Part II

MULTIVARIATE POWER-SERIES DISTRIBUTIONS

Chapter 5

MULTIVARIATE POWER-SERIES DISTRIBUTIONS

5.1: Introduction and summary:-

Noack (55) has defined a univariate power-series distribution as

(5.1.1) $Pr(\xi=x) = a_x Z^x/f(Z) \quad , x=0,1,2,\ldots$

where $a_x Z^x \geq 0$, a_x is a function of x or constant and $f(Z) = \sum_{x=0}^{\infty} a_x Z^x$ is convergent for some value of $|Z| \leq r$.

We (41) extend this definition to Multivariate power-series as

(5.1.2) $Pr(\xi_1=x_1,\ldots,\xi_k=x_k)=a_{x_1,\ldots,x_k} Z_1^{x_1}\ldots Z_k^{x_k}/f(Z_1,\ldots,Z_k)$,

$x_i=0,1,2,\ldots$ $(i=1,2,\ldots,k)$ where $a_{x_1,\ldots x_k} Z_1^{x_1}\ldots Z_k^{x_k} \geq 0$,

$a_{x_1,\ldots,x_k}$ is a function of $x_1,\ldots,x_k$ or constant and

$f(Z_1,\ldots,Z_k)=\sum_{\text{all } x_i} a_{x_1,\ldots x_k} Z_1^{x_1}\ldots Z_k^{x_k}$ is convergent.

Recurrence relations for the cumulants and the factorial-cumulants in multivariate as well as univariate cases are established and the problem of obtaining the distribution from its first two moments (or means, variances and covariances) has been solved. As illustrations and applications of these results, multivariate Poisson, negative multinomial, multinomial, multivariate logarithmic power-series, generalised Poisson and some other distributions are obtained and their properties derived.

All the above results are extended to truncated power-series distributions for multivariate and univariate cases.

5.2: Relations between cumulants and factorial-cumulants:-

r-th factorial-moment is defined by $E\{\xi(\xi-h)\ldots(\xi-rh+h)\}$ i.e. the coefficient of $\theta^r/r!$ in the expansion of $E(1+\theta h)^{\xi/h}$ if $h\neq 0$.

Hence if $F(\theta)$=factorial-moment generating function, then

$$(5.2.1)\qquad F(\theta) = E(1+\theta h)^{\xi/h}.$$

Now if $\phi(\theta)$= moment generating function$=E(e^{\theta\xi})$, then

$$(5.2.2)\qquad F(\theta)=\phi\{\log(1+\theta h)^{1/h}\} \text{ and } \phi(\theta)=F\{(e^{\theta h}-1)/h\}.$$

Hence the relation between the factorial-cumulants and cumulants can be easily seen to be the same as that bet-ween factorial-moments and moments. We note that

(5.2.3) as $h\to 0$, $F(\theta)=\phi(\theta)$ and $K_{[r]}=K_r$ where $K_{[r]}$ is the r-th factorial-cumulant and K_r is the r-th cumulant.

The relationsbetween factorial-moment generating function and the moment generating function for multivariate case is easily given by

$$(5.2.4)\quad F(\theta_1,\dots,\theta_k)=\emptyset\left\{\log(1+\theta_1 h_1)^{1/h_1},\dots,\log(1+\theta_k h_k)^{1/h_k}\right\}$$

and $\emptyset(\theta_1,\dots,\theta_k)=F\left\{(e^{\theta_1 h_1}-1)/h_1,\dots,(e^{\theta_k h_k}-1)/h_k\right\}$

for all $h_i \neq 0$.

5.3: Properties of Power-series distributions:-

We shall first establish the following theorem:

Theorem I:- If $K_{[r_1,\dots,r_k]}$ is the factorial-cumulant of the distribution given in (5.1.2), then

$$K_{[r_1,\dots,r_k]}=\prod_{i=1}^{k} Z_i^{r_i h_i}\left(Z_i^{-h_i+1}\frac{\partial}{\partial Z_i}\right)^{r_i}\log f(Z_1,\dots,Z_k)$$

and $K_{[r_1,\dots,r_{i-1},r_i+1,r_{i+1},\dots,r_k]}=Z_i\dfrac{\partial K_{[r_1,\dots,r_k]}}{\partial Z_i}-r_i h_i K_{[r_1,\dots,r_k]}$, where $(Z^t\frac{\partial}{\partial Z})^j=(Z^t\frac{\partial}{\partial Z})(Z^t\frac{\partial}{\partial Z})\dots$ j times.

Proof:- It can be proved by induction that

$$(1+\theta h)^r\frac{\partial^t}{\partial\theta^t}\Phi\left\{Z(1+\theta h)^{1/h}\right\}=Z^{rh}\left(Z^{-h+1}\frac{\partial}{\partial Z}\right)^r\Phi\left\{Z(1+\theta h)^{1/h}\right\}$$

and so we can easily establish

$$(5.3.1)\quad\frac{\partial^t}{\partial\theta^t}\log\Phi\left\{Z(1+\theta h)^{1/h}\right\}=(1+\theta h)^{-t}Z^{ht}\left(Z^{-h+1}\frac{\partial}{\partial Z}\right)^r\log\Phi\left\{Z(1+\theta h)^{1/h}\right\}.$$

Now by the use of (5.2.1), it can be easily shown that the factorial-moment generating function for (5.1.2) is

(5.3.2) $$F(\theta_1,\ldots,\theta_k)=\frac{f\left\{Z_1(1+\theta_1 h_1)^{1/h_1},\ldots,Z_k(1+\theta_k h_k)^{1/h_k}\right\}}{f(Z_1,\ldots,Z_k)}\ .$$

Therefore $K_{[r_1,\ldots,r_k]}=\left[\prod_{i=1}^{k}\frac{\partial^{t_i}}{\partial\theta_i^{t_i}}\log F(\theta_1,\ldots,\theta_k)\right]_{\text{each }\theta_i=0}$.

(5.3.3) i.e. $K_{[r_1,\ldots,r_k]}$

$$=\left[\prod_{i=1}^{k}\frac{\partial^{t_i}}{\partial\theta_i^{t_i}}\log f\left\{Z_1(1+\theta_1 h_1)^{1/h_1},\ldots,Z_k(1+\theta_k h_k)^{1/h_k}\right\}\right]_{\text{each }\theta_i=0}\ .$$

After using the relation (5.3.1), we can write (5.3.3) as

(5.3.4) $$K_{[r_1,\ldots,r_k]}=\prod_{i=1}^{k} Z_i^{r_i h_i}\left(Z_i^{-h_i+1}\frac{\partial}{\partial Z_i}\right)^{r_i}\log f(Z_1,\ldots,Z_k)\ .$$

Differentiating (5.3.4) with respect to Z_i, we can easily obtain the recurrence relation in factorial-cumulants as stated in the above theorem.

<u>Corollary 1</u>:- If each $h_i=0$, we have the relations for cumulants as

(5.3.5) $$K_{r_1,\ldots,r_k}=\prod_{i=1}^{k}\left(Z_i\frac{\partial}{\partial Z_i}\right)^{r_i}\log f(Z_1,\ldots,Z_k)$$

$$=Z_i\,\frac{\partial K_{r_1,\ldots,r_{i-1},r_i-1,r_{i+1},\ldots,r_k}}{\partial Z_i}\ .$$

<u>Corollary 2</u>:- If each $h_i=1$, we have the relations for factorial-cumulants as

(5.3.6) $$K_{[r_1,\ldots,r_k]}=\prod_{i=1}^{k} Z_i^{r_i}\left(\frac{\partial}{\partial Z_i}\right)^{r_i}\log f(Z_1,\ldots,Z_k)$$

i.e. $K_{[r_1,\dots,r_{i-1},r_i+1,r_{i+1},\dots,r_k]}=(Z_i\frac{\partial}{\partial Z_i}-r_i)K_{[r_1,\dots,r_k]}$.

<u>Corollary 3</u>:- For the univariate power-series distribution (5.1.1), the relations for factorial-cumulants and cumulants are as

(5.3.7) $K_{[r]}=Z^{rh}(Z^{-h+1}\frac{d}{dZ})^r \log f(Z)=\left\{Z\frac{d}{dZ}-h(r-1)\right\}K_{[r-1]}$ and

$$K_r=(Z\frac{d}{dZ})^r \log f(Z) = Z\,\frac{dK_{r-1}}{dZ}.$$

<u>Theorem II</u>:- The power series distribution is uniquely determined from its means, variances and covariances.

<u>Proof</u>:- Let us assume that means μ_i $i=1,2,\dots,k$, variances and covariances σ_{ij} $i,j=1,2,\dots,k$ of random variables ξ_i $i=1,2,\dots,k$ be given in terms of $y_1,y_2,\dots,y_k$ which are functions of some unknown parameters $Z_1,Z_2,\dots,Z_k$.

(5.3.8) Let $f_i=\frac{\partial}{\partial y_i}\log f(Z_1,\dots,Z_k)$ where $f(Z_1,\dots,Z_k)$ is a power-series to be determined from means, variances & covariances, $n_{ij}=\frac{\partial \mu_j}{\partial y_i}$, $m_{ij}=\frac{\partial y_j}{\partial \log Z_i}$ & $g_{ij}=\frac{\partial \log Z_j}{\partial y_i}$. Let us define the matrices $\underline{\mu}=(\mu_i):k\times 1$, $\Sigma=(\sigma_{ij}):k\times k$, $\underline{f}=(f_i):k\times 1$, $N=(n_{ij}):k\times k$, $M=(m_{ij}):k\times k$ & $G=(g_{ij}):k\times k$. Then it is evident that $G=M^{-1}$ or $M=G^{-1}$.

From the set of equations (5.3.5), we have $\mu_i=Z_i\frac{\partial}{\partial Z_i}\log f(Z_1,\dots,Z_k)=\sum_{t=1}^{k} m_{it}f_t$ i.e. $\underline{\mu}=M\underline{f}$, and

$$\sigma_{ij}=Z_i\frac{\partial\mu_j}{\partial Z_i}=\sum_{t=1}^{k} m_{it}n_{tj} \quad \text{i.e. } \Sigma=MN.$$

Hence we have

(5.3.9) $G=M^{-1}=(\Sigma N^{-1})^{-1}=N\Sigma^{-1}$ and $\underline{f}=M^{-1}\underline{\mu}=G\underline{\mu}$.

Since Σ is known & N can be obtained from μ_i, $i=1,2,\ldots,k$, G can be calculated from $G=N\Sigma^{-1}$ and then integrating g_{ij} with respect to y_i, we have

$$\log Z_j=\int g_{ij}\,dy_i + c_j(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k)$$

where $c_j(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k)$ is a constant of integration, not depending on y_i. This equation must be true for all i and so

(5.3.10) $\log Z_j=g_j(y_1,\ldots,y_k)+c_j$ or $Z_j=p_j\exp\{g_j(y_1,\ldots,y_k)\}$

where c_j is a pure constant not depending on any y_i's, $p_j=\exp(c_j)$ is a pure constant and $g_j(y_1,\ldots,y_k)$ is a function of $y_1,\ldots,y_k$ such that it contains all $\int g_{ij}\,dy_i$ $i=1,2,\ldots,k$.

Similarly from the relation $\underline{f}=G\underline{\mu}$, we arrive at

(5.3.11) $\log f(Z_1,\ldots,Z_k)=g(y_1,\ldots,y_k)+c$ or $f(Z_1,\ldots,Z_k)=$

$p\exp\{g(y_1,\ldots,y_k)\}$ where c is a pure constant, $p=\exp(c)$ and $g(y_1,\ldots,y_k)$ is a function of $y_1,\ldots,y_k$ such that it contains all $\int\sum_{t=1}^{k} g_{it}\mu_i\,dy_i$ $i=1,2,\ldots,k$.

Let us define $a_{x_1,\dots,x_k}$ as the coefficient of $\prod_{j=1}^{k} (Z_j/p_j)^{x_j}$ in the expansion of $f(Z_1,\dots,Z_k)/p$ where Z_j's and $f(Z_1,\dots,Z_k)$ are derived in (5.3.10) and (5.3.11). Then $p \prod_{j=1}^{k} p_j^{-x_j} \; a_{x_1,\dots,x_k}$ will be the coefficient of $\prod_{j=1}^{k} Z_j^{x_j}$ in the expansion of $f(Z_1,\dots,Z_k)$. Hence by the definition of the power-series distribution in (5.3.1), we can without any loss of generality take $p_j=p=1$ for all j and write the solutions as

(5.3.12) $Z_j=\exp\{g_j(y_1,\dots,y_k)\}$ & $f(Z_1,\dots,Z_k) = e^{g(y_1,\dots,y_k)}$.

Now suppose that two power-series distributions have means, variances and covariances equal to each other. Then from the expressions derived in (5.3.9), (5.3.10) & (5.3.11), or (5.3.12). We see that the two power-series are identical. Hence the theorem is proved.

Corollary 4:- The univariate power-series distribution is uniquely determined from its first two cumulants (or moments) and write the solution as

(5.3.13) $\log Z=\int(\frac{dK_1}{dy}/K_2)\,dy$ and $\log f(Z)=\int(K_1\frac{dK_1}{dy}/K_2)\,dy$.

We have noted that two successive cumulants of a power-series distribution other than the first two cumulants do not determine the power-series distribution uniquely. This is also true for the multivariate power-series distributions.

5.4: Illustrative examples:

We shall deal in this section first multivariate power-series distributions and then give certain univariate distributions also.

(i) Negative-Multinomial distribution (41):-

We shall extend the known negative-binomial distribution to obtain the general negative-multinomial distribution, which is generated by the power-series.

(5.4.1) $f(Z_1,\ldots,Z_k) = (1-Z_1-Z_2-\ldots-Z_k)^{-n}$ where

$Z_i=p_i/q$, $q=1+\sum_{i=1}^{k} p_i$, $0\leq Z_i \leq 1$, $p_i\geq 0$ & $0\leq \sum_{i=1}^{k} Z_i \leq 1$, $n>0$.

Then by the definition of (5.1.2), we have

(5.4.2) $Pr(\xi_1=x_1,\ldots,\xi_k=x_k)=\Gamma(n+\sum_{i=1}^{k} x_i)\prod_{i=1}^{k} p_i^{x_i}(\prod_{i=1}^{k} x_i!)^{-1}\,.$

$\{\Gamma(n)\}^{-1} q^{-n-\sum_{i=1}^{k} x_i}$ for each $x_i=0,1,2,\ldots.$

By the use of the cumulant relations in (5.3.5), it is easy to show that

(5.4.3) $E(\xi_i)=\mu_i=np_i, V(\xi_i)=\sigma_{ii}=np_i(1+p_i)$ & $Cov(\xi_i,\xi_j)=\sigma_{ij}=$

np_ip_j for $i\neq j$, $i,j=1,2,\ldots,k$.

Conversely, i.e. given the moments (5.4.3), we shall obtain the power-series as defined in (5.4.1).

Let ~~xxx~~ us write (5.4.3) in a matrix notation as $\underline{\mu}=n\underline{p}$, $\Sigma=n(D_p+\underline{p}\underline{p}')$ where $D_p=\text{dig.}(p_1,p_2,\ldots,p_k)$, $\underline{p}'=(p_1,\ldots,p_k)$. Also let $\underline{1}'=(1,1,\ldots,1)$: px1 and N, G are the same types of matrices as defined in (5.3.8). Hence

$N=nI$ and $\Sigma^{-1}=(D_p^{-1}-\underline{1}\,\underline{1}'/q)/n$ by using (A.1.19 d).

i.e.

(5.4.4) $G=N\Sigma^{-1}=D_p^{-1}-\underline{1}\,\underline{1}'/q$ and $\underline{f}=G\underline{\mu}=n\,\underline{1}/q$.

Therefore the equations (5.4.4) gives, $\log Z_i=\log p_i-\log q$ i.e. $Z_i=p_i/q$ and $\log f(Z_1,\ldots,Z_k)=n\log q$ i.e. $f(Z_1,\ldots,Z_k)=q^n$.

This is the same as (5.4.1) if we substitute the value of $q^{-1}=1-\sum_{i=1}^{k}Z_i$ in $f(Z_1,\ldots,Z_k)=q^n$.

(ii) <u>Multinomial Distribution:-</u>

Instead of the means, variances and covariances given in (5.4.3), we take them as

(5.4.5) $\underline{\mu}=n\underline{p}$ and $\Sigma=n(D_p-\underline{p}\,\underline{p}')$, $(0\leq p_i\leq 1)$.

Let $q_1 = 1-\sum_{i=1}^{k} p_i$. Then

(5.4.6) $N=nI$, $\Sigma^{-1}=(D_p^{-1}+\underline{1}\,\underline{1}'/q_1)/n$ and so $G=D_p^{-1}+\underline{1}\,\underline{1}'/q_1$,

and $\underline{f}=G\underline{\mu}=n\,\underline{1}/q_1$.

Therefore, the equations (5.4.6) give,

(5.4.7) $f(Z_1,\ldots,Z_k)=(1+Z_1+\ldots+Z_k)^n$ where $Z_i=p_i/q_1$,

$Z_i \geq 0$, & $\sum_{i=1}^{k} p_i \leq 1$, n is an integer and so

$$\Pr(\xi_1=x_1,\ldots,\xi_k=x_k)=n!\; q_1^{x_{k+1}} \prod_{i=1}^{k} p_i^{x_i} \Big/ \prod_{j=1}^{k+1} x_j!$$

for each $x_i=0,1,\ldots,n$, and $x_{k+1}=n-\sum_{i=1}^{k} x_i$.

This is the known multinomial distribution which is derived here as a power-series distribution.

(iii) <u>Truncated negative-multinomial distribution truncated at $x_1=x_2=\ldots=x_k=0$</u> :

In this case we have the power-series,

(5.4.8) $f(Z_1,\ldots,Z_k)=(1-Z_1-\ldots-Z_k)^{-n}-1$ where Z_i's are the same as defined in (5.4.1). Then

(5.4.9) $$\Pr(\xi_1=x_1,\ldots,\xi_k=x_k)=\Gamma(n+\sum_{i=1}^{k} x_i)\prod_{i=1}^{k} p_i^{x_i}\left(\prod_{i=1}^{k} x_i!\right)^{-1} \cdot \{\Gamma(n)\}^{-1}(q^n-1)^{-1}q^{-\sum x_i}$$ for each $x_i=0,1,\ldots\ldots$

such that the point $x_1=\ldots=x_k=0$ is omitted.

The means, variances and covariances of (5.4.9) are derived by the use of the relations (5.3.5) as

(5.4.10) $\mu_i=np_iq^n/(q^n-1)$, $\sigma_{ii}=np_iq^n/(q^n-1)+np_i^2q^n\{1-n/(q^n-1)\}/(q^n-1)$, and $\sigma_{ij}=np_ip_jq^n\{1-n/(q^n-1)\}/(q^n-1)$ for $i\neq j$, $i,j=1,2,\ldots,k$.

Conversely, i.e. to obtain the power-series distribution from the means, variances and covariances, we note that (notation same as (5.3.8)),

$\underline{\mu}=nq^n\underline{p}/(q^n-1)$, $\Sigma=nq^n[D_p+\{1-n/(q^n-1)\}\underline{p}\underline{p}']/(q^n-1)$,

$N=nq^n\{I-n\,\underline{1}\,\underline{p}'/q(q^n-1)\}/(q^n-1)$ and using (A.1.19 d),

$\Sigma^{-1}=(q^n-1)\{D_p^{-1}-(q^n-1-n)\,\underline{1}\,\underline{1}'/(q^{n+1}-q-nq+n)\}/nq^n$. Hence

(5.4.11) $G=N\Sigma^{-1}=(D_p^{-1}-\underline{1}\,\underline{1}'/q)$ and $\underline{f}=G\underline{\mu}=nq^{n-1}\,\underline{1}/(q^n-1)$.

i.e. from the above equations of G and $\underline{f}$, we have

$\log f(Z_1,Z_2,\ldots,Z_k)=\log(q^n-1)$ and $Z_i=p_i/q$.

This is the same as defined in (5.4.8).

(iv) Multivariate logarithmic-power-series distribution:-

In (iii), let us suppose that $n\to 0$. Then we have the means and variances & covariances in matrix notations as

(5.4.12) $\underline{\mu}=\underline{p}/\log q$ and $\Sigma=\{D_p+(\log q-1)\underline{p}\,\underline{p}'/\log q\}/\log q$.

To obtain the power-series from (5.4.12), we note that $N=(I-\underline{1}\underline{p}'/\log q)/\log q$ &

$\Sigma^{-1} = \log q \left\{ D_p^{-1} - (\log q - 1)\, \mathbf{1}\,\mathbf{1}' / (q \log q - q + 1) \right\}$. Therefore

(5.4.13) $G = N\Sigma^{-1} = D_p^{-1} - \mathbf{1}\,\mathbf{1}'/q$ and $\underline{f} = \mathbf{1}/q \log q$.

Hence we get the power-series as

(5.4.14) $f(Z_1,\ldots,Z_k) = -\log(1 - Z_1 - \ldots - Z_k)$ where $Z_i = p_i/q$

$i = 1,2,\ldots,k$ and so

$$\Pr(\xi_1 = x_1,\ldots,\xi_k = x_k) = \left(\sum_{i=1}^{k} x_i - 1\right)! \prod_{i=1}^{k} p_i^{x_i} \Big/ \left(\prod_{i=1}^{k} x_i!\right) q^{\Sigma x_i} \log q$$

for each $x_i = 0,1,2,\ldots$ such that $x_1 = x_2 = \ldots = x_k = 0$ is omitted.

The above distribution will be called as the multivariate logarithmic power-series distribution.

(v) Multivariate Poisson distribution:-

In this case the power-series is

(5.4.15) $f(Z_1,\ldots,Z_k) = \exp\left(\sum_i a_i Z_i + \sum_{i>j} a_{ij} Z_i Z_j + \sum_{i>j>k} a_{ijk} Z_i Z_j Z_k + \ldots\right)$

where a_i's, a_{ij}'s, a_{ijk}'s ... are constants such that

$b_{x_1,\ldots,x_k} Z_1^{x_1} \ldots Z_k^{x_k} \geq 0$, where $b_{x_1,\ldots,x_k}$ is the coefficient of $Z_1^{x_1} \ldots Z_k^{x_k}$ in the expansion of $f(Z_1,\ldots,Z_k)$.

The means, variances and covariances are

$\mu_i = \sigma_{ii} = Z_i\left(a_i + \sum_j{}' a_{ij} Z_j + \sum_{j>t}{}' a_{ijt} Z_j Z_t + \ldots\right)$ and

$\sigma_{ij} = Z_i Z_j \left(a_{ij} + \sum_t{}'' a_{ijt} Z_t + \ldots\right)$ where $\sum_j{}'$ means summation over j such that $j \neq i$, $\sum_{j>t}{}'$ is summation over j & t such that

$j \neq i$, $j > t \neq i$, and $\sum_t''$ is summation over t such that $t \neq (i,j)$,..... . The converse, i.e. to obtain the distribution from means and variances & covariances, is immediate & easy.

We may note here that in order to define the multivariate Poisson distribution in the sense of Aitken (4) and Maritz (50), we consider in the series (5.4.15), the parameters $Z_1, Z_2, \ldots$ as pseudo-parameters, and after deviation of the distribution we put $Z_i=1$.

(vi) A class of univariate discrete distributions:-

Let $a_1, a_2, \ldots$ be constants such that $\sum_{i=1}^{\infty} a_i Z^i$ is a convergent series. Then the power-series defined by

(5.4.16) $f(Z)=\exp\left(\sum_i a_i Z^i\right)$ should be such that $b_x Z^x \geq 0$

where b_x is the coefficient of Z^x in the expansion of f(Z), and $b_o=1$, $b_x=\sum_{j=1}^{x} \sum_{\pi's} \prod_{i=1} (a_i^{\pi_i}/\pi_i!)$ where $\sum_{\pi's}$ indicates the summation over the permutations $\pi_1, \pi_2, \ldots$ such that $\sum_{i=1}^{\infty} i\,\pi_i = x$ and $\sum_{i=1}^{\infty} \pi_i = j$. Then we find that

$$\Pr(\xi = x) = b_x Z^x \exp\left(-\sum_i a_i Z^i\right) .$$

In particular,

$$\Pr(\xi=0) = \exp\left(-\sum_{i=1}^{\infty} a_i Z^i\right), \quad \Pr(\xi=1)=Za_1 \Pr(\xi=0),$$

$2\,Pr(\xi=2)=2\,Z^2a_2\,Pr(\xi=0)+Za_1\,Pr(\xi=1)$,

$3\,Pr(\xi=3)=3Z^3a_3\,Pr(\xi=0)+2Z^2a_2\,Pr(\xi=1)+Za_1\,Pr(\xi=2)$, &

$4\,Pr(\xi=4)=4Z^4a_4\,Pr(\xi=0)+3Z^3a_3\,Pr(\xi=1)+2Z^2a_2Pr(\xi=2)+Za_1Pr(\xi=3)$.

Hence in general, we can write

(5.4.17) $$Pr(\xi=x)=\sum_{i=1}^{x} ia_i\,Z^i\,Pr(\xi=x-i)/x.$$

The mean and the variance of the variate are

$$\mu=\sum_i ia_iZ^i \;\&\; \sigma^2=\sum_i i^2a_iZ^i\;.$$

We can easily establish the distribution from the mean and the variance.

We note also that in order to define a class of discrete distributions in the sense of Maritz (50), we have to consider in (5.4.16), Z as a pseudo-parameter i.e. If we put Z=1 in (5.4.17) and the moments obtained from (5.4.17), we shall get all the distributions defined by Maritz and their moments.

(vii) Louis Gold's Poisson generalization as a power series (26):

Consider the power-series

(5.4.18) $$f(Z)=(e^u-e^{Zu})/(1-Z)=ue^u\int_0^1 e^{-(1-Z)ut}dt$$ where

$Z\neq 1$ and $Z>0$ and u is a positive constant. Then

(5.4.19) $$Pr(\xi=x)=(1-Z)Z^x\left(\sum_{j=x+1}^{\infty} u^j/j!\right)/(e^u-e^{Zu})$$ for x=0,1,2,...

Hence by using (5.3.7), we have

$E(\xi)=Z\{1/(1-Z)-ue^{Zu}/(e^{u}-e^{Zu})\}$ and

$V(\xi)=Z/(1-Z)^{2}-Zue^{Zu}\{(e^{u}-e^{Zu})(Z+1)-Zue^{Zu}\}/(e^{u}-e^{Zu})^{2}$.

We can easily obtain the power-series from these moments.

(a) We can note that as $Z\to 1$ in (5.4.19), we have the distribution which is known as the Poisson-binomial-exponential-limit (26) i.e.

$$Pr(\xi=x)=e^{u}\sum_{j=x+1}^{\infty}u^{j-1}/j! \text{ for } x=0,1,2,\ldots,\infty.$$

(b) We can also note that Goodman's Poisson generalization (27) is

$Pr(\xi=x)=Pr(xd\leq\eta\leq xd+d-1)$ where d is a positive integer and η is a random variable having poisson variate with parameter Z. i.e. $Pr(\xi=x)=e^{-Z}\sum_{i=0}^{d-1}Z^{xd+i}/(xd+i)!$

for $x=0,1,2,\ldots$.

This distribution cannot be derived from the power-series nor can it be derived by introducing pseudo-parameter as the case of (vi).

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