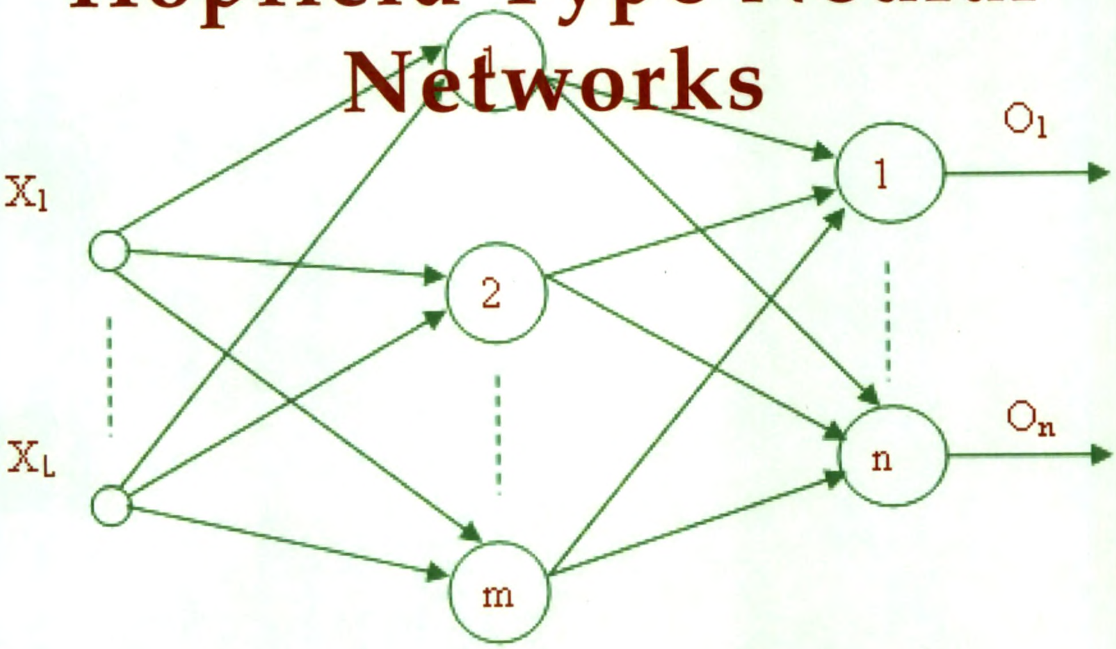
Chapter 3

Stability, Controllability and Observability of Hopfield Type Neural Networks



Chapter 3

STABILITY, CONTROLLABILITY AND OBSERVABILITY OF HOPFIELD TYPE NEURAL NETWORKS

3.1 Introduction

Hopfield Neural Networks were designed by J. J. Hopfield in the beginning of the 1980s (refer Hopfield[[32]]). He used these networks for pattern completion and to solve optimization problems. The Hopfield Neural Network (HNN) is a single layered network, the connections in it are of recurrent type, that is, feed-forward as well as feedback. In this network each neuron is connected to the other but not back to itself. It is most widely known as auto-associative networks in the literature. This single layered network of interconnected neurons can store multiple *stable* states. Since network have a set of stable states, given an input pattern the network can converge to the stable state nearest to that pattern. This implies that, we can use the network for auto-association in which a noisy or partial pattern can stablize to a nearby state corresponding to one of the originally stored pattern.

The mathematical model is derived in Section 3.2. In Section 3.3, we deal with the existence and uniqueness of solution of Hopfield model. Stability properties are studied in Section 3.4. The controllability result is proved in Section 3.5 and we conclude the chapter giving observability result in Section 3.6

3.2 Mathematical modeling for Hopfield networks

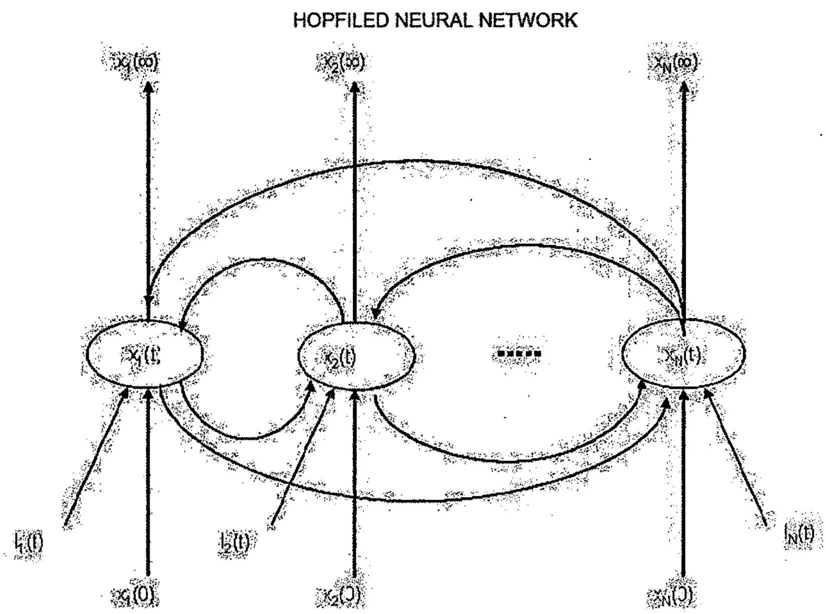


Figure 3.1: Hopfield Neural Networks with n neurons

To setup a mathematical model for the Hopfield Neural Network, consider a network having n neurons, as depicted in Figure 3.1.

Each neuron in the network is connected to the other neuron but not back to itself. Let W_{ij} be the strength of the connection (called weight) from j^{th} neuron to i^{th} neuron. Let $x_i(t)$, $i = 1, 2, ..., n$ be the information stored in the i^{th} neuron at time t, which will be interchangeably called as state, potential or impulse as in case of biological neuron.

Let $x_i(0)$, be the initial information or input provided to the i^{th} neuron at time $t = 0$. Because of the recurrent nature of the network the initial potential $x_i(0)$ passes to all other neurons and the i^{th} neuron receives back the corresponding outputs from other neurons. This process continues for all neurons and there is a stage where the potentials of neurons are unchanged as t increases. This is called the convergence of the network. We denote the converged state of the i^{th} neuron $x_i(\infty)$. For the faster convergence process we supply some external input to the i^{th} neuron which is denoted by $I_i(t)$. Transmission or non-transmission (firing or non-firing) of information from one neuron to other neuron is characterized by a function called transfer function (e.g. sigmoidal). At time t the i^{th} neuron receives a potential $W_{ij}f_j(x_j(t))$ due to connecting strength W_{ij} and transfer function f_j corresponding to the j^{th} neuron. Thus, the contribution to the i^{th} neuron over time, is given by

$$\int_0^t W_{ij}f_j(x_j(\tau))d\tau \qquad (3.2.1)$$

The potential that arrived early may decay. This forces to include a decay function in the above contribution of potential. Let us choose decay function given by

$$h(t) = \left(\frac{1}{\mu_i}\right)e^{\frac{-t}{\mu_i}} \qquad (3.2.2)$$

The constant μ_i is the positive time constant for the i^{th} neuron. A large μ_i means that only the most recent contributions to the neurons potential are effective. The function $h(t-\tau)$ is small for large negative τ, indicating that contributions from early arrived impulses are less. Assuming $I_i(t)$ to be the external input applied to the i^{th} neuron at time t, which also fades with time, it's potential contribution to the neuron is given by

$$\int_0^t h(t-\tau)I_i(\tau))d\tau \qquad (3.2.3)$$

Thus, the potential of the i^{th} neuron after receiving contributions from all n neurons except itself is given by

$$x_i(t) = \sum_{j=1 j\neq i}^{n} \int_0^t W_{ij}\left(\frac{1}{\mu_i}\right)e^{\frac{-(t-\tau)}{\mu_i}}f_j(x_j(\tau))d\tau + \int_0^t \left(\frac{1}{\mu_i}\right)e^{\frac{-(t-\tau)}{\mu_i}}I_i(\tau)d\tau \qquad (3.2.4)$$

The above equation (3.2.4) represents the state of the i^{th} neuron at time t. For the sake of convenience we transform the above integral equation into the differential equation by taking the time derivative, on both the sides

$$\frac{dx_i(t)}{dt} = \frac{d}{dt}\left\{ \sum_{j=1 j\neq i}^{n} \int_0^t W_{ij}\left(\frac{1}{\mu_i}\right) e^{\frac{-(t-\tau)}{\mu_i}} f_j(x_j(\tau))d\tau + \int_0^t \left(\frac{1}{\mu_i}\right) e^{\frac{-(t-\tau)}{\mu_i}} I_i(\tau)d\tau \right\} \tag{3.2.5}$$

Using the generalized Leibnitz rule, we get

$$\frac{dx_i(t)}{dt} = -\frac{1}{\mu_i}\int_0^t \left(\frac{1}{\mu_i}\right) e^{\frac{-(t-\tau)}{\mu_i}} \left[\sum_{j=1}^{n} W_{ij} f_j(x_j(\tau)) + I_i(\tau)\right] d\tau$$

$$+\frac{1}{\mu_i}\left[\sum_{j=1}^{n} W_{ij} f_j(x_j(t)) + I_i(t)\right]$$

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + a_i \left[\sum_{j=1}^{n} W_{ij} f_j(x_j(t)) + I_i(t)\right]$$

where, $a_i = 1/\mu_i$.

The above equation represents the state for the i^{th} neuron, which can be written down for all n neurons, so that we obtain a system of differential equations as

$$\mu_i \frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{n} W_{ij} f_j(x_j(t)) + I_i(t)$$

for $i = 1, 2, ..., n$ $\mu_i > 0$.

This is the fundamental system of differential equations that governs the dynamics of Hopfield networks. In fact, many other neural networks also have similar state equation.

The above equations can be written in the state space representation as follows:

$$\frac{dx}{dt} = Ax + Bu + HF(x) \tag{3.2.6}$$

$$x(0) = x_0$$

where, $x = [x_1, x_2, ..., x_n]^T$; $u(t) = [I_1(t), I_2(t), ..., I_n(t)]^T$

A and B are diagonal matrix of order n given by

$A = diag(-a_1, -a_2, ..., -a_n)$ and $B = diag(a_1, a_2, ..., a_n)$

where, $a_i = \frac{1}{\mu_i}$ for $i = 1, 2, ..., n$.

Also, $H = \begin{bmatrix} 0 & a_1 W_{12} & \dots & a_1 W_{1n} \\ a_2 W_{21} & 0 & \dots & a_2 W_{2n} \\ \vdots & & & \\ a_n W_{n1} & a_n W_{n2} & & 0 \end{bmatrix}; F(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \\ \vdots \\ f_n(x_n) \end{bmatrix}$.

The equation (3.2.6) is a time invariant continuous semilinear model for the Hopfield Network. We study the existence and uniqueness in the following section.

3.3 Existence and Uniqueness of Solution

By using the variation of parameter technique, the solution of equation (3.2.6) can be written as

$$x(t) = \Phi(t, 0)x_0 + \int_0^t \Phi(t, \tau)Bu(\tau)d\tau + \int_0^t \Phi(t, \tau)HF(x(\tau))d\tau \qquad (3.3.7)$$

where, $\Phi(t, \tau)$ is the state transition matrix for the homogeneous linear part of (3.2.6) (refer Chapter 2, Section 2.2.1, Equation (2.2.8)), and is given by

$$\Phi(t, \tau) = e^{A(t-\tau)} = \begin{bmatrix} e^{-a_1(t-\tau)} & 0 & \dots & 0 \\ 0 & e^{-a_2(t-\tau)} & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & e^{-a_n(t-\tau)} \end{bmatrix}$$

Thus (3.3.7) takes the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}HF(x(\tau))d\tau \qquad (3.3.8)$$

We prove the following result regarding the existence and uniqueness of solution of the Hopfield networks by using Generalized Banach Contraction Principle.

THEOREM 3.3.1 *If the transfer functions f_i's are Lipschitz continuous then there exist a unique solution for any initial condition $x_i(0) = x_{i0}$ for $i = 1, 2, ..., n$ and for any external control function $u \in L^2([0, T]; R^n)$.*

Proof : Under the assumptions given in the hypothesis equation (3.2.6) and (3.3.8) are equivalent. Hence, for the proving existence of solution of (3.2.6) we prove the solvability for (3.3.8). That is, we prove that for any initial condition $x(0) = x_0 \in R^n$ and for control $u \in L^2([0,T]; R^n)$ there exists unique solution for (3.3.7) provided F is Lipschitz continuous. If f_i's are Lipschitz continuous with constant α_i's then F is Lipschitz continuous with constant $\alpha = max(\alpha_i)$.

We employ the tools from operator theory to establish the proof.

Define an operator $K : C([0,T]; R^n) \to C([0,T]; R^n)$ given by:

$$(Kx)(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}HF(x(\tau))d\tau \tag{3.3.9}$$

Now,

$$\begin{aligned}
\| (Kx)(t) - (Ky)(t) \|_{R^n} &= \| \textstyle\int_0^t e^{A(t-\tau)}H(F(x(\tau)) - F(y(\tau)))d\tau \|_{R^n} \\
&\leq \textstyle\int_0^t \| e^{A(t-\tau)}H \| \| (F(x(\tau)) - F(y(\tau))) \|_{R^n} \, d\tau \\
&\leq p\alpha \textstyle\int_0^t \| (x(\tau) - y(\tau)) \|_{R^n} \, d\tau
\end{aligned}$$

where, $p = max_{t \in [0,T]} \| e^{A(t-t_0)}B \|$

That is,

$$\| (Kx)(t) - (Ky)(t) \|_{R^n} \leq p\alpha t \| x - y \|_{C([0,T];R^n)} \tag{3.3.10}$$

Hence,

$$\begin{aligned}
\| Kx - Ky \|_{C([0,T];R^n)} &= \sup_{t \in [0,T]} \| (Kx)(t) - (Ky)(t) \|_{R^n} \\
&\leq p\alpha T \| x - y \|_{C([0,T];R^n)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\| (K^2x - K^2y)(t) \| &= \| K(Kx)(t) - K(Ky)(t) \|_{R^n} \\
&\leq p\alpha \textstyle\int_0^t \| (Kx)(\tau) - (Ky)(\tau) \| \, d\tau \text{ by using (3.3.10)} \\
&\leq p^2\alpha^2 \textstyle\int_0^T \tau \| x - y \|_{C([0,T];R^n)} \, d\tau \\
&\leq p^2\alpha^2 \frac{T^2}{2!} \| x - y \|_{C([0,T];R^n)} \\
\| (K^3x - K^3y)(t) \| &\leq p^3\alpha^3 \frac{T^3}{3!} \| x - y \|_{C([0,T];R^n)}
\end{aligned}$$

In general,

$$\| (K^m x - K^m y)(t) \|_{C([0,T];R^n)} \leq \frac{(Tp\alpha)^m}{m!} \| x - y \|_{C([0,T];R^n)} \qquad (3.3.11)$$

As m tends to infinity the quantity $(Tp\alpha)^m/m!$ tends to zero. That is, there exists some m such that the Lipschitz constant $(Tp\alpha)^m/m!$ of K^m is less than 1 and hence K^m is a contraction mapping for some $m \geq 1$.

Therefore, by the generalized Banach Contraction Principle there exists a unique fixed point for the operator K which is solution to the system (3.3.7). Moreover, the solution can be computed by using the following successive approximation algorithm.

$$x^{n+1}(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}HF(x^n(\tau))d\tau \qquad (3.3.12)$$

$x^0(t)$ - arbitrary.

3.4 Stability Analysis of Hopfield networks

In this section, we deal with the stability of the Hopfield network (3.2.6), which is equivalent to (3.3.7). We will show that slightly perturbed initial condition or external inputs do not lead to a drastic change in the solution. That is, the system is able to tolerate the noisy initial conditions and converge to the stable state. We prove that the Hopfield system is Bounded Input Bounded Output (BIBO) stable and also prove its asymptotic stability.

THEOREM 3.4.1 *The system (3.2.6) is BIBO stable on* $[0, T]$.

Proof : We again use operator theory to show this. Define the solution operator

$$S : R^n \times C([0,T]; R^n) \to C([0,T]; R^n)$$

by

$$S(x_0, u) = x(.)$$

where, $x(.)$ is the unique solution of (3.3.7) corresponding to the initial state x_0 and external input u. Let (x_0, u) be the pair of initial condition and external input to the system and $(\tilde{x_0}, \tilde{u})$ be slightly changed initial condition and external input.

Let $S(x_0, u) = x$ and $S(\tilde{x_0}, \tilde{u}) = \tilde{x}$.

Therefore,

$$S(x_0, u) - S(\tilde{x_0}, \tilde{u}) = e^{At}(x_0 - \tilde{x_0}) + \int_0^t e^{A(t-\tau)} B(u(\tau) - \tilde{u}(\tau))d\tau$$

$$+ \int_0^t e^{A(t-\tau)} H(F(x(\tau)) - F(\tilde{x}(\tau)))d\tau$$

Hence,

$$\| S(x_0, u) - S(\tilde{x_0}, \tilde{u}) \| \leq \| e^{At} \| \| x_0 - \tilde{x_0} \| + \int_0^t \| e^{A(t-\tau)} \| \| B \| \| u(\tau) - \tilde{u}(\tau) \| d\tau$$

$$+ \int_0^t \| e^{A(t-\tau)} \| \| H \| \| F(x(\tau)) - F(\tilde{x}(\tau)) \| d\tau$$

$$\| x(t) - \tilde{x}(t) \| \leq a_1 \| x_0 - \tilde{x_0} \| + \textstyle\int_0^t a_1 b \| u(\tau) - \tilde{u}(\tau) \| d\tau$$

$$+ \int_0^t p\alpha \| x(\tau) - \tilde{x}(\tau) \| d\tau$$

where, $a_1 = sup \| e^{At} \|, b = \| B \|$

By using Gronwal's inequality we get

$$\| x(t) - \tilde{x}(t) \| \quad \leq [a_1(Tb \| u - \tilde{u} \| + \| x_0 - \tilde{x_0} \|)]e^{p\alpha T}$$

$$\text{That is, } \| x - \tilde{x} \| \quad \leq e^{p\alpha T}[\dot{a}_1(Tb \| u - \tilde{u} \| + \| x_0 - \tilde{x_0} \|)]$$

This implies that the solution operator S is Lipschitz continuous with respect to the initial state and the external stimulus $u(.)$. Therefore, bounded inputs results in bounded outputs and hence the theorem.

The next result shows that the Hopfield Neural Networks are asymptotically stable.

THEOREM 3.4.2 *The state of the Hopfield network is bounded as* $t \to \infty$ *for bounded input* x_0 *and control* $u(.)$.

Proof : We know that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)} HF(x(\tau))d\tau. \tag{3.4.13}$$

Hence,

$$\| x(t) \| \leq \| e^{At}x_0 \| + \int_0^t \| e^{A(t-\tau)} \| \| B \| \| u(\tau) \| d\tau + \int_0^t \| e^{A(t-\tau)} \| \| H \| F(x(\tau)) \| d\tau$$

Since A is the diagonal matrix given by $A = diag(-a_1, -a_2, ..., -a_n)$.

We have,

$$e^{At} = \begin{bmatrix} e^{-a_1 t} & 0 & \dots & 0 \\ 0 & e^{-a_2 t} & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & e^{-a_n t} \end{bmatrix}$$

$$\| e^{At} \|^2 = \sum_{i=1}^{n} e^{-2a_i t}$$

Thus, there exists a $\lambda \geq 0$, such that

$$\| e^{At} \| \leq e^{-\lambda t} \forall t. \tag{3.4.14}$$

Using (3.4.14) in (3.4.13) and using Gronwal's inequality we get,

$$\| x(t) \| \leq \left\{ \| x_0 \| e^{-\lambda t} + \alpha_0 \int_0^t e^{-\lambda(t-\tau)} h d\tau + \int_0^t e^{-\lambda(t-\tau)} b \| u(\tau) \| d\tau \right\} e^{h\alpha e^{-\lambda t}}$$

where, $h = \| H \|; \alpha_0 = \| F(0) \|$.

Taking limit as $t \longrightarrow \infty$, we get $\| x(t) \| < \infty$ for $t \longrightarrow \infty$, if $\| x_0 \|$ and $\| u \|$ are finite. Thus, $\| x(t) \|$ is bounded as $t \longrightarrow \infty$ for bounded input and control. Hence the system is asymptotically stable.

3.5 Controllability of Hopfield network

In this section, we investigate the controllability property of the Hopfield network (3.2.6). The controllability aspect of Hopfield and recurrent networks are studied by Anke Meyer Base [51] and Levin, A. U. [57].

We give sufficient conditions for the controllability of (3.2.6). Here we do not assume that the transfer functions $f_i's$ are of sigmoidal type. However, sigmoidal type functions are also included in the class of our functions.

THEOREM 3.5.1 *Suppose that the transfer functions $f_i's$ satisfy:*

1. $|f_i(x) - f_i(y)| \leq \alpha_i|x - y| \forall x, y \in R$ *for some constant* $\alpha_i > 0$

2. $|f_i(x)| \leq M_i \forall x \in R$ *for some positive constant* M_i

Then, the Hopfield network (3.2.6) is controllable on any finite interval $[0, T]$.

Proof : We will first show that the linear part of (3.2.6) i.e.

$$\dot{x} = Ax + Bu(t)$$

$$x(0) = x_0 \tag{3.5.15}$$

is controllable , and subsequently using fixed-point argument we prove that the non-linear system is also controllable.

Since B matrix is the identity matrix in R^n, the rank of the Kalman's controllability matrix

$$Q = [B|AB|A^2B|...|A^nB]$$

is n. Hence the linear system is controllable by the condition given in chapter 2, (refer Gopal [30]). Also, the controllability Grammian $W(0, T)$ for the system (3.2.6) is given by

$$W(0,T) = \int_0^T e^{A(T-\tau)}BB^*e^{A^*(T-\tau)}d\tau$$

$$= \begin{pmatrix} \frac{a_1}{2}[1 - e^{-2a_1T}] & 0 & \dots & 0 \\ 0 & \frac{a_2}{2}[1 - e^{-2a_2T}] & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & \frac{a_n}{2}[1 - e^{-2a_nT}] \end{pmatrix}$$

This controllability Grammian is invertible, hence the control

$$u(t) = B^*e^{A^*(T-\tau)}W^{-1}(0,T)(x_f - e^{AT}x_0) \tag{3.5.16}$$

$$= \begin{pmatrix} \frac{2e^{-a_1(T-\tau)}}{1-e^{-a_1T}}(x_{f1} - e^{-a_1T}x_{01}) \\ \frac{2e^{-a_2(T-\tau)}}{1-e^{-a_2T}}(x_{f2} - e^{-a_2T}x_{02}) \\ \vdots \\ \frac{2e^{-a_n(T-\tau)}}{1-e^{-a_nT}}(x_{fn} - e^{-a_nT}x_{0n}) \end{pmatrix}$$

steers the linear system (3.5.15) from $x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{bmatrix}$ to $x_f = \begin{bmatrix} x_{f1} \\ x_{f2} \\ \vdots \\ x_{fn} \end{bmatrix}$ during $[0, T]$.

Now we define a control for the nonlinear system

$$\dot{x} = Ax + Bu(t) + HF(x)$$
$$x(0) = x_0 \tag{3.5.17}$$

as

$$u(t) = B^* e^{A^*(T-\tau)} W^{-1}(0,T) \left[x_f - e^{AT} x_0 - \int_0^T e^{A(T-s)} HF(x(s)) ds \right] \tag{3.5.18}$$

and show that the above control (3.5.18) steers the nonlinear system (3.5.17) from x_0 to x_f.

We know that the state of (3.5.17) satisfies

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + \int_0^t e^{A(t-\tau)} HF(x(\tau)) d\tau \tag{3.5.19}$$

With the control given in (3.5.18) the state (3.5.19) becomes

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} BB^* e^{A^*(T-\tau)} W^{-1}(0,T) \left[x_f - e^{AT} x_0 - \int_0^T e^{A(T-s)} HF(x(s)) ds \right] d\tau$$
$$+ \int_0^t e^{A(t-\tau)} HF(x(\tau)) d\tau$$

At, $t = 0$ we get $x(0) = x_0$ which is obvious and putting $t = T$ we get

$$x(T) = e^{AT} x_0 + \int_0^T e^{A(T-\tau)} BB^* e^{A^*(T-\tau)} W^{-1}(0,T) (x_f - e^{AT} x_0 - \int_0^T e^{A(T-s)} HF(x(s)) ds) d\tau$$
$$+ \int_0^T e^{A(T-\tau)} HF(x(\tau)) d\tau$$
$$= e^{AT} x_0 + WW^{-1} \left[x_f - e^{AT} x_0 - \int_0^T e^{A(T-s)} HF(x(s)) ds \right] d\tau$$
$$+ \int_0^T e^{A(T-\tau)} HF(x(\tau)) d\tau$$
$$= x_f$$

Thus, the control defined by (3.5.18) is a steering control provided (3.5.19) possess a solution. For proving the existence of solution for (3.5.19) we define an operator $S : C([0,T]; R^n) \to C([0,T]; R^n)$ given as

$$(Sx)(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}BB^*e^{A^*(T-\tau)}W^{-1}(0,T)\left[x_f - e^{AT}x_0 - \int_0^T e^{A(T-s)}HF(x(s))ds\right]d\tau$$

$$+ \int_0^t e^{A(t-\tau)}HF(x(\tau))d\tau$$

If there exists a fixed point for the operator S then the fixed point will be the solution of the nonlinear equation (3.5.19). We use Schauder's fixed-point theorem for this purpose. First we show that S maps a closed bounded ball S_b into itself, *i.e.*$S(S_b) \subset S_b$

We know that,

$$\| Sx \|_C = \sup \| (Sx)(t) \|_{R^n}$$

$$\leq sup \| e^{At} \| \| x_0 \| + \int_0^t \| e^{A(t-\tau)} \| \| B \| \| B^* \| \| e^{A^*(T-\tau)} \| \| W^{-1}(0,T) \|$$

$$(\| x_f \| + \| e^{AT} \| x_0 \| + \int_0^T \| e^{A(T-s)}H \| Mds)d\tau + \int_0^t e^{A(t-\tau)}HMd\tau$$

where, $M = max\{M_i\}$ for $i = 1, 2, ..., n$.

By assumption, there exists M_0 such that

$$\| Sx \|_C \leq M_0$$

Choose $S_b = \{x(.) \in C([0,T]; R^n) : \| x \| \leq M_0\}$, which would imply S maps S_b into itself. Since F is Lipschitz continuous it follows that S is also continuous. Similarly, it can be shown that S is compact operator from S_b to S_b (refer Joshi & George [36]).

Therefore, by using Schauder's fixed point theorem S has fixed point x. Hence, the system is controllable.

The iterative formula for computing the steering control which steers the initial state x_0 to the desired final state x_f is given by

$$u^{k+1}(t) = B^*e^{A^*(T-\tau)}W^{-1}(0,T)(x_f - e^{AT}x_0 - \int_0^T e^{A(T-s)}HF(x(s))ds) \quad (3.5.20)$$

where,

$$x^{n+1}(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}HF(x^n(\tau))d\tau$$

3.6 Observability of Hopfield Networks

Suppose the associated output equation to the Hopfield Neural Network when the network reach stable state is

$$y(x) = Cx(t) \tag{3.6.21}$$

Definition : The Neural Networks represented by (3.2.6) is said to be completely observable if from the knowledge of observations $y(t)$ on the interval $[0, T]$ it is possible to find initial state x_0.

We prove the observability of the neural network represented by (3.2.6) with Lipschitz continuous nonlinearity f.

THEOREM 3.6.1 *Suppose that*

1. $rank[C|CA|CA^2|...|CA^{n-1}] = n$
2. f_i *is Lipschitz continuous, and that there exist constant* $\alpha_i > 0$ *such that* $|f_i(x) - f_i(y)| \leq \alpha_i |x - y|$ *for all* $x, y \in R$
3. $\alpha = max(\alpha_i) << 1$

Then the Hopfield network (3.2.6) is completely observable. Moreover, the initial state x_0 *can be computed from the following iterative scheme:*

$$x_0^{k+1} = M^{-1} \int_0^T e^{A^T s} C^T \{y(s) - \int_0^s Ce^{A(s-\tau)} Bu(\tau) d\tau ds - \int_0^s Ce^{A(s-\tau)} HF((Sx_0^k)(\tau)) d\tau\} ds$$

starting from any x_0^0, *where,*

$$M = \int_0^T e^{A^T t} C^T C e^{At} dt$$

.

Proof : We have seen that system of equations

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + a_i \left[\sum_{j=1}^{n} W_{ij} f_j(x_j(\tau)) + I_i(\tau) \right], \; i = 1, 2, ..., n \tag{3.6.22}$$

and

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t e^{A(t-\tau)}HF(x(\tau))d\tau \quad (3.6.23)$$

are equivalent.

Using (3.5.19) we have

$$y(t) = Cx(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t Ce^{A(t-\tau)}HF(x(\tau))d\tau \quad (3.6.24)$$

From theorem (3.3.1), we have for each x_0 in R^n, there exists a unique solution $(Sx_0)(.) = x(.)$ in $C([0,T], R^n)$. Hence (3.6.24) becomes

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t Ce^{A(t-\tau)}HF(Sx_0)(\tau)d\tau \quad (3.6.25)$$

Multiplying both sides of (3.6.25) by $e^{A^Tt}C^T$, we get

$$e^{A^Tt}C^Ty(t) = e^{A^Tt}C^TCe^{At}x_0 + \int_0^t e^{A^Tt}C^TCe^{A(t-\tau)}Bu(\tau)d\tau$$

$$+ \int_0^t e^{A^Tt}C^TCe^{A(t-\tau)}HF(Sx_0)d\tau \quad (3.6.26)$$

Integrating from 0 to T we get

$$\int_0^T e^{A^Tt}C^Ty(t)dt = \int_0^T e^{A^Tt}C^TCe^{At}x_0 + \int_0^T \int_0^s e^{A^Ts}C^TCe^{A(s-\tau)}Bu(\tau)d\tau ds$$

$$+ \int_0^T \int_0^s e^{A^Ts}C^TCe^{A(s-\tau)}HF(Sx_0)d\tau ds \quad (3.6.27)$$

We know that the first integral on the right is given by

$$M(0,T) = \int_0^T e^{A^Tt}C^TCe^{At}dt \quad (3.6.28)$$

is the observability Grammian. From hypothesis (1) it is clear that $M(0,T)$ is invertible, refer Brockett [14]. This has been shown by Anke Mayer Base [51] by using Kalman's rank condition, with $C = I_n$. Pre-multiplying (3.6.26) by the inverse of observability Grammian we get

$$M^{-1}\int_0^T e^{A^Tt}C^Ty(t)dt = x_0 + M^{-1}\int_0^T \int_0^s e^{A^Ts}C^TCe^{A(s-\tau)}Bu(\tau)d\tau ds$$

$$+M^{-1}\int_0^T \int_0^s e^{A^Ts}C^TCe^{A(s-\tau)}HF(Sx_0)d\tau ds \quad (3.6.29)$$

Define an operator $N : R^n \to R^n$ by

$$Nx_0 = M^{-1}\int_0^T e^{A^Tt}C^Ty(t)dt - M^{-1}\int_0^T e^{A^Ts}C^T\int_0^s Ce^{A(s-\tau)}Bu(\tau)d\tau ds$$

$$-M^{-1}\int_0^T e^{A^Ts}C^T\int_0^s Ce^{A(s-\tau)}HF(Sx_0)d\tau ds \qquad (3.6.30)$$

Using the above definitions (3.6.29) becomes

$$x_0 = Nx_0 \qquad (3.6.31)$$

For x_0 and $\tilde{x}_0$ in R^n, we have

$$Nx_0 - N\tilde{x}_0 = M^{-1}\int_0^T\int_0^s e^{A^Ts}C^TCe^{A(s-\tau)}H(F(S\tilde{x}_0) - F(Sx_0))d\tau ds$$

Therefore,

$$\| Nx_0 - N\tilde{x}_0 \| \leq \beta \| x_0 - \tilde{x}_0 \|$$

where, $\beta = m_1 a^3 b\alpha e^{\alpha b\alpha T}\frac{T^2}{2}$ with, $m_1 = \| M^{-1} \|$.

Since for sufficiently small α_i the value of β is strictly less than 1 and hence N is contraction.

Therefore, by Generalized Banach Contraction principle, (3.6.31) has unique solution x_0 and the solution can be computed from the iterates

$$x_0^{k+1} = Nx_0^k \qquad (3.6.32)$$

starting from arbitrary x_0^0.

This proves the observability of the Hopfield Neural Networks and hence the initial state can be computed from

$$x_0^{k+1} = M^{-1}\int_0^T e^{A^Ts}C^T\{y(s) - \int_0^s Ce^{A(s-\tau)}Bu(\tau)d\tau ds - \int_0^s Ce^{A(s-\tau)}HF((Sx_0^k)(\tau))d\tau\}ds$$

starting from any x_0.

The observability of Neural Network can also be established without the condition 3. in the above theorem if f is uniformly bounded.

THEOREM 3.6.2 *Suppose that*

1. $rank[C|CA|CA^2|...|CA^{n-1}] = n$

2. *the transfer functions f_i is continuous.*
3. *f_i is uniformly bounded, that is there exists a constant $k_i > 0$ such that $|f_i(x) \leq k_i$ for all $x \in R$.*

Then the Hopfield network is completely observable.

Proof : We have shown in the previous theorem that the observability of the Hopfield system (3.2.6) with (3.6.21) is equivalent to the solvability of the operator equation (3.6.31). We now show that N defined by (3.6.30) has the fixed point by using Schauder's fixed point theorem.

We have

$$\| Nx_0 \|_{R^n} \leq \| M^{-1} \int_0^T e^{A^T t} C^T y(t) dt \| + \| M^{-1} \| \int_0^T \| e^{A^T s} \| \| C^T \|$$
$$\int_0^s \| C \| \| e^{A(s-\tau)} \| \| B \| F((Sx_0)(\tau)) \| d\tau ds$$

Let

$$\begin{aligned} m_2 &= \| M^{-1} \textstyle\int_0^T e^{A^T t} C^T y(t) dt \| \\ m_1 &= \| M^{-1} \| \\ c &= \| C \| \\ k &= max\, \{k_i\} \end{aligned}$$

Therefore,

$$\| Nx_0 \|_{R^n} \leq m_2 + m_1 a^2 c^2 bk \frac{T^2}{2} = m_3$$

Let B_{m_3} be the compact ball in R^N defined by

$$B_{m_3} = \left\{ x \in R^N : \| x \| \leq m_3 \right\}$$

Since the solution operator S and F are continuous, N is also continuous. Moreover, N is a finite rank operator. Hence N is compact.

Therefore by the Schauder's fixed point theorem there exists a fixed point for this operator. Hence the system is completely observable.

3.7 Summary

In this chapter, we have defined a class of nonlinear functions, which on applying as transfer function to the Hopfield networks we obtain controllability and stability of the network. Also, such networks are observable. The study of controllability and stability of Neural Networks are important for understanding the behavior of network see Haykin [31] and Levin and Narendra [57]. Observe that the Hopfield Neural Network is represented by a semilinear dynamical system in which the nonlinearity depends only on the state variable. The controllability and observability have been obtained by using fixed point theorems and stability properties has been stuided using the spectral properties of the linear part.

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