Chapter 2

Pairing and Seniority in Sn Isotopes

In the shell model, the nucleus is considered as a system of nucleons moving in a common potential well. Many properties of such idealized systems can be obtained from the study of the motion of one particle, the other particles showing their effect through the Pauli principle only, namely, in forbidding the occupation of some states by the particle considered. Obviously, a common single particle central interaction cannot replace completely the actual mutual interactions in a system of many particles. We are therefore interested in solving the Schrödinger equation

$$(H_0 + H_{residual})\Psi(r_1, r_2, ..., r_A) = E \cdot \Psi(r_1, r_2, ..., r_A) \tag{2.1}$$

where H_0 is the Hamiltonian of the central field

$$H_0 = \sum[T_i + U(r_i)] \tag{2.2}$$

and $H_{residual}$ may include corrections to the single particle potential $U(r_i)$ as well as additional two-body interactions between the particles.

2.1 Two Particle Configurations

We now discuss the effects of residual interactions which affect the energies of multi-particle configurations. If we consider two identical nucleons with angular momentum j_1 and j_2, they can be coupled to the total angular momentum J. The easiest method to calculate J is the "m-scheme", which is shown for the configuration $|(7/2)^2 J\rangle$ in tab. 2.1.

The states of two non-interacting particles moving in a central field are generally degenerate. The residual interactions affect the energies of two-particle configuration, leading to a difference in the energies $\triangle E$ which is given by

$$\triangle E(j_1 j_2 J) = \langle j_1 j_2 JM|V_{12}|j_1 j_2 JM\rangle \tag{2.3}$$

j_1=7/2	j_2=7/2		
m_1	m_2	M	J
7/2	5/2	6	
7/2	3/2	5	
7/2	1/2	4	
7/2	-1/2	3	6
7/2	-3/2	2	
7/2	-5/2	1	
7/2	-7/2	0	
5/2	3/2	4	
5/2	1/2	3	
5/2	-1/2	2	4
5/2	-3/2	1	
5/2	-5/2	0	
3/2	1/2	2	
3/2	-1/2	1	2
3/2	-3/2	0	
1/2	-1/2	0	0

*Only positive total M values are shown. The table is symmetric for M<0

Table 2.1: "m scheme" for the configuration $|(7/2)^2 J\rangle$*

In the following, we consider the simple δ-function interaction, which can be written as

$$V_{12}(\delta) = -V_0 \cdot \delta(r_1 - r_2) \tag{2.4}$$

Using the polar coordinates and performing angular momentum algebra calculations [3], we obtain for the energy shifts in the identical particle configuration $|j_1 j_2 J\rangle$

$$\Delta E(j^2 J) = -V_0 \cdot F_R(nl) \cdot A(j^2 J) \quad (J \ even) \tag{2.5}$$

where

$$F_R(nl) = \frac{1}{4\pi} \int \frac{1}{r^2} \cdot R_{nl}^4(r)\, dr \tag{2.6}$$

and

$$A(j^2 J) = \frac{(2j+1)^2}{2} \begin{pmatrix} j & j & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2 \tag{2.7}$$

Note that for J=0 the energy lowering is largest and the energy spacings decrease with J. For the 2-particle configuration $|7/2, 7/2, J\rangle$ the relative energy values $\Delta E/[V_0 F_R(nl)]$ are 4.0 (0^+ state), 0.952(2^+ state), 0.467(4^+ state) and 0.233(6^+ state). This property is identical to that defined for a pairing interaction.

2.2 Geometrical Analysis of δ-Function Residual Interaction

It is also possible to approach this entire subject from an alternate view point and actually derive the typical behaviour of the 3J-symbol from a simple geometrical analysis. Using a semi-classical concept, we can determine an angle θ between the angular momentum vectors j_1 and j_2 of the two particles, as illustrated schematically in fig. 2.1. In this case,

$$cos\theta = \frac{J^2 - j_1^2 - j_2^2}{2|j_1||j_2|} \tag{2.8}$$

Note that θ=180° corresponds to J=0.

In the following, we make use of some trignometric equations. Finally we can approximate the 3-J symbol for identical particles by

$$\begin{pmatrix} j & j & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2 \approx \frac{sin^2\frac{\theta}{2}}{\pi \cdot j^2 \cdot sin\theta} \tag{2.9}$$

For the angular dependence of the δ-function residual interaction we obtain

$$\Delta E(j^2 J) = \frac{-V_0 \cdot F_R}{\pi} tan\frac{\theta}{2} \quad (T = 1, J \ even) \tag{2.10}$$

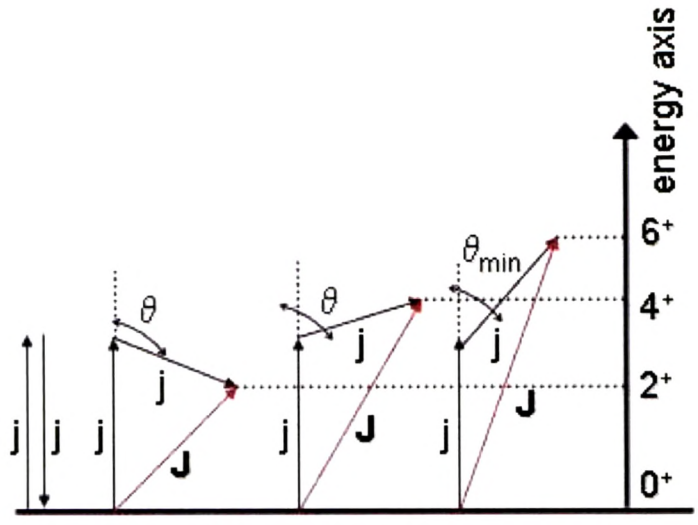

Figure 2.1: Schematic illustration of geometrical interpretation of short-range residual interaction for two particle configuration j_1 and j_2.

This extremely simple result expresses the energy shifts in different J states for a δ-interaction between two identical particles in equivalent orbits. It was derived for large j, J, but is remarkably accurate even for low spins. This property is identical to that defined for a pairing intercation.

For the 2-particle configuration $|(7/2)^2, J\rangle$ the angle θ between the angular momentum vectors is 171° (0^+ state), 143° (2^+ state), 111° (4^+ state) and 72° (6^+ state). For ^{112}Sn the geometrical analysis of the partial level scheme is shown in fig. 2.2. In the T=1 case, the empirical energy distribution follows the expected energy pattern quite well.

2.3 Generalized Seniority Scheme

The shell model [7] has been used for many years to describe the structure of nuclei, especially those that are fairly light or moderately near closed shells. With the steady improvement of computers, the size of the model space that can be accomodated has grown, expanding the region of nuclei that can be treated. Even so, some nuclear properties are not well explained with the valence space comprising a single major shell, suggesting the need for still larger model spaces. It is clear, however, that when the size of the single-particle valence space is increased, some truncation of the configuration space is necessary if calculations are to be carried out. One possibility is to use the generalized-seniority [5] or broken-pair [8] approx-

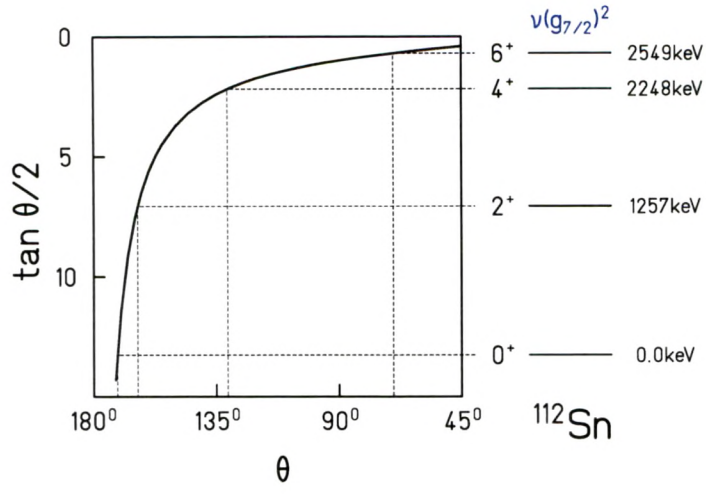

Figure 2.2: A geometrical analysis of the $|(g_{7/2})^2, J\rangle$ multiplet in ^{112}Sn. The angular dependence of the δ-function residual interaction strength is shown on the left with the semiclassical angle for the two identical particles in the equivalent orbits (T=1).

imation. This method has been widely used to approximate the shell model [9], especially when dealing with semi-magic nuclei. In the following, a brief overview of the generalized-seniority approach is presented [3].
The seniority scheme was introduced by Racah [10, 11] for the classification of states in atomic spectra. His aim was to find an additional quantum number in order to distinguish between states of electron ℓ^n configurations which have the same values of S, L and J (and M). This problem arises most frequently in LS-coupling, which is the prevalent coupling scheme of atomic electrons, than in jj-coupling. The seniority scheme for j^n configurations of identical nucleons was introduced by Racah [12] and independently by Flowers [13].

The scheme introduced by Racah is based on the idea of pairing of particles into J=0 pairs. Loosely speaking, the seniority quantum number ν , is equal to the number of unpaired particles in the j^n configuration, where n is the number of valence nucleons. The doubly-magic core nucleus plays the role of the vacuum. In the special case of j^2 configurations, there is complete pairing in the J=0 state and its seniority is $\nu = 0$. In all other j^2 states, with J=2, 4,.....2j-1, there are no pairs coupled to J=0 and the seniority is ν=2. Hence, the low energy states of (semi-magic) nuclei are states of low generalized seniority. The shell-model problem can now be solved in a truncated space of low generalized seniority.

Of particular interest are the energy levels in semi-magic nuclei with ν=2 and J=2, 4,...., 2j-1. They appear in all j^n configurations with n even and 2j+1>n>0. If the two-body interaction V=$\sum_{i<k}^{n} V_{ik}$ is diagonal in the seniority scheme, they should have the same spacing

(also between them and the J=0, ν=0 ground state) in all nuclei. The relevant formulae for the energy differences are given below

$$\begin{aligned} E(j^n, \nu = 2, J) - E(j^n, \nu = 0, J = 0) &= \langle j^2 J|V|j^2 J\rangle + \frac{n-2}{2}V_0 - \frac{n}{2}V_0 \\ &= \langle j^2 J|V|j^2 J\rangle - V_0 \end{aligned} \tag{2.11}$$

The energies of the ν=2 states are independent of n. For the spacing within the ν=2 states one obtains

$$\begin{aligned} E(j^n, \nu = 2, J) - E(j^n, \nu = 2, J') &= \left[\langle j^2, J|V|j^2, J\rangle + \frac{n-2}{2}V_0\right] \\ &- \left[\langle j^2, J'|V|j^2, J'\rangle + \frac{n-2}{2}V_0\right] \\ &= \langle j^2, J|V|j^2, J\rangle - \langle j^2, J'|V|j^2, J'\rangle \end{aligned} \tag{2.12}$$

Thus, all the energy differences of seniority ν=0 and ν=2 states in the n-particle configuration are identical to those in the two-particle system and are independent of n. It is also important to note, that the two-body interaction matrix elements of seniority ν states in the j^n configuration (n even) are related to the matrix elements in a j^ν configuration.
The Sn nuclei provide a classic example of eq. 2.11 and its generalization to the multi-j case: the entire known set of ν=2 levels, J=0^+, 2^+,... is virtually constant across an entire major shell (fig. 2.3)

For the E2 transition rates, induced by the operator Q=r^2Y_2, from the first 2^+ state to the 0^+ ground state in even-even nuclei one obtains

$$\begin{aligned} \left\langle j^n, J = 2 \parallel Q \parallel j^n, J = 0\right\rangle^2 &= \left[\tfrac{n\cdot(2j+1-n)}{2\cdot(2j-1)}\right]\cdot\left\langle j^2, J = 2 \parallel Q \parallel j^2, J = 0\right\rangle^2 \\ &= \frac{(2j+1)^2}{2\cdot(2j-1)}\cdot f\cdot(1-f) \\ &\cdot \left\langle j^2, J = 2 \parallel Q \parallel j^2, J = 0\right\rangle^2 \end{aligned} \tag{2.13}$$

where f=n/(2j+1) is the fractional filling of the shell. One should notice that matrix elements in the configuration j^n are linked to those in the configuration j^ν which is the power of the seniority scheme. For shells that are not too filled, so that (2j$\pm$1)$\gg$n, this becomes

$$\left\langle j^n, J = 2 \parallel Q \parallel j^n, J = 0\right\rangle^2 \approx \frac{n}{2}\left\langle j^2, J = 2 \parallel Q \parallel j^2, J = 0\right\rangle^2 \tag{2.14}$$

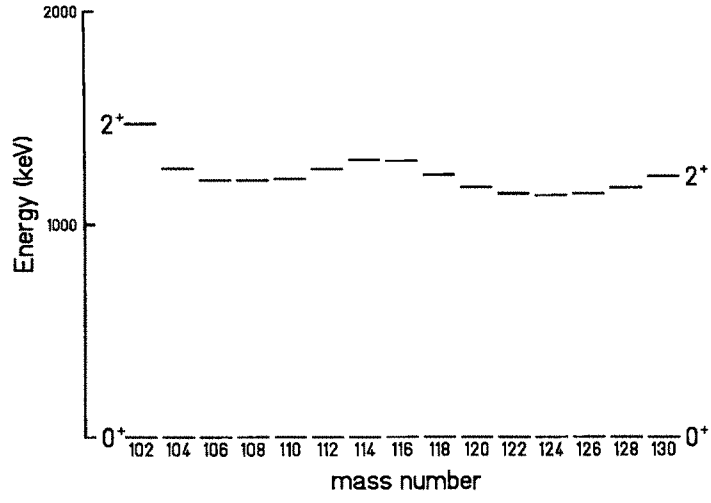

Figure 2.3: Partial level schemes [NNDC] in the even-A Sn nuclei for mass numbers between A=102 and A=130.

This expression increases as the number of particles n in the shell. The reduced transition probability is defined as

$$B(E2; J_i \rightarrow J_f) = \frac{1}{2J_i+1}\left\langle J_f \parallel Q \parallel J_i \right\rangle^2 \tag{2.15}$$

In the j^n configuration the B(E2) value is just propotional to the number of particles n in the shell, for small n. For large n, $n \rightarrow$ 2j+1, it falls off, vanishing, as it must, at the closed shell. For j, n$\gg$2, we see that, as given in the general case (eq. 2.13) above,

$$B(E2; 2_1^+ \rightarrow 0_1^+) \approx f \cdot (1-f) \tag{2.16}$$

This behaviour is commonly observed in real nuclei, with B(E2;$2_1^+ \rightarrow 0_1^+$) values rising to mid-shell, and falling thereafter. Data beautifully illustrating this are shown for Z=50, N=50-82 in (fig. 2.4). For transitions that do not change seniority, the expression is [assuming ν=2]

$$\begin{aligned} \left\langle j^n, J \parallel Q \parallel j^n, J' \right\rangle &= \left(\tfrac{2j+1-2n}{2j-3}\right)\left\langle j^2, J \parallel Q \parallel j^2, J' \right\rangle \\ &= \frac{2j+1}{2j-3} \cdot [1-2f] \cdot \left\langle j^2, J \parallel Q \parallel j^2, J \right\rangle \end{aligned} \tag{2.17}$$

It has an interesting behaviour as a function of n. For low ν the numerator goes simply as (1-2f). It therefore has opposite signs in the first and second halves of the shell and hence

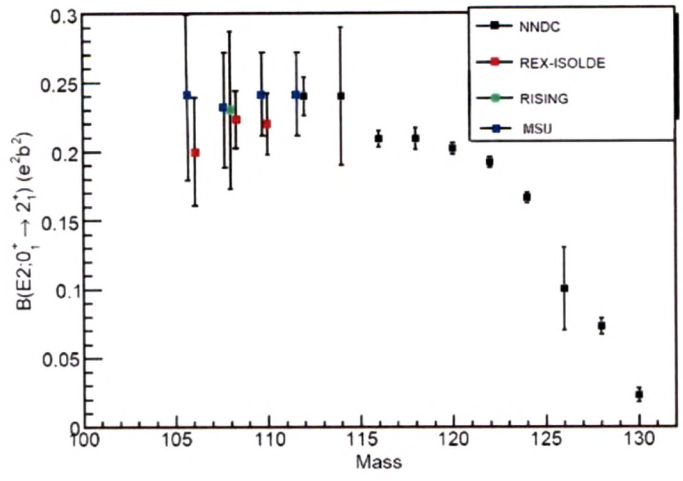

Figure 2.4: B(E2 ↑) values in the even-A Sn nuclei for mass numbers between A=106 and A=130. The data are from NNDC and [6, 14, 15, 16].

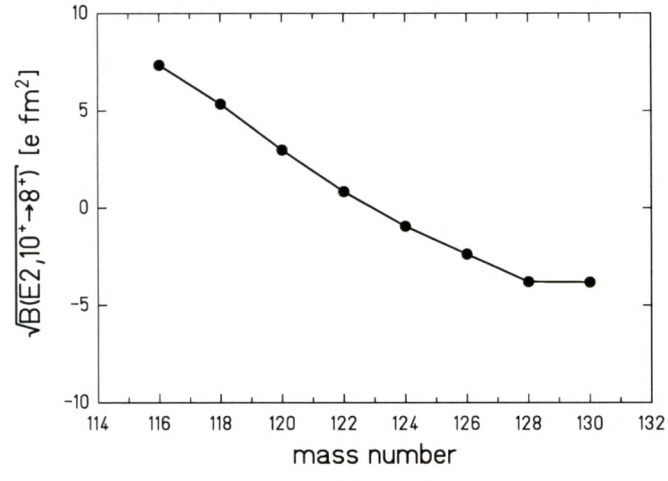

Figure 2.5: Measured E2 transition matrix elements for the $(\nu h_{11/2})^n$ $10^+ \rightarrow 8^+$ transitions in even-A Sn isotopes [17, 18].

must vanish identically at mid-shell. This is why nuclei are prolate at the begining of the shell and (sometimes) oblate at the shell.
Fig. 2.5 shows the square root of the reduced transition probabilities $B(E2, 10^+ \rightarrow 8^+)$ versus the mass number in even-A Sn isotopes. The square root leaves ambiguity about the sign of the E2 matrix element, but in practice this causes no difficulty because opposite signs are required in the bottom and top halves of the subshell.