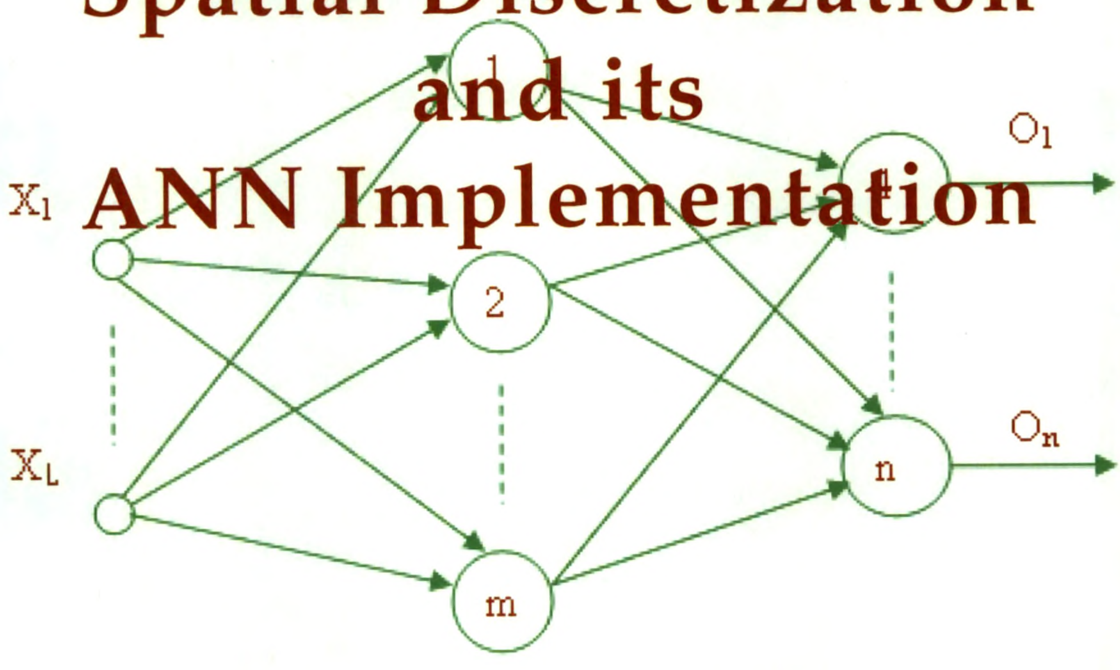
Chapter 7

Controllability of Parabolic Semilinear Dynamical Systems using Spatial Discretization and its ANN Implementation



Chapter 7

CONTROLLABILITY OF PARABOLIC SEMILINEAR DYNAMICAL SYSTEMS USING SPATIAL DISCRETIZATION AND ITS ANN IMPLEMENTATION

7.1 Introduction

In this chapter, we develop the steering controller for one dimensional semilinear parabolic thermal system. The linear parabolic version of such thermal systems is already been studied by Alotaibi S. and et. al. (refer [1]).

We establish the controller for one dimensional semilinear parabolic system by discretizing the system with respect to the space variable. By doing so, the one dimensional parabolic system reduces into a finite dimensional state space form. We can take care of both distributed controls and boundary controls in this setup. We implement the steering control by using Artificial Neural Networks (ANN).

The problem of controlled flow rate, which is common in thermal systems, is nonlinear in nature. Not much literature is available in the computation of steering controllers for such systems. We here make an attempt to obtain the controllers by approximating the thermal system by a finite dimensional system.

Consider a conductive bar of length L. It has a source at the end $x = L$, to be

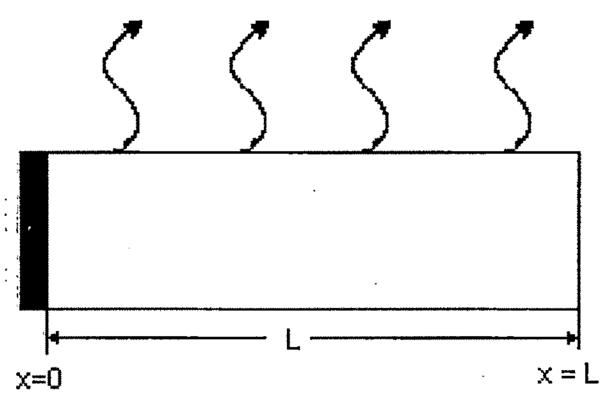


Figure 7.1: One dimensional conducting rod of length L

used whenever required, as shown in the Figure 7.1. This source can be used for rising or lowering the temperature.

The temperature distribution is governed by the partial differential equation

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} - \zeta(U - U_\infty) \tag{7.1.1}$$

where, $U = U(x,t)$ is the state of the system representing the temperature distribution along the bar, here t is time and x is the longitudinal coordinate measured from one end. U_∞ is the temperature of the surroundings.

Also, here α is the thermal diffusivity and

$$\zeta = hP/\rho c A_c$$

where, h is the convection heat transfer coefficient, A_c is the constant cross-sectional area of the bar, P is the perimeter of the cross-section, ρ is the density, and c is the specific heat.

The environment temperature, U_∞ or the temperature of the end $U(L,t)$ can be used for controlling the temperature of the rod. When U_∞ is used for controlling, it is referred to as distributed control problem, whereas, if $U(L,t)$ is used as temperature control source then it is called boundary control with the control variable being introduced through the boundary condition.

7.2 Controllability of Thermal Systems

In this chapter, we develop the controllability results concerning the distributed control as well as boundary control for the phenomenon with nonlinear behavior. In practise the heat flow depends on some nonlinear phenomenon so we take the model as

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} - \zeta(U - U_\infty) + g(U, U_\infty). \tag{7.2.2}$$

where, g is a nonlinear function of U and U_∞

The parabolic system is said to be controllable if from the given heat profile at time $t = 0$ the system can be brought to the desired heat profile in the finite time $t = t_f$. In the sequel we will assume, that the system initially is at uniform temperature U_0 and at the end, $x = 0$ of the rod, $(\partial U/\partial x)(0,t) = 0$.

System with Distributed Control :

In this section, we will analyze the system with the control variable being the ambient temperature U_∞ with a fixed boundary condition $U(L,t) = U_L$.

Using another variable $y = U - U_L$ the equation (7.2.2) becomes,

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial^2 y}{\partial x^2} - \zeta y + \zeta y_\infty + f(y, y_\infty) \tag{7.2.3}$$

where, $y_\infty = U_\infty + U_L$ and $f(y, y_\infty) = g(y + U_L, y_\infty + U_L)$.

The homogeneous boundary and initial conditions are given by $(\partial y/\partial x)(0,t) = 0$, $y(L,t) = 0$ and $y(x,0) = 0$. For convenience, the initial temperature distribution has been taken to be zero.

Denoting, $A = \alpha\frac{\partial^2}{\partial x^2} - \zeta$, $B = \zeta$, $u = y_\infty$ in equation (7.2.3) we get the system in the form

$$\frac{\partial y}{\partial t} = Ay + Bu + f(y,u) \tag{7.2.4}$$

The system (7.2.4) is the semilinear dynamical system in infinite dimension.

The controllability of the linear part of (7.2.4) can be established by the following result

LEMMA 7.2.1 *The linear part of the system (7.2.4) given by*

$$\frac{\partial y}{\partial t} = \alpha\frac{\partial^2 y}{\partial x^2} - \zeta y + \zeta y_\infty \tag{7.2.5}$$

is controllable.

Proof: Equation (7.2.5) can be written in the form

$$\frac{\partial y}{\partial t} = Ay + Bu \tag{7.2.6}$$

where, $A = \alpha\frac{\partial^2}{\partial x^2} - \zeta$, $B = \zeta$, $u = y_\infty$

Since A is self adjoint, it has real eigen values β_m, with $m = 0, 1, 2, \ldots$, and a complete orthonormal set of eigenfunctions $\phi_m(x)$ which forms a spatial basis for the solution of system(7.2.6).

Kalmka (refer [40]) has proved that the system such as (7.2.6), is approximately controllable if and only if the inner product

$$< \beta_m, \phi_m >= \int_0^L \beta_m \phi_m dx \neq 0 \text{ for all } m. \tag{7.2.7}$$

For our system, the eigenvalues and eigenfunctions are:

$$\beta_m = -\frac{(2m+1)^2\pi^2}{4L^2} - \zeta,$$

$$\phi_m = \sqrt{\frac{2}{L}} cos \frac{(2m+1)\pi x}{2L}.$$

which satisfies inequality (7.2.7) for all m, hence the system is state controllable.

For such system as (7.2.4), since the linear part is controllable, the Lipschitz and uniformly bounded nonlinear function f allows to infer that the semilinear system (7.2.4) will be approximately controllable refer George, R. K. [27], Anil [3].

For the realization of controller using ANN we however, require finite dimnsional approximation, which we have obtained using the technique of spatial discretization.

7.3 Finite-dimensional spatial approximation :

The approximate system is obtained by applying spatial discretization to the system (7.2.4). That is, for our system dividing the domain $[0, L]$, corresponding to x variable, into n equal parts of size Δx.

For each x_i corresponding to $i = 0, 1, ..., n$, we get $y_i = y(x_i, t)$.

Hence, a finite difference spatial discretization of equation (7.2.3) for i^{th} node $i > 0$, will be given as

$$\frac{dy_i}{dt} = \alpha \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x_i^2} - \zeta y_i + \zeta y_\infty(t) + f(y_i, y_\infty)$$

That is,

$$\frac{dy_i}{dt} = \frac{\alpha}{\Delta x^2} y_{i-1} - \left(2\frac{\alpha}{\Delta x^2} + \zeta\right) y_i + \frac{\alpha}{\Delta x^2} y_{i+1} + \zeta y_\infty(t) + f(y_i, y_\infty) \tag{7.3.8}$$

If we take $\sigma = \alpha/\Delta x^2$, (7.3.8) will be given as

$$\frac{dy_i}{dt} = \sigma y_{i-1} - (2\sigma + \zeta) y_i + \sigma y_{i+1} + \zeta y_\infty + f(y_i, y_\infty) \tag{7.3.9}$$

for $i = 1, 2, ..., n + 1$.

Here $i = 1$ indicates the end $x = 0$ and $i = n + 1$ indicates the right end $x = L$

of the rod.

Collecting the equations for all the nodes we get

$$\frac{dy}{dt} = Ay + Bu + f_n(y, u) \tag{7.3.10}$$

where, $y(t) = [y_1, y_2, ..., y_n]^T \in R^n$ and $u(t) = y_\infty \in R$,

$$A = \begin{bmatrix} -(\sigma+\zeta) & \sigma & 0 & \cdots & 0 \\ \sigma & -(2\sigma+\zeta) & \sigma & & \cdots \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & & & \\ 0 & \cdots & 0 & \sigma & -(2\sigma+\zeta) \end{bmatrix} \in R^{n\times n} \tag{7.3.11}$$

$$B = \zeta[1, ..., 1]^T \in R^n$$

and the nonlinear Lipschitz function

$$f_n = (f_1, f_2, ..., f_n)$$

where, $f_i = f(y_i, u)$.

The boundary conditions used at two ends are $y_0 = y_1$ and $y_{n+1} = 0$, respectively. Here, the boundary conditions are applied so as to make A non-singular.

The system (7.3.10) is finite dimensional semilinear system, which is approximation of the parabolic semilinear system (7.2.3).

THEOREM 7.3.1 *Suppose that the nonlinear function f is Lipschitz continuous and uniformly bounded. Then the finite dimensional discretized system (7.3.10) is controllable.*

Proof: The corresponding linear portion of the system is

$$\frac{dy}{dt} = Ay + Bu \tag{7.3.12}$$

For system (7.3.12), the controllability matrix $U_c = [B|AB|A^2B|...|A^{n-1}B] \in R^{n\times n}$.

If U_c has rank n, then the finite dimensional discretized system (7.3.10) is controllable.

It can be shown here that

$$\det U_c = (-1)^{\lfloor n/2 \rfloor} \sigma^{n(n-1)/2} \zeta^n$$

hence $\mathrm{rank}(U_c) = n$. Since f_n is Lipschitz and uniformly bounded, the semilinear system (7.3.10) is controllable (refer Remark 4.4.5) with the steering signal applied from the environment.

System with Boundary Control :

Now suppose, for the system (7.2.2), we use $U(L,t) = U_L(t)$ as controller for steering the initial heat profile of the rod to the desired profile. Using the constant outside temperature, U_∞ as reference and defining, $y = U - U_\infty$, equation (7.2.2) becomes

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial^2 y}{\partial x^2} - \zeta y + g(y) \tag{7.3.13}$$

with the initial and boundary conditions $(\partial y/\partial x)(0,t) = 0$, $y(L,t) = U_L(t) - U_\infty$ and $y(x,0) = 0$.

Again, as in case of distributed control, we spatially discretize system (7.3.13) taking n parts of length Δx to obtain the finite dimensional approximation.

The obtained finite dimensional approximation in case of boundary control is given by equations:

$$\frac{dy_i}{dt} = \sigma y_{i-1} - (2\sigma + \zeta) y_i + \sigma y_{i+1} + g(y_i) \tag{7.3.14}$$

for $i = 1, 2, ..., n+1$.

These equations in the matrix form can be written as

$$\frac{dy}{dt} = Ay + Bu + G_n(y) \tag{7.3.15}$$

where, $y(t) = [y_1, y_2, ..., y_n]^T \in R^n$, $u(t) = y(L,t) \in R$, A is same as given by equation (7.3.11) and $B = [0, ..., 0, \sigma]^T \in R^n$, $G_n(y) = [g_1(y_1), g_2(y_2), ..., g_n(y_n)]^T$ with $g_i(y_i) = g(y_i)$.

The adiabatic condition on the left end, that is $(\partial y/\partial t)(0,t) = 0$, is same as before. However, the temperature at the right end is acting as the control variable.

THEOREM 7.3.2 *Suppose that the nonlinear function g is Lipschitz continuous and uniformly bunded then the approximate n-dimensional system (7.3.15) is controllable.*

Proof: The controllability matrix, U_c for the linear system is given by

$$U_c = \begin{bmatrix} 0 & \dots & \dots & 0 & \sigma^n \\ 0 & \dots & 0 & \sigma^{n-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \sigma^3 & \dots & \dots \\ 0 & \sigma^2 & -2\sigma^2(2\sigma+\zeta) & \dots & \dots \\ \sigma & -\sigma(2\sigma+\zeta) & \sigma^3+\sigma(2\sigma+\zeta)^2 & \dots & \dots \end{bmatrix} \in R^{n \times n} \qquad (7.3.16)$$

Here, also $\text{rank}(U_c) = n$.

Since, U_c is full rank it is indicated that the state of the linear system is controllable even when the control is applied through boundary. Now, by applying the result from Chapter 4, we obtain that nonlinear system (7.3.15) is controllable as G_n is Lipschitz and bounded.

For the simulation of controller demonstrated in the the next section for distributed as well as boundary control we take the value of $\sigma = 1$ and $\zeta = 0.0118$.

Neural Network implementation of Steering Control :

In this section, we demonstrate the implementation of the distributed and boundary control using Artificial Neural Network. We consider the spatial discritization with $n = 4$ for our simulations. Thus, our system would be 5-dimensional systems. We first consider the linear systems for the boundary and distributed control, that is, assuming the $f(y) = 0$. In the next phase we simulate the semilinear systems for both type of controllers.

In the simulation for the continuous version of the problems (distributed as well as boundary) difficulty is faced in computing the controllability matrix and its inverse due to the special form of the matrices. Hence, we mainly concentrate on the discrete forms of the linear as well as semilinear systems for the distributed and boundary control.

In our simulations, the input-output patterns required for training the Neural Network are generated using definitions. In all the simulations our aim is to steer the initial temperature profile in the neighborhood of zero to the desired temperature profile

$$y(x,T) = x(x-1).$$

The desired temperature distribution, D for 5-dimensional discretized problems is,

$$D = \begin{bmatrix} 0 & 0.1875 & 0.2500 & 0.1875 & 0 \end{bmatrix}^T, \text{, with} L = 1.$$

ANN controller for Linear Systems :

The following linear system is considered with $n = 5, \sigma = 0.5$ and $\zeta = 0.0118$.

$$\dot{x} = Ax + Bu$$

where,

$$A = \begin{bmatrix} -0.5118 & 0.5000 & 0 & 0 & 0 \\ 0.5000 & -1.0118 & 0.5000 & 0 & 0 \\ 0 & 0.5000 & -1.0118 & 0.5000 & 0 \\ 0 & 0 & 0.5000 & -1.0118 & 0.5000 \\ 0 & 0 & 0 & 0.5000 & -1.0118 \end{bmatrix}$$

in both the case for the steering control.

The control matrix B will be different in the case of distribute and boundary control. To obtain the corresponding discrete systems the discretization is done using the sampling rate $ST = 1$. With this sampling factor the equivalent linear discrete system for the assumed parameters is given by

$$x(k+1) = Fx + Gu$$

where,

$$F = e^{A \times ST}$$

and

$$G = \int_0^{ST} e^{A}\tau B d\tau.$$

Thus, we have

$$F = \begin{bmatrix} 0.6658 & 0.2548 & 0.0574 & 0.0091 & 0.0011 \\ 0.2548 & 0.4684 & 0.2065 & 0.0494 & 0.0080 \\ 0.0574 & 0.2065 & 0.4604 & 0.2054 & 0.0484 \\ 0.0091 & 0.0494 & 0.2054 & 0.4593 & 0.1974 \\ 0.0011 & 0.0080 & 0.0484 & 0.1974 & 0.4109 \end{bmatrix}.$$

and G is discretized form of B.

Linear System with Distributed control :

The control matrix B for the continuous system in case of the distributed control is

$$B = \begin{bmatrix} 0.0118 & 0.0118 & 0.0118 & 0.0118 & 0.0118 \end{bmatrix}^T$$

The control matrix G for the corresponding discrete linear system is given by

$$G = \begin{bmatrix} 0.0117 & 0.0117 & 0.0117 & 0.0114 & 0.0095 \end{bmatrix}^T$$

The Neural Network with the architecture $N^2_{30,10}$ is trained to act as distributed controller for the linear 5-dimensional system which brings the initial temperature to the desired temperature profile in 10-time steps.

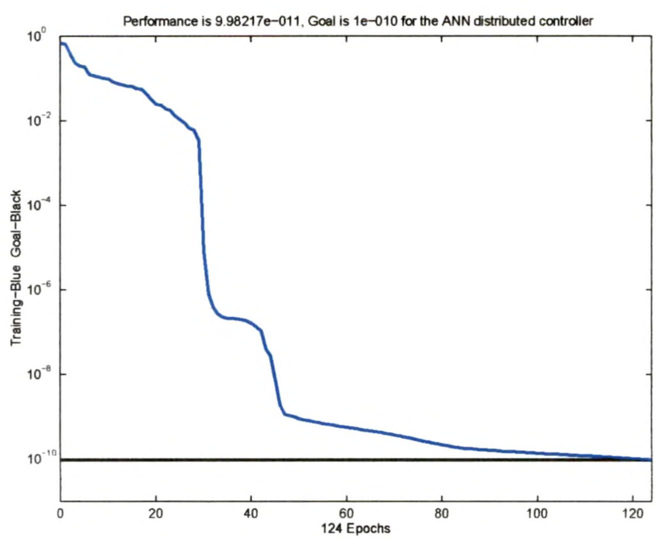


Figure 7.2: Convergence of Training for distributed ANN controller

100 input-output patterns are generated for training the NN controller using Back-propagation algorithm.

Input: The 5-dimensional vector giving the initial temperature.

Output: The 10-dimensional control vector produced using definition to steer the state from the initial state to the desired final temperature in 10 time steps. Each component in the control vector gives the scalar controller in each step.

The training of Neural Network converges in 124 epochs. The trained NN controller is checked for the initial temperature in the neighborhood to zero to be steered to the desired temperature

$$D = \begin{bmatrix} 0 & 0.1875 & 0.2500 & 0.1875 & 0 \end{bmatrix}^T.$$

For example, for the initial temperature distribution

$$\begin{bmatrix} 0 & .1 & .15 & .1 & 0 \end{bmatrix}^T$$

the steering signal is given by NN is

$$10^3 * \begin{bmatrix} -3.8235 \\ -2.2701 \\ -0.2228 \\ 2.1024 \\ 3.9658 \\ 3.7619 \\ -0.4880 \\ -6.5395 \\ 3.3228 \\ -0.4431 \end{bmatrix}^T$$

This NN signal steers the given initial temperature to the final temperature

$$\begin{bmatrix} 0.0032 & 0.1906 & 0.2528 & 0.1898 & 0.0015 \end{bmatrix}^T$$

The comparison between the desired and the reached temperature distribution is as shown in the figure 7.3.

The MATLAB program for the simulation is given in Appendix-A, NNgPARABOLIC_d.m.

Linear System with Boundary control :

The control matrix B for the continuous system in case of the distributed control is

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.5000 \end{bmatrix}^T$$

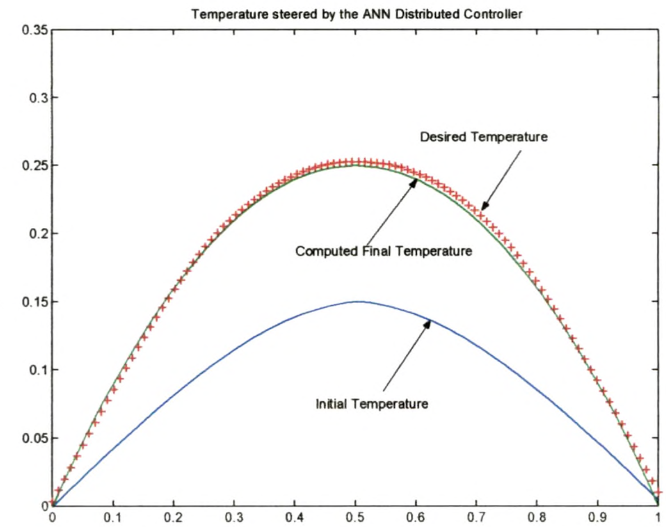


Figure 7.3: The final temperature steered by the ANN distributed controller

For the corresponding discrete linear system G is given by

$$G = \begin{bmatrix} 0.0001 & 0.0012 & 0.0103 & 0.0680 & 0.3247 \end{bmatrix}^T$$

The NN with the architecture $N^2_{30,10}$ is trained to act as boundary controller.

The I/O training patterns are:

Input: 5-dimensional vector which indicated temperature perturbed about zero.

Output: The 10-dimensional control signal

This signal steers the initial perturbed temperature to the desired temperature

$$D = \begin{bmatrix} 0 & 0.1875 & 0.2500 & 0.1875 & 0 \end{bmatrix}^T.$$

After, the successful training for the NN boundary controller in 679 epochs, it is checked with the initial temperature distribution

$$\begin{bmatrix} 0 & 0.1 & 0.1 & 0.1 & 0 \end{bmatrix}^T$$

The steering NN signal generated is

$$1.0e+003*\begin{bmatrix} -3.8235 \\ -2.2700 \\ -0.2227 \\ 2.1025 \\ 3.9660 \\ 3.7621 \\ -0.4880 \\ -6.5396 \\ 3.3228 \\ -0.4431 \end{bmatrix}$$

This NN signal steers the given temperature profile to

$$\begin{bmatrix} 0.0023 & 0.1898 & 0.2522 & 0.1895 & 0.0015 \end{bmatrix}^T$$

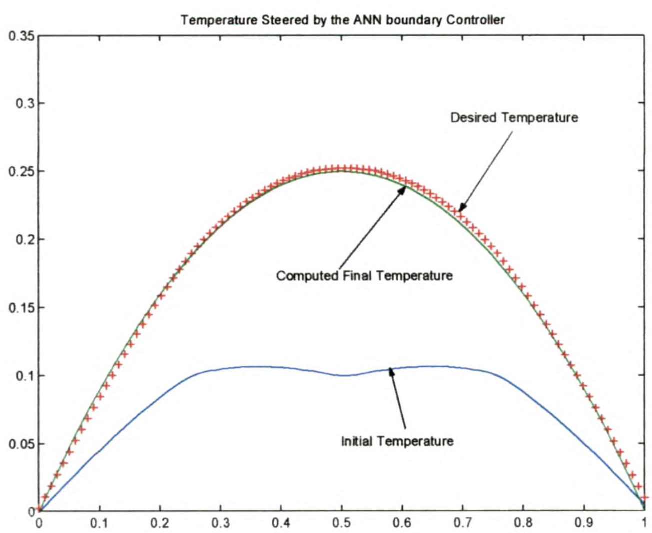


Figure 7.4: The final temperature distribution obtained in linear system by the ANN boundary controller

The Figure 7.4 shows the comparison of the desired and steered temperature distribution.

The MATLAB program for the simulation is given in Appendix-A, NNgPARABOLIC_b.m.

7.4 Simulation for Semilinear Thermal Systems :

The above simulation results convey that the linear parts of the corresponding semilinear systems are controllable using the ANN controllers in the boundary as well distributed forms.

For the simulation of semilinear system we consider the nonlinear function $f = \frac{1}{6}sin(x)$. This function f is Lipschitz, hence, it satisfies our condition for controllability.

The controllability of semilinear system is multistep problem as it requires solving of the coupled equations giving the evolution of state and control in each iteration (refer Chapter 4).

The Neural Networks controllers for the semilinear systems are trained as N-step controllers. Here, N denotes the number of steps required by the coupled equations to converge, starting from arbitrary (x^0, u^0).

Semilinear System with Distributed control :

For the distributed control of the semilinear system with the nonlinear function f as $\frac{1}{6}sin(x)$ it is observed that the solution converges to the desired one in 12-iterations for arbitrary (x^0, u^0). Hence, the NN with the architecture $N^2_{30,84}$ is trained to act as 7 time step steering control.

50 input-output pairs are generated using definitions for the training of the NN.

Input: The input vector is the 5-dimensional vector which signifies any x^0 in the neighborhood of zero. It is generated using the random function of MATLAB.

Output: The output produced by the NN is 84-dimensional vector where, the chunk of seven components contribute to the steering signal for steering the state to the next state, the next seven component steers the current state to the next next step and last seven components contribute to the signal that steers the system to the desired temperature profile.

The training of the NN as distributed control converged in 156 epochs. The validation is done for arbitrary initial temperature distribution. For example, for the arbitrary

$$x^0 = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.1 & 0 \end{bmatrix}^T$$

The NN distributed control signal is a 84 dimensional vector produced by the Neural Network, This NN signal steers the x^0 to the temperature profile

$$\begin{bmatrix} 0.0023 & 0.1898 & 0.2522 & 0.1895 & 0.0015 \end{bmatrix}^T$$

as desired.

The computations are in MATLAB program NNgPARABOLIC_nld.m shown in Appendix-A.

Semilinear System with Boundary control :

Neural Network with the architecture $N^2_{5,10,56}$ is trained to act as boundary Controller. As before 50 input-output patterns are generated using definitions to train the network.

The network is trained for seven time steps and it is observed for such a setup the coupled equations converge in 8 iterations. In this case, the Neural Network acts as 7 time step controller, with the output vector having 56 components. The eight chunks of consecutive seven components steers the temperature nearer and nearer to the desired temperature.

For example, for the arbitrary

$$x^0 = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.1 & 0 \end{bmatrix}^T$$

The NN boundary control signal is vector with 56-components. This NN signal steers the x^0 to the temperature profile

$$\begin{bmatrix} -0.0000 & 0.1875 & 0.2500 & 0.1875 & -0.0001 \end{bmatrix}^T$$

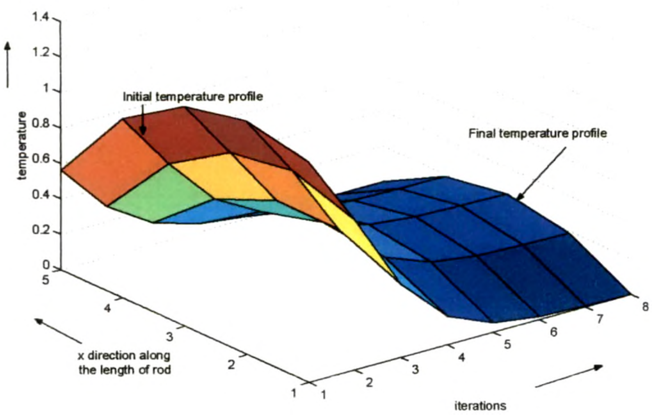


Figure 7.5: The evolution of state temperature distribution surface

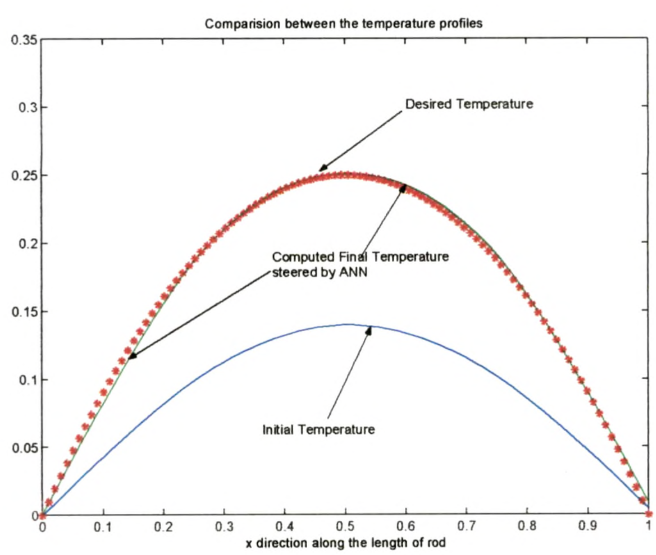


Figure 7.6: The temperature distribution steered in semilinear system by the ANN boundary controller

as desired.

The Figure 7.5 shows the gradual change in the temperature in each iteration, that is evolution of state over the time and the Figure 7.6 gives the comparison between the steered final temperature distribution and the desired temperature distribution along the rod.

The MATLAB program NNgPARABOLIC_nlb.m shown in Appendix-A is used for computation.

7.5 Summary

In this chapter, it is shown that the ANN algorithms cam be implemented as steering controllers for thermal systems with boundary controller or distributed controller. It has been proved in the literature that parabolic system is approximately controllable. To implement the ANN controllers we first discretize the parabolic equation into a finite-dimensional discretized equation. Controllability of this approximate system is established both for the boundary control and distributed control. After generating sufficient data for training, an ANN based controller has been developed for linear system as well as semilinear system with boundary control and distributed control.

The simulation results demonstrates that ANN can be easily used as controllers for the automated thermal systems. Hence, in the automation industry the ANN controllers have a role to play.

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