Appendix D

$\epsilon_{\alpha\beta}^{(L)}$ Matrix Elements for Electric and Magnetic Multipole Operators

The electric multipole operator T^{E_L} with multipolarity 2^L, in pn space is defined in terms of proton and neutron effective charges e_p and e_n and its explicit form is,

$$\begin{aligned} T^{E_L} &= e_p \left(\sum_{i=\text{proton}} r_i^L Y^L(\theta_i, \phi_i) \right) \\ &\quad + e_n \left(\sum_{i=\text{neutron}} r_i^L Y^L(\theta_i, \phi_i) \right) \end{aligned} \tag{D.1}$$

The T^{E_L} belongs to category [B] operator (5.2). Writing T^{E_L} in second quantized form and evaluating the reduced matrix elements of $Y^L(\theta, \phi)$ in oscillator single particle orbits, expression for $\epsilon_{\alpha\beta}^{(k=L)}(T^{E_L})$ matrix element is derived,

$$\begin{aligned} \epsilon_{\alpha\beta}^{(L)}(T^{E_L}) &= e_x \langle n_\alpha \ell_\alpha || r^L || n_\beta \ell_\beta \rangle \\ &\times (-1)^{j_\alpha + \frac{1}{2}} \sqrt{\frac{(2j_\alpha+1)(2j_\beta+1)}{16\pi}} \begin{pmatrix} j_\alpha & L & j_\beta \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \left\{ 1 + (-1)^{\ell_\alpha + \ell_\beta + L} \right\}, \\ &\text{for } x = p \text{ and } (\alpha, \beta) \in p \text{ or for } x = n \text{ and } (\alpha, \beta) \in n \end{aligned}$$

$$= 0, \text{ for } (\alpha \in p \text{ and } \beta \in n) \text{ or } (\alpha \in n \text{ and } \beta \in p) \quad \text{(D.2)}$$

The radial matrix elements $\langle n_\alpha \ell_\alpha || r^L || n_\beta \ell_\beta \rangle$ are given by ([Ha-79]),

$$\begin{aligned}
\langle n_f \ell_f | r^k | n_i \ell_i \rangle &= \int R_{n_f \ell_f}(r) r^k R_{n_i \ell_i}(r) r^2 dr \\
&= \sum_{\mu_f=0}^{n_f} \sum_{\mu_i=0}^{n_i} (-1)^{\mu_f+\mu_i} \left(\frac{1}{2\nu}\right)^{k/2} \binom{n_f}{\mu_f} \binom{n_i}{\mu_i} \\
&\times \sqrt{\frac{(2\ell_f + 2n_f + 1)!!\,(2\ell_i + 2n_i + 1)!!}{n_f!\; n_i!\; 2^{n_i+n_f}}} \\
&\times \frac{(\ell_f + \ell_i + 2\mu_f + 2\mu_i + k + 1)!!}{(2\ell_f + 2\mu_f + 1)!!\,(2\ell_i + 2\mu_i + 1)!!} \\
&\times \begin{cases} 1 & \text{if } \ell_1 + \ell_2 + k \text{ is even} \\ \sqrt{\frac{2}{\pi}} & \text{if } \ell_1 + \ell_2 + k \text{ is odd} \end{cases}
\end{aligned} \quad \text{(D.3)}$$

In Eq. (D.3), $\nu = \frac{1}{b^2} = \frac{m\omega}{\hbar}$; $\nu \simeq 0.96 A^{-1/3}\ fm^{-2}$.

The Magnetic multipole operator T^{M_L} with multipolarity 2^L in pn space is defined in terms of magnetic g-factors g_ℓ^p, g_s^p, g_ℓ^n and g_s^n

$$\begin{aligned}
T^{M_L} &= \sum_{i=\text{proton}} \left[\vec{\nabla}_i r_i^L Y_m^L(\theta_i \phi_i)\right] \cdot \left[g_l^p \frac{2\vec{\ell}_i}{L+1} + g_s^p \vec{s}_i\right] \\
&+ \sum_{i=\text{neutron}} \left[\vec{\nabla}_i r_i^L Y_m^L(\theta_i \phi_i)\right] \cdot \left[g_l^n \frac{2\vec{\ell}_i}{L+1} + g_s^n \vec{s}_i\right]
\end{aligned} \quad \text{(D.4)}$$

Writing $\tilde{g}$ as

$$\tilde{g}^x = \frac{2}{L+1} g_\ell^x \quad \text{(D.5)}$$

where $x = p$ or n and evaluating the gradient $\nabla'_\mu r^L Y_m^L$ using Eqs. 5.7.1 - 5.7.2 of [Ed-74] and Eqs. (A.2.19, A.2.20, A.2.21) of [Br-77] one gets

$$
\begin{aligned}
T^{M_L} &= \sum_{i=\text{proton}} \sqrt{L(2L+1)}r_i^{L-1}\left\{Y^{L-1}(\theta_i,\phi_i)\left[\tilde{g}^p\vec{j}_i + (g^p_{s_i} - \tilde{g}^p)\vec{s}_i\right]^1\right\}^L_M \\
&+ \sum_{i=\text{neutron}} \sqrt{L(2L+1)}r_i^{L-1}\left\{Y^{L-1}(\theta_i,\phi_i)\left[\tilde{g}^n\vec{j}_i + (g^n_{s_i} - \tilde{g}^n)\vec{s}_i\right]^1\right\}^L_M .
\end{aligned}
\tag{D.6}
$$

The $\epsilon^{(L)}_{\alpha\beta}(T^{M_L})$ matrix elements of T^{M_L} operator can be derived by evaluating the reduced matrix elements of $\left\{Y^{L-1}(\theta,\phi)\left[\tilde{g}\vec{j} + (g_s - \tilde{g})\vec{s}\right]\right\}^L$ and the result is,

$$
\begin{aligned}
\epsilon^{(L)}_{\alpha\beta}(T^{M_L}) &= \langle n_\alpha\ell_\alpha||r^{L-1}||n_\beta\ell_\beta\rangle\sqrt{L(2L+1)} \\
&\times\left\{\left[\frac{2g^x_\ell}{L+1}(-1)^{L+j_\alpha+j_\beta}\sqrt{j_\beta(j_\beta+1)(2j_\beta+1)}\right.\right. \\
&\times\left.\left\{\begin{array}{ccc} L-1 & 1 & L \\ j_\beta & j_\alpha & j_\beta \end{array}\right\}\langle j_\alpha||Y^{L-1}(\theta,\phi)||j_\beta\rangle\right] \\
&+\left[\left(g^x_s - \frac{2g^x_\ell}{L+1}\right)\sqrt{\frac{3}{2}(2j_\alpha+1)(2j_\beta+1)}\right. \\
&\times\left.\left.\left\{\begin{array}{ccc} \ell_\alpha & \ell_\beta & L-1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j_\alpha & j_\beta & L \end{array}\right\}\langle \ell_\alpha||Y^{L-1}(\theta,\phi)||\ell_\beta\rangle\right]\right\} , \\
&\quad \text{for } x=p \text{ and } (\alpha,\beta)\in p \text{ or for } x=n \text{ and } (\alpha,\beta)\in n \\
&= 0, \text{ for } (\alpha\in p \text{ and } \beta\in n) \text{ or } (\alpha\in n \text{ and } \beta\in p) .
\end{aligned}
\tag{D.7}
$$

In order to use the above equation, one should have the expressions for the double barred matrix elements $\langle \ell_\alpha||Y^L(\theta,\phi)||\ell_\beta\rangle$ and $\langle j_\alpha||Y^L(\theta,\phi)||j_\beta\rangle$ and they are,

$$
\langle \ell_\alpha||Y^L(\theta,\phi)||\ell_\beta\rangle = (-1)^{\ell_\alpha}\sqrt{\frac{(2\ell_\alpha+1)(2\ell_\beta+1)(2L+1)}{4\pi}}\left(\begin{array}{ccc} \ell_\alpha & L & \ell_\beta \\ 0 & 0 & 0 \end{array}\right)
$$

$$
\begin{aligned}
\langle \ell_\alpha \frac{1}{2} j_\alpha || Y^L(\theta,\phi) || \ell_\beta \frac{1}{2} j_\beta \rangle &= (-1)^{j_\alpha+1/2}\sqrt{\frac{(2j_\alpha+1)(2j_\beta+1)(2L+1)}{16\pi}} \\
&\times \begin{pmatrix} j_\alpha & L & j_\beta \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \left\{1+(-1)^{\ell_\alpha+\ell_\beta+L}\right\} . \quad \text{(D.8)}
\end{aligned}
$$

The expression for the radial matrix element $\langle n_\alpha \ell_\alpha || r^{L-1} || n_\beta \ell_\beta \rangle$ appearing in (D.7) is given by (D.3).