $\varphi^{-1}(\overline{f};\{a,b\})$ and

$\varphi^{-1}(\overline{g};\{a,c\})$ are dual points of $DP(X)$ say $(f;\{p,q\})$ and $(g;\{r,s\})$ respectively. Then using similar arguments as above we can show that $\{p,q\}\cap\{r,s\}$ is a singleton say p and the choice of p does not depend upon the choice of b and c. Define $\overline{F}:Y\to X$ by $\overline{F}(a)=p$. Arguing as above, one can show that $\overline{F}(K)\subseteq H$.

We now prove that $\overline{F}\circ F$ is identity on X. Let $p\in X$ and $q\in X-\{p\}$. Clearly $\varphi(f;\{p,q\})$ is a dual, say $(\overline{f};\{a,b\})$. Then $F(p)\in\{a,b\}$. Assume $F(p)=a$. Suppose $\overline{F}(a)\neq p$. Then $\overline{F}(a)=q$. Choose $r\in X-\{p,q\}$. Then there exists $c\in Y$ such that $\varphi(g;\{p,r\})$ is a dual point say $(\overline{g};\{a,c\})$. Since $\overline{F}(a)\in\{p,r\}$ and $\overline{F}(a)\neq p$, therefore $\overline{F}(a)=r$, a contradiction as $\overline{F}(a)=q\neq r$. Therefore we conclude that $\overline{F}(a)=p$. This proves $\overline{F}\circ F$ is identity on X. Similarly, we can prove that $F\circ\overline{F}$ is identity on Y. Hence $F:X\to Y$ is a bijective map which preserves closed nowhere dense sets. Also, by the definition of the map F, it follows that if $\varphi(f;H)=(\overline{f};K)$, then $F(H)=K$ and hence $\wp(\varphi(f))=\{F(A)\mid A\in\wp(f)\}$.

3. Topology of X and the order structure of $DP(X)$.

We shall see an easy proof of the fact that if topological spaces X and Y are homeomorphic topological spaces then $DP(X)$ and $DP(Y)$ are order isomorphic. To find when the converse is true, we shall use the following known fact:

Note. Let X be a countably compact T_3 space without isolated points. Then $A \subseteq X$ is closed if and only if whenever $B \subseteq A$ and $Cl_X B$ is nowhere dense in X then $Cl_X B \subseteq A$ **[22]**.

In fact, we shall use the given order isomorphism between $DP(X)$ and $DP(Y)$ to construct a cln-bijection F between X and Y and then use the fact stated in the above note to prove that both F and F^{-1} are closed maps.

Theorem 3.3.1. *Let X and Y be countably compact T_3 spaces without isolated points. Then $DP(X)$ and $DP(Y)$ are order isomorphic if and only if X and Y are homeomorphic.*

Proof. Suppose X and Y are homeomorphic and let $h: X \to Y$ be a homeomorphism. Define φ: $DP(X) \to DP(Y)$ by $\varphi(f) = f \circ h^{-1}$. The map φ thus defined is a bijective order preserving map from $DP(X)$ to $DP(Y)$. Hence $DP(X)$ and $DP(Y)$ are order isomorphic.

Conversely, suppose $DP(X)$ and $DP(Y)$ are order isomorphic. Then by Lemma 3.2.3 there is a cln-bijection $F: X \to Y$. We shall show that F is a closed map. By symmetry, it will follow that F^{-1} is also a closed map and hence F will be a homeomorphism. Let $M \subseteq X$ be such that $Cl_X M$ is nowhere dense. First observe that $F(Cl_X M) = Cl_Y F(M)$. Clearly, $M \subseteq Cl_X M$ and hence $F(M) \subseteq F(Cl_X M)$. Since $Cl_X M$ is a closed nowhere dense set and F is a cln-bijection, $Cl_Y F(M)$ is a closed nowhere dense set satisfying $Cl_Y F(M) \subseteq F(Cl_X M)$. For the reverse containment, observe that

$F^{-1}(Cl_Y F(M))$ is a closed nowhere dense set containing M which implies that $F(Cl_X M) \subseteq Cl_Y F(M)$. This proves $F(Cl_X M) = Cl_Y F(M)$, whenever $Cl_X M$ is a nowhere dense set.

We complete the proof by showing that F is a closed map. Let C be a closed subset of X. Suppose $B \subseteq F(C)$ and $Cl_Y B$ is a nowhere dense set. Then Y being countably compact T_3 space without isolated points, to establish that the map F is closed, it is sufficient to show that $Cl_Y B \subseteq F(C)$. Now,

$$B \subseteq F(C) \Rightarrow F^{-1}(B) \subseteq C.$$

Also C is closed implies $Cl_X F^{-1}(B) \subseteq C$ and hence

$$F^{-1}(Cl_Y B) = Cl_X F^{-1}(B) \subseteq C.$$

This proves $Cl_Y B \subseteq F(C)$ and hence the required result.

Note. A cln-bijection between non-countably compact spaces without isolated points need not be a homeomorphism is justified by Example 3.2.2. In fact, the map $f : Q \cup \{p\} \to Q \cup \{q\}$ defined in that example is a cln-bijection but we know that the spaces $Q \cup \{p\}$ and $Q \cup \{q\}$ considered there are not homeomorphic.