CHAPTER - I

INTRODUCTION

The theory of orthogonal series is not always a classical section of the analysis. It has its origin since last 200 years, which originated during the discussion of the problem of vibrating string considered by Euler in 1753 in connection with the work by Daniel Bernoulli. During their discussion they had advanced the theory of vibrating strings to the stage where the partial differential equation $y_{tt}=a^2y_{xx}$ was known and the solution of the boundary value problem had been found from the general solution of that equation. Thus, they have led to the possibility of representing an arbitrary function by a trigonometrical series. The problem of what functions can be represented by trigonometric series arose again later during the researches by French mathematical physicist J.B. Fourier.

The last several years have been a period of intensive development in the theory of Fourier series. Advances have also been made in the theory of Fourier series w.r.t. general orthogonal systems, during the last thirty years. But the less attention has been paid to the theory of orthogonal series and so the present work is based on the orthogonal series.

The researches of some of the mathematicians like Fejèr, Hardy, Hilbert, Hobson, Lebesgue, F. Riesz, M. Riesz, Weyl, Alexits, Kaczmarz, Steinhaus, Menchoff, Zygmund, Lorentz, Meder and Tandori are mainly in the subject of convergence and summability problems of orthogonal series. In India Prof. C.M. Patel[1], A.R. Sapre[2], S.C. Bhatnagar[3] and R.K. Patel[4] also have worked in this direction. We would like to discuss some of the problems connected with the convergence and summability of orthogonal series. We begin with number of definitions and concepts relevant to the body work of our thesis.

1.2 Throughout the thesis we shall make use of either Stieltjes-Lebesgue integral or the Lebesgue integral. The notion of the orthogonality is introduced by means of the Stieltjes Lebesgue integral. Let $\mu(x)$ be a positive bounded and monotone increasing function in the closed interval $[a, b]$. Such a function is called the distribution function.[5]

A real function f(x) is called L_μ-integrable, if it is μ-measurable and

$$\int_a^b |f(x)| \, d\mu(x) < \infty. \tag{1.2.1}$$

If $\mu(x)$ is absolutely continuous and $\rho(x) = \mu'(x)$, then

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1) Patel [59]

2) Sapre [73]

3) Bhatnagar [14]

4) Patel [62]

5) Freud [22]

for any L_μ-integrable function f(x) the relation

$$(1.2.2) \qquad \int_a^b f(x)d\mu(x) = \int_a^b f(x)\,\varrho(x)dx$$

is valid. In this case we shall say that f(x) is an $L_{\varrho(x)}$-integrable function and $\varrho(x)$ a covering function or weight function. If in particular $\varrho(x)=1$, then we shall say in accordance with the usual terminology that f(x) is L-integrable.

A function f(x) is called L_μ^2 or $L_{\varrho(x)}^2$-integrable, if it is L_μ or $L_{\varrho(x)}$-integrable respectively and if, furthermore,

$$\int_a^b f^2(x)d\mu(x) < \infty \quad \text{or} \quad \int_a^b f^2(x)\varrho(x)dx < \infty$$

holds. We shall talk about an L^2-integrable function, if $\varrho(x) = 1$.

<u>ORTHOGONALITY</u> : A finite or denumerably infinite system $\{\phi_n(x)\}$ of L_μ^2-integrable function is said to be orthogonal with respect to the distribution $d\mu(x)$ in the interval $[a,b]$, if

$$(1.2.3) \qquad \int_a^b \phi_m(x)\,\phi_n(x)d\mu(x)=0, \quad m\neq n$$

holds and none of the functions $\phi_n(x)$ vanishes almost everywhere.

A system $\{\phi_n(x)\}$ is said to be orthonormal, (ONS) if in addition to the condition (1.2.3) the condition

$$\int_a^b \phi_n^2(x)\,d\mu(x)=1, \quad n=0,1,2,\ldots\ldots$$

is also satisfied. Every orthogonal system $\{\psi_n(x)\}$ can be converted into an ONS by means of multiplying every one of its members by a suitably chosen constant factor. For, since none of the functions $\psi_n(x)$ can vanish almost every where, the functions

$$\phi_n(x) = \frac{\psi_n(x)}{\left\{ \int_a^b \psi_n^2(x)\,d\mu(x) \right\}^{\frac{1}{2}}}$$

exist and it is immediately evident that they constitute an ONS with respect to $d\mu(x)$. If, in particular $\mu(x)=x$ i.e. $\mu'(x)=\varrho(x) \equiv 1$, then $\{\phi_n(x)\}$ is simply an ONS in the ordinary sense.

<u>ORTHOGONALIZATION</u> : A system of functions $\{f_n(x)\}$ is called linearly independent in $[a,b]$, if the validity of the relation of the form

$$\sum_{k=0}^{n} a_k f_k(x)=0$$

for μ-almost every $x \in [a,b]$ necessarily implies the relation

$$a_0=a_1 = \ldots\ldots = a_n=0.$$

Every orthogonal system $\{\phi_n(x)\}$ is linearly independent.[1)]

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1) Alexits. ([4], p.4)

Conversely any linearly independent system of functions can be converted into an ONS $\{\phi_n(x)\}$ such that $\phi_n(x)$ are linear combinations of the functions $f_o(x)$, $f_1(x)$, ... $f_n(x)$. The process of constructing an ONS of functions from a given sequence of linearly independent functions is known as Gram-Schmidt process of orthogonalization.[1]

ORTHOGONAL SERIES AND ORTHOGONAL EXPANSION.

Any series

$$\sum_{n=o}^{\infty} C_n \psi_n(x) \tag{1.2.4}$$

constructed from the functions of an orthogonal system and an arbitrary set of real numbers C_o, C_1, ... is called an orthogonal series. However, if the coefficients C_n in the series (1.2.4) are representable in the form

$$C_n = \frac{1}{\int_a^b \psi_n^2(x)d\mu(x)} \int_a^b f(x)\,\psi_n(x)d\mu(x),\quad n=0,1,2\ldots$$

according to Fourier's manner, then we shall say that the series

$$\sum_{n=o}^{\infty} C_n \psi_n(x)$$

1) Schmidt [75]

is the orthogonal expansion of the function f(x) and we shall express this relation by

$$f(x) \sim \sum_{n=0}^{\infty} C_n \psi_n(x).$$

In this case we shall call the numbers C_0, C_1....... the expansion coefficients of the function f(x).

The orthogonal expansion and orthogonal series differ from each other due to the following minimum property established by Gram.[1)]

Let f(x) denote an L_μ^2-integrable function and $\{\phi_n(x)\}$ an arbitrary ONS. Among all the expressions of the form

$$S_n(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

the integral

$$I(S_n) = \int_a^b [\, f(x) - S_n(x) \,]^2 \, d\mu(x)$$

attains for $S_n(x) = s_n(x)$ the least value, where

$$s_n(x) = \sum_{k=0}^{n} C_k \phi_k(x), \quad C_k = \int_a^b f(t)\phi_k(t) d\mu(t)$$

The above result of Gram gives rise to the important property of expansion coefficients known as Bessel's inequality

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1) Gram [23]

$$\sum_{n=o}^{\infty} C_n^2 \leq \int_a^b f^2(x)d\mu(x).$$

Bessel's inequality implies that the expansion coefficients C_n of an L_μ^2-integrable function converge to zero as n is indefinitely increased.

The most fundamental theorem in the theory of orthogonal series is the Riesz Fischer theorem proved nearly simultaneously and independently by Riesz[1] and Fischer[2]. The above theorem was later on generalized by Fomin[3] as follows.

Let $\{\phi_k(x)\}$ be an ONS on the interval $[a,b]$, $\phi_k \in L^q[a,b]$, $k=o,1,2....$, $1<q\leq\infty$ and let $\frac{1}{p}+\frac{1}{q}=1$ if $q<\infty$ and $p=1$ if $q=\infty$. If there exists an increasing sequence $\nu_k \longrightarrow \infty$ such that

$$\sum_{k=o}^{\infty}\left(\frac{1}{\nu_k}-\frac{1}{\nu_{k+1}}\right)\int_a^b\left|\sum_{m=o}^{k} a_m\nu_m\phi_m(x)\right|^p dx < \infty$$

with a_k real, then there exists a function $f\in L^p[a,b]$ such that

$$a_k=\int_a^b f(x)\phi_k(x)dx \quad \text{for } k=o,1,2........$$

$\mu(n)$ - LACUNARY ORTHOGONAL SERIES

Let $\mu(x)\leq x$ denote a positive function concave from below, defined for $x\geq 1$ and increasing monotonely to infinity. We shall call the orthogonal series

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1) Riesz [70]
2) Fischer [20]
3) Fomin [21]

$$\sum_{n=o}^{\infty} C_n \phi_n(x)$$

$\mu(n)$ - lacunary, if the number of non-vanishing coefficients C_k with $n < k \leq 2n$ does not exceed $\mu(n)$. Furthermore, we shall say that the coefficients have the positive number sequence $\{q_n\}$ as a majorant, if the relation

$$C_n = O(q_n)$$

holds.

1.3 A little divergence at this point will be made from our main theme so as to define the various summability methods which are to be used in the body work of our thesis.

CESÀRO SUMMABILITY

Let

$$\sum_{n=o}^{\infty} u_n$$

be a given series and $A_n^{\alpha} = \binom{n+\alpha}{n}$ be given by

$$\sum_{n=o}^{\infty} A_n^{\alpha} x^n = \frac{1}{(1-x)^{1+\alpha}} \quad (\alpha \neq -1, -2, \ldots\ldots\ldots).$$

We write

$$\sigma_n^o = s_n^o = s_n = u_o + u_1 + \ldots\ldots + u_n$$

and

$$s_n^{\alpha} = \sum_{k=o}^{n} A_{n-k}^{\alpha-1} s_k = \sum_{k=o}^{n} A_{n-k}^{\alpha} u_k .$$

Then the quotient

$$\sigma_n^\alpha = \frac{s_n^\alpha}{A_n^\alpha}$$

is called the n^{th} Cesàro mean of the sequence $\{s_n\}$ or simply (C,α)-mean. The series

$$\sum_{n=0}^{\infty} u_n$$

is said to be (C,α)-summable to s[1] if $\sigma_n^\alpha \longrightarrow s$

as $n \longrightarrow \infty$.

The series

$$\sum_{n=0}^{\infty} u_n$$

with partial sums s_n is said to be strongly (C,α)-summable with index k to the sum s, if

$$\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} | s_\nu - s |^k \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

For $\alpha=1$, this gives the definition of strong summability (H,k)[2].

RIESZ SUMMABILITY

Let $\{\lambda_n\}$ be a positive, strictly increasing sequence of numbers with $\lambda_0=0$ and $\lambda_n \longrightarrow \infty$. The series

$$\sum_{n=0}^{\infty} u_n$$

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1) Cesàro [16], Chapman [17], Knopp ([32], [33])

2) Zygmund ([98], p.180), Bary ([12], p.2), Moricz [53]

is said to be (R, λ_n, α)-summable[1] $(\alpha > 0)$ to the sum s, if

$$\sigma_n^{\alpha}(\lambda) = \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)^{\alpha} u_k \longrightarrow s \quad \text{as} \quad n \longrightarrow \infty .$$

Here $\sigma_n^{\alpha}(\lambda)$ is called the n^{th} (R, λ_n, α)-mean of the series $\sum u_n$.

In particular for $\alpha=1$

$$\sigma_n^{1}(\lambda) = \sigma_n(\lambda) = \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right) u_k = \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) s_k$$

defines the n^{th} $(R, \lambda_n, 1)$- mean of the series

$$\sum u_n .$$

Obviously, the $(R, \lambda_m, 1)$ summation introduced by M.Riesz is a generalization of the $(C, 1)$-summation process for $\lambda_n = n$. For $\lambda_n = \log(n+1)$, the Riesz summability is known as Riesz logarithmic summability.

<u>EULER SUMMABILITY</u>.

Let

$$\sum_{n=0}^{\infty} u_n$$

be an infinite series with the sequence of partial sums $\{s_n\}$. A sequence to sequence transformation given by the equation

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1) Das [18], Lorentz [43] .

$$\tau_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k, \quad n=0,1,2\ldots\ldots\ldots$$

defines the sequence $\{\tau_n^q\}$ of (E, q)-means ($q > 0$) known as Euler means.[1]

In place of $\tau_n^{(1)}$ we simply write τ_n.

The series

$$\sum_{n=0}^{\infty} u_n$$

is said to be (E, q)-summable to the sum s, if

$$\lim_{n \to \infty} \tau_n^{(q)} = s.$$

The series

$$\sum_{n=0}^{\infty} u_n$$

is said to be strong summable (E, q) to the sum s(x), if

$$\frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (s_k - s)^2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

NÖRLUND SUMMABILITY

Let $\{p_n\}$ be a sequence of non-negative real numbers. The series

$$\sum_{n=0}^{\infty} u_n$$

with partial sums s_n is said to be (N, p_n)-summable[2] to s, if

1) Hardy ([24] , p.180)

2) Hardy ([24] , p.64)

$$t_n = \frac{1}{P_n} \sum_{k=o}^{n} p_{n-k}\, s_k \longrightarrow s \quad \text{as} \quad n \longrightarrow \infty,$$

where $P_n = p_o+p_1+ \ldots\ldots\ldots +p_n$, $p_o > o$, $p_n \geq o$. We then write (N,p_n) lim $s_n=s$ or $(N,p_n) \sum u_n=s$.

The transforms t_n are called the Nörlund means of the sequence $\{s_n\}$ or of the series

$$\sum_{n=o}^{\infty} u_n .$$

It is wellknown that the method (N,p_n) is regular, if and only if

$$\lim_{n\to\infty} \frac{p_n}{P_n} = o .$$

The sequence $\{p_n\}$ will be said to belong to the class M^{α}, for a certain real $\alpha \geq o$, if

(i) $o < p_n < p_{n+1}$ for $n= o,1,2,\ldots\ldots\ldots$

(or) $o < p_{n+1} < p_n$ for $n=o,1,2,\ldots\ldots\ldots$

(ii) $p_o+p_1+\ldots\ldots\ldots +p_n = P_n \uparrow \infty$

(iii) $\lim_{n\to\infty} \frac{np_n}{P_n} = \alpha$[1].

Obviously, if $\{p_n\} \in M^{\alpha}$, then the method (N,p_n) is regular.

Let

$$S_n = \frac{1}{P_n} \sum_{k=o}^{n} \frac{P_k}{k+1}$$

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1) Meder [48]

The sequence $\{p_n\}$ will be said to belong to the class BVM^{α}, if $\{p_n\} \in M^{\alpha}$ and if $\{S_n\}$ is a sequence of bounded variation i.e.

$$\sum_{n=1}^{\infty} | s_n - s_{n-1} | < \infty.$$ [1]

If for some sequence $\{p_n\}$, conditions (i) and (ii) are satisfied and moreover, if

$$\lim_{n\to\infty} \frac{n \Delta p_{n-1}}{p_n} = 1-\alpha , \text{ where } \alpha \geq 0, \ \Delta p_{n-1} = p_{n-1} - p_n,$$

then we shall say that the sequence $\{p_n\}$ belongs to the class $\overline{M}^{\alpha}$. [2]

A sequence $\{p_n\}$ is said to belong to the class M^*, if

(a) $p_n > 0$

(b) $\{p_n\}$ is convex or concave

(c) $0 < \underline{\lim}_{n\to\infty} \frac{np_n}{P_n} \leq \overline{\lim}_{n\to\infty} \frac{np_n}{P_n} < +\infty$. [2]

A series

$$\sum_{n=0}^{\infty} u_n$$

is said to be strongly (N,p_n)-summable to s, with $\{p_n\} \in M^{\alpha}$, $\alpha > 0$ $(p_n \uparrow)$, if

$$\frac{1}{n+1} \sum_{k=0}^{n} (T_k - s)^2 \longrightarrow 0 \text{ as } n \longrightarrow +\infty$$

where
$$T_k = \frac{1}{p_k} \sum_{\nu=0}^{k} (p_{k-\nu} - p_{k-\nu-1}) s_{\nu}$$

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1) Meder [48]

2) Meder [49]

and $p_{-1}=0$.[1]

GENERALIZED NÖRLUND MEAN

Let $p=\{p_n\}$ and $q=\{q_n\}$ be non-negative sequences of real numbers. We write

$$r_n = \sum_{\nu=0}^{n} p_{n-\nu} q_\nu$$

and assume that r_n is non zero for all values of n.

The n^{th} generalized Nörlund mean of the sequence of partial sums $\{s_n\}$ of the series

$$\sum_{n=0}^{\infty} u_n$$

is given by

$$T_n^{(p,q)} = \frac{1}{r_n} \sum_{k=0}^{n} p_{n-k} q_k s_k, \quad n=0,1,2\ldots..$$

The method (N,p,q) reduces to the Nörlund method when $q_n=1$ and to the method $(\overline{N},q)$ when $p_n=1$.

An increasing sequence of natural numbers

$$n_1 < n_2 < \ldots < n_k < \ldots\ldots$$

is said to satisfy the condition (L) if the series

$$\sum \frac{1}{n_k}$$

1) Meder [49]

satisfies the condition (L), i.e.

$$\sum_{k=m}^{\infty} \frac{1}{n_k} = O\left(\frac{1}{n_m} \right).^{1)}$$

1.4 LEBESGUE FUNCTIONS

The concept of Lebesgue function was introduced by Lebesgue.[2] He investigated the influence of these functions on the divergence of Fourier series. In the case of trigonometric system the Lebesgue functions $L_n(x)$ are constants and are therefore called the Lebesgue constants.

CESÀRO KERNEL AND LEBESGUE FUNCTIONS.

The sums

$$K_n(t,x) = \sum_{k=o}^{n} \phi_k(t)\phi_k(x) \quad \text{and} \quad K_n^{\alpha}(t,x) = \sum_{k=o}^{n} \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}} \phi_k(t)\phi_k(x)$$

$(\alpha > -1)$ are called the n^{th}-kernel and n^{th} (C,α)-kernel respectively of the ONS $\{\phi_n(x)\}$, whereas

$$L_n(x) = \int_a^b |K_n(t,x)| \, d\mu(t) \quad \text{and} \quad L_n^{\alpha}(x) = \int_a^b |K_n^{\alpha}(t,x)| \, d\mu(t)$$

are called the n^{th} Lebesgue function and n^{th} Lebesgue (C,α)-function of the ONS $\{\phi_n(x)\}$ respectively.

RIESZ KERNEL AND LEBESGUE FUNCTION

The sum

$$U_n^{\alpha}(t,x) = \sum_{k=o}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)^{\alpha} \phi_k(t)\phi_k(x)$$

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1) Bary [11]

2) Lebesgue [37]

and the integral

$$V_n^{\alpha}(x) = \int_a^b |U_n^{\alpha}(t,x)| \, d\mu(t), \qquad \alpha > o$$

are respectively called the $n^{th}(R, \lambda_n, \alpha)$- kernel and n^{th} Lebesgue (R, λ_n, α) function of the ONS $\{\phi_n(x)\}$.

EULER KERNEL AND LEBESGUE FUNCTION

The n^{th} (E,q)- kernel $E_n^{(q)}(t,x)$, $(q > o)$ and n^{th} Lebesgue (E,q)-function $F_n^{(q)}(x)$, $(q > o)$ of the ONS $\{\phi_n(x)\}$ are defined by

$$E_n^{(q)}(t,x) = \frac{1}{(1+q)^n} \sum_{\nu=o}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x)$$

$$= \frac{1}{(1+q)^n} \sum_{\nu=o}^{n} \phi_\nu(t)\phi_\nu(x) \cdot \sum_{r=\nu}^{n} \binom{n}{r} q^{n-r}$$

and

$$F_n^{(q)}(x) = \int_a^b |E_n^{(q)}(t,x)| \, d\mu(t).$$

NÖRLUND KERNEL AND LEBESGUE FUNCTION

We define the $n^{th}(N, p_n)$-kernel and n^{th} Lebesgue (N, p_n)-function for the ONS $\{\phi_n(x)\}$ by

$$G_n(t,x) = \sum_{k=o}^{n} \frac{P_{n-k}}{P_n} \phi_k(t)\phi_k(x)$$

and

$$H_n(x) = \int_a^b |G_n(t,x)| d\mu(t)$$

respectively.

POLYNOMIAL-LIKE AND CONSTANT-PRESERVING ORTHONORMAL SYSTEMS.

The concept of the polynomial-like orthogonal system was introduced by Alexits.[1)]

An ONS $\{\phi_n(x)\}$ is called polynomial-like, if its n^{th} kernel $K_n(t,x)$ has the following structure :

$$K_n(t,x) = \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(n)} \phi_{n+i}(t)\phi_{n+j}(x)$$

where p and r are natural numbers independent of n and the constants $|\gamma_{i,j,k}^{(n)}|$ have a common bound independent of n, while the measurable functions $F_k(t,x)$ satisfy the condition

$$F_k(t,x) = O\left(\frac{1}{|t-x|}\right)$$

for every $x \in [a,b]$. Here the function $\phi_{n+i}(x)$ with negative index is considered to be identically equal to zero.

It is clear that the system of orthonormal polynomials $\{p_n(x)\}$ and the trigonometrical systems are polynomial-like.

The ONS $\{\phi_n(x)\}$ is said to be constant-preserving if $\phi_0(x)$= constant. In this case beside C_0 all the expansion coefficients of the constant function f(x)=C vanish and therefore, we have for the n^{th} partial sum $s_n(x)$ of its expansion

$$s_n(x) = C \int_a^b \phi_0(t)\phi_0(x)d\mu(t) = C.$$

i.e. a representation preserving constancy.

1) Alexits [3]

SINGULAR INTEGRALS

The concept of singular integral is due to Lebesgue[1], and it possesses important convergence properties.

The partial sums $s_n(x)$ of the expansion of an $L_{\varrho(x)}$-integrable function in the functions of an ONS $\{\phi_n(x)\}$ are of the form

(1.4.1) $$I_n(f,x)=\int_a^b f(t)\ \psi_n(t,x)\ \varrho(t)dt$$

where $\psi_n(t,x)$ denotes the sum

$$\sum_{k=0}^{n} \phi_k(t)\phi_k(x).$$

The n^{th} sums

$$t_n(x) = \sum_{k=0}^{n} \alpha_{nk} s_k(x)$$

of an expansion summed by a linear summation process are also representable by the integral (1.4.1), where $\psi_n(t,x)$ denotes the sum

$$\psi_n(t,x)=\sum_{k=0}^{n} \alpha_{nk}\phi_k(t)\phi_k(x).$$

The integral $I_n(f,x)$ is said to be singular (with simgular point x) if, for an arbitrary number $\delta>0$ and for an arbitrary subinterval $[\alpha,\beta]$ of $[a,b]$

1) Lebesgue [37]

(1.4.2) $\lim_{n\to\infty}\int_I \psi_n(t,x)\rho(t)dt=1$ and $\lim_{n\to\infty}\int_J \psi_n(t,x)\rho(t)dt=0,$

where $I=[a,b]\cap[x-\delta,\ x+\delta]$, $J=[\alpha,\beta]-[x-\delta,\ x+\delta]$.

(1.4.3) $\underset{t\in[a,b]-[x-\delta,\ x+\delta]}{\text{ess lub}} |\psi_n(t,x)|\leq\psi(\delta)$

where $\psi(\delta)$ is a number depending on δ and x but independent of n.

If $\psi_n(t,x)$ satisfies uniformly the conditions (1.4.2) and (1.4.3) in an x-set E, then the integral $I_n(f,x)$ is said to be uniformly singular on E.

Let the function f(x) be defined in the interval [a,b]. Then, the continuity modulus[1] of the function f(x) in the interval [a,b] is defined as

$$\omega(f,\delta,a,b)=\sup_{\substack{|t-x|\leq\delta\\ t,x\in[a,b]}}|f(t)-f(x)|.$$

We denote by $\omega(\delta)$ a majorant function of $\omega(f,\delta,a,b)$, i.e. a function satisfying the condition

$$\omega(\delta)\geq\omega(f,\delta,a,b).$$

1.5 Some of the important and general results of real analysis which are to be used quite frequently in the course of the proofs of our theorems shall be referred in this section.

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1) Bary ([11], p.37).

B-Levy's Theorem[1]: If $\{f_n(x)\}$ is a monotone increasing sequence of L_μ-integrable functions and furthermore

$$\left| \int_a^b f_n(x)d\mu(x) \right| \leq C, \quad (n=0,1,2,\ldots\ldots)$$

then the limiting function

$$f(x) = \lim_{n\to\infty} f_n(x)$$

is also L_μ-integrable and the relation

$$\lim_{n\to\infty} \int_a^b f_n(x)d\mu(x) = \int_a^b f(x)d\mu(x)$$

holds.

If, in particular $u_0(x)$, $u_1(x)$, ... are L_μ-integrable functions such that

$$\sum_{n=0}^{\infty} \int_a^b |u_n(x)|\, d\mu(x) < \infty ,$$

then the series

$$\sum_{n=0}^{\infty} u_n(x)$$

is (absolutely) convergent almost everywhere.

Kronecker's lemma: If $\{\lambda_n\}$ is positive, monotone increasing and tending to infinity, then the convergence of the series

$$\sum_{n=0}^{\infty} u_n \lambda_n^{-1}$$

1) Alexits ([4], p.11)

implies the estimate

$$s_n = o(\lambda_n), \text{ where } s_n = \sum_{k=1}^{n} u_k$$

1.6 Convergence and summability of orthogonal series.

The problem of convergence of orthogonal series was originally started by Jerosch and Weyl[1] who pointed out that the condition

$$C_n = \ (n^{-\frac{3}{4}-\epsilon}), \ \epsilon > 0$$

is sufficient for the convergence of the series

$$(1.6.1) \qquad \sum_{n=0}^{\infty} C_n \phi_n(x) .$$

Further, Weyl[2] has improved this condition by showing that the condition

$$\sum_{n=1}^{\infty} C_n^2 \sqrt{n} < \infty$$

is sufficient for the convergence of the series (1.6.1). Later on Hobson[3] has modified the above condition to

$$\sum_{n=1}^{\infty} C_n^2 n^{\epsilon} < \infty, \ \epsilon > 0$$

and Plancherel[4] has tackled the same problem with the condition

$$\sum_{n=2}^{\infty} C_n^2 \log^3 n < \infty .$$

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1) Jerosch and Weyl [26]
2) Weyl [92]
3) Hobson [25]
4) Plancherel [65]

The chain of ideas in this direction continued and finally a masterpiece work regarding the convergence of the orthogonal series (1.6.1) was carried out nearly simultaneously and independently of one another by Rademacher[1] and Menehoff.[2] They have shown that the series (1.6.1) is convergent almost everywhere in the interval of orthogonality if the condition

$$\sum_{n=1}^{\infty} c_n^2 \log^2 n < \infty$$

is satisfied. Further generalizations of this theorem were given by Salem[3], Talalyan[4], Walfisz[5] and Kantorovitch[6] who proved the following result :

$$\int_a^b \max_n \left\{ \sum_{k=2}^{n} \frac{c_k \phi_k(x)}{\log k} \right\}^2 d\mu(x) = O(1) \sum_{k=0}^{n} c_k^2$$

The theorem of Rademacher and Menchoff is the best of its kind is obvious from the following fundamental theorem of convergence theory given by Menchoff.[2]

If w(n) is an arbitrary positive monotone increasing sequence of numbers with $w(n)=o(\log n)$, then there exists an everywhere divergent orthogonal series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)$$

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1) Rademacher [67]
2) Menchoff [50]
3) Salem [71]
4) Talalyan [81]
5) Walfisz [91]
6) Kantorovitch [31]

whose coefficients satisfy the condition

$$\sum_{n=1}^{\infty} c_n^2 w^2(n) < \infty .$$

Another theorem which needs to be mentioned in this direction is due to Tandori[1], who proved that if $\{c_n\}$ is a positive monotone decreasing sequence of numbers for which

$$\sum_{n=1}^{\infty} c_n^2 \log^2 n = \infty$$

holds, then there exists in $[a,b]$ an ONS $\{\psi_n(x)\}$ dependent on $\{c_n\}$ such that the orthogonal series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)$$

is divergent everywhere in $[a,b]$.

The question of convergence of orthogonal expansion is also smoothed by means of Lebesgue functions introduced by Lebesgue[2], who investigated the influence of these functions on the divergence of Fourier series. The effect of Lebesgue functions on the convergence of Fourier series was investigated by Kolmogoroff-Seliverstoff[3] and Plessner[4] who showed that under the condition

$$(1.6.2) \qquad \sum_{n=1}^{\infty} (a_n^2+b_n^2)\log n < \infty ,$$

the Fourier series

1) Tandori [83]
2) Lebesgue [37]
3) Kolmogoroff A. and Seliverstoff G. [35]
4) Plessner [66]

(1.6.3) $$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

is convergent almost everywhere.

Further, it was proved by Plessner that the condition (1.6.2) is equivalent to

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{f(x+t)-f(x-t)}{t}\right|^2 dtdx<\infty$$

where f(x) is the function whose Fourier series is (1.6.3).

Over an above the question of convergence, Menchoff and Kaczmarz have discussed the Cesàro summability of orthogonal series (1.6.1). The fundamental theorem concerning the Cesàro summability of orthogonal series was at first proved by Menchoff[1] and independently also by Kaczmarz[2]. They have shown that if $\{w(n)\}$ denotes a positive monotone increasing sequence of numbers whose terms are of order of magnitude $w(n)=o\,(\log\log n)$, then there exists an orthogonal series

$$\sum_{n=1}^{\infty} a_n \psi_n(x)$$

which is nowhere A-summable, although its coefficients satisfy the condition

$$\sum_{n=1}^{\infty} a_n^2 w^2(n)<\infty .$$

1) Menchoff ([51], [52])

2) Kaczmarz [27]

The concept of Lebesgue functions in the convergence and summability theory of orthogonal series was generalized by Kaczmarz[1], Tandori[2], Meder[3], Zinovev[4], Alexits[5] and Osilenker[6]. Kaczmarz[1] has shown that if

$$L_n(x)=O(\lambda_n)$$

where $\lambda(n)\leq\lambda(n+1)$

and $$\sum_{1}^{\infty} c_n^2\ \lambda^2(n) <+\infty\ ,$$

then the series (1.6.1) converges almost everywhere. The analogous result for $(C,\alpha>0)$-summability was also established by him.

The order of Lebesgue functions which plays an important role in the convergence theory of orthogonal series was estimated by Moricz[7], Olevskii[8], Ratajski[9] and Alexits.[10]

This chain of ideas was extended in the field of functional series also by Alexits and Sharma[11], Tandori[12] and Moricz[13]. Alexits and Sharma have proved that, if

$$\sum_{k=0}^{\infty} a_k^2 < \infty$$

and the Lebesgue functions

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1) Kaczmarz [28]
2) Tandori ([82], [85], [88])
3) Meder [47]
4) Zinovev [94]
5) Alexits [4]
6) Osilenker [58]
7) Moricz [53]
8) Olevskii ([56], [57])
9) Ratajski ([68], [69])
10) Alexits ([4], p. 179, 206)
11) Alexits and Sharma [10]
12) Tandori ([87], [89])
13) Moricz [55]

$$L_n^1(x)=\int_E |K_n^1(t,x)|\,d\mu(t) \quad \text{where} \quad K_n^1(t,x)=\sum_{k=o}^{n}(1-\frac{k}{n+1})f_k(t)f_k(x)$$

of the sequence of μ-integrable functions $\{f_n(x)\}$ on μ-measurable set $E\subset X$, which is measurable with a positive measure μ, satisfy the condition $L_n^1(x)=O(\lambda_n)$ uniformly on the measurable set E of finite measure, then the sums

$$\sigma_n(x) = \sum_{k=o}^{n}(1-\frac{k}{n+1})a_k f_k(x)$$

have the order of magnitude $O_x(\lambda_n^{\frac{1}{2}})$ on E almost everywhere. Moreover, they have proved that, if the Lebesgue functions $L_n^1(x)$ are uniformly bounded on the measurable set E of finite measure and

$$\sum a_n^2<\infty,$$

then the series

$$\sum a_n f_n(x)$$

is (C,1)-summable almost everywhere.

Moricz has generalized these theorems of Alexits and Sharma by estimating the order of Lebesgue function corresponding to general summation process.

1.7 Sunouchi[1)] has discussed the convergence of

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1) Sunouchi [76]

$$(1.7.1) \qquad \sum_{n=1}^{\infty} \frac{|s_n(x)-\sigma_n(x)|^k}{n}, \quad k>1$$

under the restriction of boundedness of the functions $\phi_n(x)$.

In chapter II we discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{(s_n(x)-t_n(x))^2}{n}$$

and also generalize this result as follows :

If $p_o > o$, $p_n \geq o$, $np_n = O(P_n)$ and $|\phi_n(x)| \leq K$,

then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x)-t_n(x)|^q}{n} dx = O(1) \sum_{n=1}^{\infty} |c_n|^q n^{q-2}, \quad q \geq 2 .$$

The convergence of the series of the type (1.7.1) with Euler and Riesz means was carried out by Patel[1].

Moreover, we have proved in this chapter the analogous result for Nörlund summability of $\mu(n)$-lacunary orthogonal series proved by Alexits[2] and also generalized the result for generalized Nörlund summability proved by Patel R.K. and Patel C.M.[3]

The approximation of summability means to their generating function for the orthogonal series

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1) Patel [62]
2) Alexits ([4], p.130)
3) Patel and Patel [63]

(1.7.2) $$\sum_{n=0}^{\infty} c_n \phi_n(x)$$

has been studied by Alexits and Kralik[1], Leindler[2] and Bolgov and Efimov[3]. Leindler[2] has proved the following theorem.

THEOREM A : If

(1.7.3) $$\sum_{n=1}^{\infty} c_n^2 n^{2\beta} < \infty, \quad 0 < \beta < 1$$

then

$$\sigma_n(x) - f(x) = o_x(n^{-\beta})$$

holds almost everywhere in (a, b).

In Chapter III we generalize the above result to Nörlund means as follows :

If $\{p_n\} \in \bar{M}^{\alpha}$, $\alpha > \frac{1}{2}$, then under the condition (1.7.3), the relation

$$t_n(x) - f(x) = o_x(n^{-\beta})$$

holds almost everywhere in (a, b).
A similar result for Euler means is also proved in this chapter.

Chapter IV is devoted in estimating the order of certain summability means where we extend the following theorem of Alexits[4] to Riesz and Nörlund summability.

1) Alexits and Kralik [5]
2) Leindler [41]
3) Bolgov and Efimov [15]
4) Alexits ([4], p.185)

THEOREM B : If the Lebesgue functions

$$L_{2^n}(x) = \int_a^b \left| \sum_{k=o}^{2^n} \phi_k(t)\phi_k(x) \right| dt \tag{1.7.4}$$

of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E\subset [a,b]$, then the condition

$$\sum_{n=o}^{\infty} c_n^2 < \infty \tag{1.7.5}$$

implies the $(C,\alpha > o)$-summability of the orthogonal series (1.7.2) almost everywhere on E.

We mention here one of the result proved by us :

If the Lebesgue functions (1.7.4) of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E\subset [a,b]$, then the relation (1.7.5) implies that the estimate

$$\sigma_n(\lambda, x) = o_x(n)$$

holds almost everywhere on E.

Chapter V deals with the order of Lebesgue functions for polynomial-like ONS, corresponding to Euler and Riesz summation processes. Moreover, we have also discussed in this chapter the Euler and Riesz summability of orthogonal series. These results are the extensions of the following results proved by Alexits.[1)]

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1) Alexits ([4], p.206,267)

<u>THEOREM C</u> : If the ONS $\{\phi_n(x)\}$ is polynomial-like and the condition

$$\sum_{k=o}^{n} \phi_k^2(x) = O_x(n)$$

is fulfilled in the set E, then the relation

$$L_n^1(x) = O_x(1)$$

holds almost everywhere in E.

<u>THEOREM D</u> : Let $\{\phi_n(x)\}$ be a complete constant-preserving polynomial-like ONS with respect to the weight function $\varrho(x)$. Suppose that the functions $F_k(t,x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t=x$ and that the two conditions

$$\sum_{k=o}^{n} \phi_k^2(x) = O(n)$$

and

(1.7.6) $$o < \varrho(x) \leq \text{const.}$$

are also satisfied in the subinterval $[c,d]$ of $[a,b]$. If the $L^2_{\varrho(x)}$- integrable function $f(x)$ is continuous in $[c,d]$, then its expansion

(1.7.7) $$f(x) \sim \sum_{n=o}^{\infty} c_n \phi_n(x)$$

is uniformly (C,1)-summable in every inner subinterval of $[c,d]$, the sum being $f(x)$.

We are stating below two of the theorems proved by us.

(i) If the ONS $\{\phi_n(x)\}$ is polynomial-like and the condition

$$\phi_n(x) = O_x(1)$$

is fulfilled in the set E, then the relation

$$F_n^{(q)}(x) = O_x(1)$$

holds almost everywhere in E.

(ii) Let $\{\phi_n(x)\}$ be a complete constant-preserving polynomial-like ONS with respect to the weight function $\varrho(x)$. Suppose that the functions $F_k(t,x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t=x$ and that the two conditions

$$\phi_n(x) = O(1)$$

and (1.7.6) are also satisfied in the subinterval $[c,d]$ of $[a,b]$. If the $L^2_{\varrho(x)}$-integrable function $f(x)$ is continuous in $[c,d]$, then its expansion (1.7.7) is uniformly (E,q)-summable $(q > 0)$ in every inner subinterval of $[c,d]$, the sum being $f(x)$.

1.8 Strong approximation of orthogonal series

In this section we discuss the strong summability of orthogonal series. The strong $(C,1)$-summability of Fourier series, conjugate Fourier series and orthogonal series has been investigated by several authors such as

Alexits[1], Bernstein[2], Alexits and Kralik[3], Alexits and Leindler[4], Fine[5], Sun Yong Sheng[6] and Sunouchi[7].

Alexits[8] has proved the following theorem :

<u>THEOREM A</u> : Let $\{\phi_n(x)\}$ be a constant-preserving polynomial--like ONS with respect to the weight function $\varrho(x)$ satisfying the conditions

$$(1.8.1) \qquad \sum_{k=0}^{n} \phi_k^2(x) = O(n)$$

and $\quad 0 \leqslant \varrho(x) \leqslant \text{const.}$

uniformly in the sub-interval $[c,d]$ of $[a,b]$.

Let $s_n(x)$ denote the n^{th} partial sum of the expansion of an $L^2_{\varrho(x)}$-integrable and on $[c,d]$ continuous function $f(x)$ with the continuity modulus $\omega(f,\delta,\ c,d)$. If $\omega(f,\delta,\ c,d)$ possess a majorant function $\omega(\delta)$ such that $\omega(\delta)/\delta^{\frac{1}{2}-\gamma}$ with some fixed $\gamma > 0$ increases monotonely to infinity as $\delta \to 0$, then the relation

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |f(x)-s_\nu(x)| = O\left[\omega\left(\frac{1}{n}\right)\right]$$

holds uniformly on every interval $[c+\epsilon, d-\epsilon] \subset (c,d)$.

In Chapter VI we generalize this theorem of Alexits transferring to the strong $(C,\alpha>0)$-summability as follows :

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1) Alexits([3], [4], p.295)
2) Bernstein [13]
3) Alexits and Kralik ([6], [8])
4) Alexits and Leindler [9]
5) Fine [19]
6) Sun Yong Sheng [80]
7) Sunouchi [78]
8) Alexits ([4], p.295)

Let $\{\phi_n(x)\}$ be a constant-preserving polynomial-like ONS satisfying the condition (1.8.1) uniformly in the subinterval $[c,d]$ of $[a,b]$. Let $s_n(x)$ denote the n^{th}-partial sum of the expansion of an L^2-integrable and on $[c,d]$ continuous function f(x) with the continuity modulus $\omega(f,\delta,c,d)$. If $\omega(\delta)/\delta^{\frac{1}{2}-\gamma}$ with some fixed $\gamma > 0$ increases monotonely to infinity as $\delta \to 0$, then the relation

$$\frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x)-f(x)| = O[\omega(\tfrac{1}{n})]$$

holds uniformly on every interval $[c+\epsilon, d-\epsilon] \subset (c,d)$. In this chapter, we have shown that the analogous result for strong Euler summability is also valid. Moreover, we have also extended in this last chapter the results of Sunouchi[1] and Maddox[2] to strong (N,p_n)-summability.

1) Sunouchi [78]

2) Maddox [44]