CHAPTER - III

ON DEGREE OF APPROXIMATION OF CERTAIN SUMMABILITY MEANS TO THEIR GENERATING FUNCTIONS

3.1 Let $\{\phi_n(x)\}$ $(n=0,1,2,\ldots\ldots)$ be an orthonormal system (ONS) of L^2-integrable functions defined in the closed interval $[a,b]$. We consider the orthogonal series

$$(3.1.1) \qquad \sum_{n=0}^{\infty} c_n\phi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty$$

By the Riesz-Fischer theorem, the series (3.1.1) converges in L^2 to a square integrable function f(x) given by

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \qquad \text{a.e.}$$

Moreover, the Fourier coefficients of f(x) with respect to $\{\phi_n(x)\}$ are the numbers c_n, i.e.

$$c_n = \int_a^b f(x)\phi_n(x)dx\,, \qquad n=0,1,2\ldots\ldots$$

The notations $s_n(x)$, $t_n(x)$ and $\{p_n\} \in M^{\alpha}$ have the same meaning as considered in Chapter II.

If for some sequence $\{p_n\}$ the conditions

i) $o < p_n < p_{n+1}$ for $n=o,1,2\ldots\ldots.$

or $o < p_{n+1} < p_n$ for $n=o,1,2\ldots.$

ii) $p_o + p_1 + \ldots\ldots + p_n = P_n \uparrow \infty$

iii) $\lim_{n \to \infty} \frac{n \Delta p_{n-1}}{p_n} = 1-\alpha$, where $\alpha \geq o$, $\Delta p_{n-1} = p_{n-1} - p_n$

are satisfied, then we shall say that the sequence $\{p_n\}$ belongs to the class $\bar{M}^{\alpha}$.[1]

A sequence $\{p_n\}$ is said to belong to the class M^*, if

(a) $p_n > o$ $(n=o,1,2\ldots\ldots)$,

(b) $\{p_n\}$ is convex or concave,

(c) $o < \underline{\lim}_{n \to \infty} \frac{np_n}{P_n} < \overline{\lim}_{n \to \infty} \frac{np_n}{P_n} < +\infty$. [1]

The n^{th} Euler mean or (E, 1)- mean of the sequence of partial sums $\{s_n(x)\}$ of the orthogonal series (3.1.1) is defined as

$$\tau_n(x) = \frac{1}{2^n} \sum_{k=o}^{n} \binom{n}{k} s_k(x), \quad n=o,1,2\ldots\ldots$$

The n^{th} (C, 1)- mean $\sigma_n(x)$ of the orthogonal series (3.1.1) have been approximated by Meder,[2] Tandori[3], Alexits and Kralik[4] and Leindler[5]. Leindler[6] approximated the de la

1) Meder [49]
2) Meder [45]
3) Tandori [84]
4) Alexits and Kralik [5]
5) Leindler ([40], [41])
6) Leindler ([38], [40], [41])

Vallèe Poission mean of the orthogonal series (3.1.1).The Riesz means were also approximated by Leindler.[1] Later on the results of the above said papers were generalized to strong approximation of $(C,\alpha>o)$- means by Sunouchi[2] and Leindler[3]. Moreover, Bolgov and Efimov[4] have generalized the above results to the means generated by triangular matrices. In order to state their theorem we consider the following things.

Let

$$U_n(x) = \sum_{k=o}^{n} a_{nk}s_k(x),$$

denote the n^{th}- mean for the linear method of summation, which are computed from the partial sums of the series (3.1.1) and from a triangular matrix (a_{nk}), $k=o,1,\ldots n$, $n=o,1,\ldots$ for which

$$(3.1.2) \qquad \sum_{k=o}^{n} a_{nk}=1, \quad n=o,1,\ldots\ldots, \quad a_{nk}=o, \quad k>n,$$

(3.1.3) there exists $p>1$ such that for all $n=o,1,\ldots\ldots$

$$(n+1)^{1-\frac{1}{p}}\|a_n\|_p < M$$

where M is an absolute constant and

$$\|a_n\|_p = \left\{\sum_{k=o}^{n} |a_{nk}|^p\right\}^{\frac{1}{p}} \quad \text{for } 1< p<\infty$$

$$\|a_n\|_\infty = \max_{o<k<n} |a_{nk}|$$

1) Leindler [39]

2) Sunouchi([78],[79])

3) Leindler [42]

4) Bolgov and Efimov [15]

In this case, we say that $(a_{nk}) \in A^p$. We note that this method of summability defined by matrices satisfying (3.1.2) and (3.1.3) is regular in the sense of Toeplitz.

Let $\{n_m\}$ be a sequence of natural numbers which satisfies

$$(3.1.4) \qquad 1 < \gamma \leq \frac{n_{m+1}}{n_m} \leq \gamma_1 \;, \; m=o,1, \;\ldots\ldots, \; n_o=1$$

and let $l(n)$ be a positive nondecreasing function such that

$$(3.1.5) \qquad \frac{l(n_{m+1})}{l(n_m)} \leq K < \gamma^{\alpha} (o < \alpha \leq 1).$$

We shall say that $l(n)$ belongs to the class Λ^{α}, denoted by $l(n) \in \Lambda^{\alpha}$ if it is a positive non-decreasing function which satisfies (3.1.5) and is such that

$$(3.1.6) \qquad \frac{l(n)}{n^{\alpha}} \downarrow o.$$

If we have $1 < K_1 \leq \frac{l(n_{m+1})}{l(n_m)}$, then we shall denote the corresponding class by $\tilde{\Lambda}^{\alpha}$. Observe that $n^{\beta} \in \Lambda^{\alpha}$ if $\beta < \alpha$.

Bolgov and Efimov have proved the following theorem.

<u>THEOREM A</u> : Let T denote the linear method of summation determined by the matrix $(a_{nk}) \in A^p$ $(p > 1)$. Let $U_n(x)$ denote the corresponding means of the orthogonal series (3.1.1) and let $\nu(n)$ denote a positive non-decreasing function such that the condition

$$(3.1.7) \qquad \sum_{n=1}^{\infty} c_n^2 \, v^2(n) < \infty$$

implies the summability a.e. of the series using T. If the function $l(n) \in \Lambda^{1-\frac{1}{p}}$, then for an orthogonal series whose coefficients satisfy

$$(3.1.8) \qquad \sum_{n=1}^{\infty} c_n^2 \, v^2(n) l^2(n) < \infty ,$$

the T-means satisfy the following relation almost everywhere on $[a, b]$:

$$(3.1.9) \qquad |U_n(x) - f(x)| = o_x\left(\frac{1}{l(n)}\right),$$

where f(x) is the function of the Riesz-Fischer theorem to which the series (3.1.1) converges. If the function $l(n) \in \tilde{\Lambda}^{1-\frac{1}{p}}$, then the condition

$$(3.1.10) \qquad \sum_{n=1}^{\infty} c_n^2 l^2(n) < \infty$$

implies (3.1.9) almost everywhere.

Further Leindler[1)] proved the following theorem.

<u>THEOREM B</u>: If

$$(3.1.11) \qquad \sum_{n=1}^{\infty} c_n^2 \, n^{2\beta} < \infty \qquad (0 < \beta < 1),$$

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1) Leindler [41]

then

$$\sigma_n(x)-f(x) = o_x(n^{-\beta})$$

holds a.e. in (a,b).

In this chapter, we generalize the above result of Leindler to Nörlund means as follows :

By taking $v(n)=$ loglogn and $l(n)=n^{\beta}$ $(o<\beta<\frac{1}{2})$, we obtain from Theorem A that the relation

$$|t_n(x)-f(x)| = o_x(n^{-\beta})$$

holds almost everywhere in (a, b). But we can obtain the same result under the weaker hypothesis as stated below :

THEOREM 1 : If $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha > \frac{1}{2}$, then under the condition (3.1.11), the relation

$$t_n(x)-f(x) = o_x(n^{-\beta})$$

holds almost everywhere in (a, b).

REMARK : First of all we see that for the Nörlund means $t_n(x)$ as defined above, we have

(a) $a_{nk} = \dfrac{p_{n-k}}{P_n}$, k=o, 1, ..., n, n=o, 1, 2.......

for which

(i) $\displaystyle\sum_{k=o}^{n} a_{nk} = \sum_{k=o}^{n} \frac{p_{n-k}}{P_n} = 1$, n=o, 1, 2.....

and $a_{nk}=o,\ k>n.$

(ii) Now, for $o<p_n\uparrow$,

$$(n+1)^{1-\frac{1}{p}}\|a_n\|_p=(n+1)^{1-\frac{1}{p}}\left\{\sum_{k=o}^{n}\left(\frac{p_{n-k}}{P_n}\right)^p\right\}^{\frac{1}{p}}\leqslant$$

$$\leqslant (n+1)\frac{p_n}{P_n}<M \text{ as } \{p_n\}\in\overline{M}^{\alpha},\ \alpha>\tfrac{1}{2},$$

where M is an absolute constant.

Thus for $o<p_n\uparrow$, $(a_{nk})\in A^p$ with $p>1$.

On the other hand, if $o<p_n\downarrow$, then for p=2

$$(n+1)^{1-\frac{1}{p}}\|a_n\|_p=(n+1)^{\frac{1}{2}}\|a_n\|_2$$

$$=(n+1)^{\frac{1}{2}}\left\{\sum_{k=o}^{n}\frac{p_{n-k}^2}{P_n^2}\right\}^{\frac{1}{2}}$$

$$=\left\{\frac{(n+1)}{P_n^2}\sum_{k=o}^{n}p_k^2\right\}^{\frac{1}{2}}$$

$$=O(1)\left\{\frac{(n+1)}{P_n^2}\sum_{k=o}^{n}\frac{P_k^2}{k^2}\right\}^{\frac{1}{2}}$$

$$=O\left(\left(\frac{1}{2\alpha-1}\right)^{\frac{1}{2}}\right).$$

i.e. for $o<p_n\downarrow$, $(a_{nk})\in A^2$.

Hence, for Nörlund means $(a_{nk})\in A^2$.

Taking $\nu(n)=\log\log n$,

$$\sum_{n=2}^{\infty} c_n^2\ \nu^2(n) = \sum_{n=2}^{\infty} c_n^2(\log\log n)^2 < \infty$$

implies the (N,p_n)- summability of the orthogonal series (3.1.1) with $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha > \frac{1}{2}$.[1]

Also, $l(n)= n^{\beta}(o<\beta<\frac{1}{2})$ is a positive non-decreasing function satisfying the condition (3.1.5) with $n_m=2^m$, $m=o,1,2.......$, $n_o=1$ and $l(n)\in \Lambda^{\frac{1}{2}}$. Then

$$\sum_{n=4}^{\infty} c_n^2\ \nu^2(n)l^2(n)= \sum_{n=4}^{\infty} c_n^2(\log\log n)^2\ n^{2\beta} < \infty, (o<\beta<\tfrac{1}{2}) \quad \text{implies}$$

$$\left|t_n(x)-f(x)\right| = o_x(n^{-\beta})$$

almost everywhere on (a,b).

Now, we see that the condition (3.1.11) is weaker than the condition (3.1.8). Considering

$$c_n = (n^{\beta+\frac{1}{2}}(\log n)^{\frac{1}{2}}\ \log\log n)^{-1}$$

we observe that the condition (3.1.11) is true but

$$\sum_{n=4}^{\infty} c_n^2\ \nu^2(n)l^2(n) = \sum_{n=4}^{\infty} (n^{\beta+\frac{1}{2}}(\log n)^{\frac{1}{2}}\ \log\log n)^{-2}(\log\log n)^2\ n^{2\beta}$$

$$= \sum_{n=4}^{\infty} (n\log n)^{-1} = \infty .$$

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1) Meder [49]

i.e. The condition (3.1.11) need not imply the condition (3.1.8). But it is easily seen that the condition (3.1.8) implies the condition (3.1.11). Moreover, in the condition(3.1.11) the range of β is larger than in the condition (3.1.8).

Considering the second part of the Theorem A, we see that $l(n)=n^{\beta}\in\lambda^{\frac{1}{2}}$ $(o<\beta<\frac{1}{2})$ and the conditions (3.1.10) and (3.1.11) are same but in condition (3.1.11) the range of β is larger than in condition (3.1.10).

A similar result for Euler means is also proved :

THEOREM 2 : If

$$\sum_{n=1}^{\infty} c_n^2 n^{2\beta+1} < \infty, \quad (o<\beta<\tfrac{1}{2}),$$

then the relation

$$\tau_n(x)-f(x) = o_x(n^{-\beta})$$

holds almost everywhere in (a,b).

3.2 We need following lemmas to prove the above theorems.

LEMMA-1 [1]: If $\{p_n\}\in M^{\alpha}$, $\alpha>\frac{1}{2}$, then

$$\lim_{n\to\infty} \frac{n}{P_n^2}\sum_{k=o}^{n}\frac{P_k^2}{(k+1)^2} = \frac{1}{2\alpha-1}$$

LEMMA-2 [2]: If $\{p_n\}\in M^{*}$, then

1) Meder [48]

2) Meder [49]

$$\sup_{k,n\geq k} \frac{|p_{n-k}P_n - p_n P_{n-k}|}{k p_n p_{n-k}} < +\infty .$$

PROOF : We shall distinguish two cases : (1) $\{p_n\}$ is convex and bounded (2) $\{p_n\}$ is concave or convex and unbounded.

Passing to the first case, we notice that $\{p_n\}$ is then non-increasing. Therefore,

$$o < P_n p_{n-k} - P_{n-k} p_n$$

$$= k p_n p_{n-k} \left(\frac{P_n}{k p_n} - \frac{P_{n-k}}{k p_{n-k}} \right)$$

$$< k p_n p_{n-k} \frac{P_{2k}}{k p_{2k}} = O(k p_n p_{n-k})$$

for $n=k,\ k+1,\ k+2,\ \ldots\ldots,\ 2k.$

For the remaining values of n we write

$$o < P_n p_{n-k} - P_{n-k} p_n$$

$$= (P_{n-k} + p_{n-k+1} + p_{n-k+2} + \cdots\cdots + p_n) p_{n-k} - P_{n-k} p_n$$

$$= P_{n-k}(p_{n-k} - p_n) + p_{n-k}(p_{n-k+1} + p_{n-k+2} + \cdots\cdots + p_n)$$

$$= P_{n-k}\left[(p_{n-k} - p_{n-k+1}) + (p_{n-k+1} - p_{n-k+2}) + \cdots\cdots\cdots + (p_{n-1} - p_n)\right] +$$

$$+ p_{n-k}(p_{n-k+1} + p_{n-k+2} + \cdots\cdots\cdots + p_n).$$

The proof runs further after the following estimate :

$$\frac{P_n p_{n-k} - P_{n-k} p_n}{P_n P_{n-k}} = \frac{P_{n-k}}{P_n P_{n-k}} \Big[(p_{n-k} - p_{n-k+1}) + (p_{n-k+1} - p_{n-k+2}) + \ldots\ldots + (p_{n-1} - p_n) \Big] + \frac{p_{n-k} p_{n-k+1}}{P_n P_{n-k}} \Big[1 + \frac{p_{n-k+2}}{p_{n-k+1}} + \frac{p_{n-k+3}}{p_{n-k+1}} + \ldots\ldots + \frac{p_n}{p_{n-k+1}} \Big]$$

$$\leqslant \frac{P_n}{n p_n} \Big[\frac{(n-k+1)(p_{n-k} - p_{n-k+1})}{p_{n-k+1}} \cdot \frac{p_{n-k+1}}{P_{n-k}} \cdot \frac{n}{n-k+1} + \frac{(n-k+2)(p_{n-k+1} - p_{n-k+2})}{p_{n-k+2}} \cdot \frac{p_{n-k+2}}{P_{n-k}} \cdot \frac{n}{n-k+2} + \ldots\ldots + \frac{n(p_{n-1} - p_n)}{p_n} \cdot \frac{p_n}{P_{n-k}} \cdot \frac{n}{n} \Big] + \frac{p_{n-k+1}}{P_n} \Big[1 + \frac{p_{n-k+2}}{p_{n-k+1}} + \frac{p_{n-k+3}}{p_{n-k+1}} + \ldots\ldots + \frac{p_n}{p_{n-k+1}} \Big].$$

Since $\{p_n\} \in M^*$ and non-increasing, we get from above inequality

$$\frac{P_n p_{n-k} - P_{n-k} p_n}{P_n P_{n-k}} = O(k)$$

for n=2k+1, 2k+2, 2k+3, ..., which completes the proof in the first case.

Passing to the second case, we remark that $\{p_n\}$ is then non-decreasing. Hence

$$\frac{|P_n\, p_{n-k} - P_{n-k} p_n|}{P_n P_{n-k}} \leqslant \frac{P_{n-k}}{P_n P_{n-k}} (p_n - p_{n-k}) + \frac{p_{n-k+1} + p_{n-k+2} + \ldots\ldots + p_n}{P_n}$$

$$= \frac{p_{n-k}}{(n-k+1)p_{n-k}} \left[\frac{(n+1)(p_n - p_{n-1})}{p_n} \cdot \frac{n-k+1}{n+1} + \right.$$

$$+ \frac{n(p_{n-1} - p_{n-2})}{p_{n-1}} \cdot \frac{n-k+1}{n} \cdot \frac{p_{n-1}}{p_n} + \ldots\ldots + \frac{(n-k+2)(p_{n-k+1} - p_{n-k})}{p_{n-k+1}} \cdot$$

$$\left. \frac{(n-k+1)}{n-k+2} \cdot \frac{p_{n-k+1}}{p_n} \right] +$$

$$+ \left(1 + \frac{p_{n-1}}{p_n} + \frac{p_{n-2}}{p_n} + \ldots\ldots + \frac{p_{n-k+1}}{p_n}\right) = O(k)$$

for $n=k, k+1, k+2 \ldots\ldots$ which completes the proof of the lemma.

REMARK : Lemma 2 holds if we replace the class M^* by the class $\overline{M}^{\alpha}$ with $\alpha > o$.

LEMMA-3[1]: Writing

$$W_{nk} = \frac{1}{2^n} \sum_{i=o}^{k-1} \binom{n}{i} - \frac{2k}{n+1}$$

we have

$$W_{nk} < o \quad \text{for} \quad [\tfrac{n}{3}]+2 \leq k \leq n.$$

LEMMA-4[1] :

$$\left\{ \frac{1}{2^n} \sum_{i=o}^{k-1} \binom{n}{i} \right\}^2 \leq A \frac{k^2}{n^2} \quad \text{for} \quad 1 \leq k \leq [\tfrac{n}{3}] + 1,$$

A being an absolute constant and $n=1, 2, 3 \ldots\ldots$.

LEMMA-5[2]: For any value of $q > o$, the following evaluation

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1) Meder [46]

2) Ziza [95]

is valid.

$$\max_{0 \leq k \leq n} \binom{n}{k} q^k \leq B_q \frac{(1+q)^n}{\sqrt{n}}, \qquad n=1,2,3\ldots\ldots$$

where the constant B_q does not depend on n.

3.3 PROOF OF THEOREM 1 : We have

$$\sum_{n=1}^{\infty} 2^{2n\beta} \int_a^b (s_{2^n}(x)-f(x))^2\, dx$$

$$= \sum_{n=1}^{\infty} 2^{2n\beta} \sum_{k=2^n+1}^{\infty} c_k^2$$

$$= \sum_{n=1}^{\infty} 2^{2n\beta} \sum_{m=n}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} c_k^2$$

$$= \sum_{m=1}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} c_k^2 \sum_{n=1}^{m} 2^{2n\beta}$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} c_k^2 2^{2m\beta}$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} c_k^2 k^{2\beta} < \infty .$$

Consequently, by B. Levy's theorem, it follows that the series

$$\sum_{n=1}^{\infty} 2^{2n\beta}(s_{2^n}(x)-f(x))^2$$

converges almost everywhere in (a, b) and hence, it follows that the relation

$$(3.3.1) \qquad s_{2^n}(x)-f(x) = o_x(2^{-n\beta})$$

holds almost everywhere in (a, b).

Now, we proceed to prove that the relation

$$s_{2^n}(x)-t_{2^n}(x) = o_x(2^{-n\beta})$$

holds almost everywhere in (a, b).

We have

$$s_n(x)-t_n(x) = \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{P_n}\sum_{r=o}^{n} p_{n-r}s_r(x)$$

$$= \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{P_n}\sum_{r=o}^{n} p_{n-r}\sum_{k=o}^{r} c_k\phi_k(x)$$

$$= \frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{r=o}^{n} p_{n-r} - \frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{r=k}^{n} p_{n-r}$$

$$= \frac{1}{P_n}\sum_{k=o}^{n} c_k\phi_k(x)\sum_{r=o}^{k-1} p_{n-r}$$

$$= \frac{1}{P_n}\sum_{k=o}^{n}\left\{\sum_{i=n-k+1}^{n} p_i\right\} c_k\phi_k(x).$$

Therefore,

$$(3.3.2) \quad \sum_{n=1}^{\infty} 2^{2n\beta}\int_a^b (s_{2^n}(x)-t_{2^n}(x))^2 dx = \sum_{n=1}^{\infty}\frac{2^{2n\beta}}{P^2_{2^n}}\sum_{k=o}^{2^n}\left\{\sum_{i=2^n-k+1}^{2^n} p_i\right\}^2 c_k^2$$

If $o < p_n \uparrow$, then from (3.3.2), we have

$$\sum_{n=1}^{\infty} 2^{2n\beta} \int_a^b (s_{2^n}(x)-t_{2^n}(x))^2\, dx$$

$$\leqslant \sum_{n=1}^{\infty} 2^{2n\beta} \frac{p_{2^n}^2}{P_{2^n}^2} \sum_{k=o}^{2^n} k^2 c_k^2$$

$$= O(1) \sum_{n=1}^{\infty} \frac{2^{2n\beta}}{2^{2n}} \sum_{k=1}^{2^n} k^2 c_k^2$$

$$= O(1) \sum_{n=1}^{\infty} \frac{2^{2n\beta}}{2^{2n}} \sum_{m=o}^{n-1} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2$$

$$= O(1) \sum_{m=o}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2 \sum_{n=m+1}^{\infty} \frac{2^{2n\beta}}{2^{2n}}$$

$$= O(1) \sum_{m=o}^{\infty} \frac{2^{2m\beta}}{2^{2m}} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2$$

$$= O(1) \sum_{m=o}^{\infty} 2^{2m\beta} \sum_{k=2^m+1}^{2^{m+1}} c_k^2$$

$$= O(1) \sum_{m=o}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} c_k^2 k^{2\beta} < \infty .$$

Thus, the series (3.3.2) is convergent, if $o < p_n \uparrow$.

In case $o < p_n \downarrow$, then from (3.3.2) we have

$$\sum_{n=1}^{\infty} 2^{2n\beta} \int_a^b (s_{2^n}(x)-t_{2^n}(x))^2\, dx$$

$$\leq \sum_{n=1}^{\infty} 2^{2n\beta} \sum_{k=o}^{2^n} \frac{p^2_{2^n-k+1}}{P^2_{2^n}} k^2 c_k^2$$

$$= O(1) \sum_{n=1}^{\infty} 2^{2n\beta} \sum_{k=1}^{2^n} \frac{k^2}{(2^n-k+1)^2} c_k^2$$

$$= O(1) \sum_{n=1}^{\infty} 2^{2n\beta} \sum_{m=o}^{n-1} \sum_{k=2^m+1}^{2^{m+1}} \frac{k^2}{(2^n-k+1)^2} c_k^2$$

$$= O(1) \sum_{m=o}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2 \sum_{n=m+1}^{\infty} \frac{2^{2n\beta}}{(2^n-k+1)^2}$$

$$= O(1) \sum_{m=o}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2 \sum_{n=m+1}^{\infty} \frac{2^{2n\beta}}{2^{2n}} < \infty$$

Thus, the series (3.3.2) is convergent, if $o < p_n \downarrow$.

Hence, by B. Levy's theorem, we conclude that the series

$$\sum_{n=1}^{\infty} 2^{2n\beta}(s_{2^n}(x)-t_{2^n}(x))^2 < \infty$$

almost everywhere in (a, b) and therefore, the relation

(3.3.3) $$s_{2^n}(x)-t_{2^n}(x) = o_x(2^{-n\beta})$$

holds almost everywhere in (a,b).

Now, it remains to prove that the estimate

$$\sum_{k=2^m+1}^{2^{m+1}} k^{2\beta+1}(t_k(x)-t_{k-1}(x))^2 = o_x(1) \tag{3.3.4}$$

holds almost everywhere in (a,b).

Assuming $p_{-1}=P_{-1}=o$, we can write

$$t_n(x)-t_{n-1}(x)=\frac{1}{P_n}\sum_{k=o}^{n} p_{n-k}s_k(x)-\frac{1}{P_{n-1}}\sum_{k=o}^{n-1} p_{n-1-k}s_k(x)$$

$$=\frac{1}{P_n}\sum_{k=o}^{n} p_{n-k}s_k(x)-\frac{1}{P_{n-1}}\sum_{k=o}^{n} p_{n-1-k}s_k(x)$$

$$=\frac{1}{P_n}\sum_{r=o}^{n} c_r\phi_r(x)\sum_{k=r}^{n} p_{n-k}-\frac{1}{P_{n-1}}\sum_{r=o}^{n} c_r\phi_r(x)\sum_{k=r}^{n} p_{n-1-k}$$

$$=\frac{1}{P_n}\sum_{r=o}^{n} P_{n-r}\, c_r\phi_r(x)-\frac{1}{P_{n-1}}\sum_{r=o}^{n} P_{n-r-1}c_r\phi_r(x)$$

$$=\frac{1}{P_nP_{n-1}}\sum_{r=o}^{n} (p_{n-r}P_n-p_nP_{n-r})c_r\phi_r(x)$$

$$=\frac{1}{P_nP_{n-1}}\sum_{r=1}^{n} (p_{n-r}P_n-p_nP_{n-r})\, c_r\phi_r(x).$$

Consequently

$$\sum_{n=1}^{\infty} n^{2\beta+1}\int_a^b (t_n(x)-t_{n-1}(x))^2\,dx =$$

$$= \sum_{n=1}^{\infty} n^{2\beta+1} \sum_{k=1}^{n} \left(\frac{p_{n-k}P_n - p_n P_{n-k}}{P_n P_{n-1}}\right)^2 c_k^2$$

$$= \sum_{k=1}^{\infty} c_k^2 \sum_{n=k}^{\infty} \left(\frac{p_{n-k}P_n - p_n P_{n-k}}{P_n P_{n-1}}\right)^2 n^{2\beta+1}$$

Decomposing the inner sum of the last expression in two sums from $n=k$ to $n=2k$ and from $n=2k+1$ to $n=+\infty$ and applying first Lemma 2 and then Lemma 1, we obtain

$$\sum_{n=1}^{\infty} n^{2\beta+1} \int_a^b (t_n(x)-t_{n-1}(x))^2 dx =$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{2k} \frac{p_n^2 p_{n-k}^2}{P_n^2\, P_{n-1}^2}\, n^{2\beta+1} +$$

$$+ O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=2k+1}^{\infty} \frac{p_n^2 p_{n-k}^2}{P_n^2 P_{n-1}^2}\, n^{2\beta+1}$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2\, k^{2\beta}\, \frac{k}{P_k^2} \sum_{n=0}^{k} p_n^2 + O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=2k+1}^{\infty} n^{2\beta-3}$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 k^{2\beta} < \infty .$$

Hence, by B.Levy's theorem it follows that the series

$$\sum_{n=1}^{\infty} n^{2\beta+1} (t_n(x)-t_{n-1}(x))^2$$

converges almost every where in (a,b) and therefore, the relation (3.3.4) holds almost everywhere in (a,b).

Consequently, for $2^m < n < 2^{m+1}$

$$\left|t_n(x)-t_{2^m}(x)\right| = \left|\sum_{k=2^m+1}^{n} (t_k(x)-t_{k-1}(x))\right| =$$

$$= \left|\sum_{k=2^m+1}^{n} k^{\beta+\frac{1}{2}} (t_k(x)-t_{k-1}(x))\frac{1}{k^{\beta+\frac{1}{2}}}\right|$$

$$\leqslant \left\{\sum_{k=2^m+1}^{2^{m+1}} k^{2\beta+1}(t_k(x)-t_{k-1}(x))^2 \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k^{2\beta+1}}\right\}^{\frac{1}{2}}$$

$$= o_x(2^{-m\beta})$$

$$= o_x(n^{-\beta})$$

holds almost everywhere in (a,b)

i.e. The relation

$$(3.3.5) \qquad \left|t_n(x)-t_{2^m}(x)\right| = o_x(n^{-\beta})$$

holds for $2^m < n < 2^{m+1}$ almost everywhere in (a,b).

Hence, it follows from (3.3.1), (3.3.3) and (3.3.5) that the relation

$$\left|t_n(x)-f(x)\right| = o_x(n^{-\beta})$$

holds almost everywhere in (a,b).

This proves the theorem completely.

3.4 PROOF OF THEOREM 2 : As proved in Theorem 1, the estimate

(3.4.1) $$s_{2^n}(x)-f(x)= o_x(2^{-n\beta})$$

holds almost everywhere in (a,b).

Now, we prove that the estimate

$$s_{2^n}(x)- \tau_{2^n}(x)= o_x(2^{-n\beta})$$

holds almost everywhere in (a,b).

We have

$$s_n(x)- \tau_n(x) = \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{2^n}\sum_{r=o}^{n} \binom{n}{r}s_r(x)$$

$$= \sum_{k=o}^{n} c_k\phi_k(x) - \frac{1}{2^n}\sum_{r=o}^{n} \binom{n}{r} \sum_{k=o}^{r} c_k\phi_k(x)$$

$$= \frac{1}{2^n}\sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=o}^{n} \binom{n}{r}- \frac{1}{2^n}\sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=k}^{n} \binom{n}{r}$$

$$= \frac{1}{2^n} \sum_{k=o}^{n} c_k\phi_k(x) \sum_{r=o}^{k-1} \binom{n}{r}$$

Hence $$\sum_{n=1}^{\infty} 2^{2n\beta} \int_a^b (s_{2^n}(x)- \tau_{2^n}(x))^2 \, dx=$$

$$= \sum_{n=1}^{\infty} 2^{2n\beta} \sum_{k=o}^{2^n} c_k^2 \left\{\frac{1}{2^{2^n}} \sum_{r=o}^{k-1} \binom{2^n}{r} \right\}^2$$

Effecting lemmas 3 and 4, we obtain

$$\sum_{n=1}^{\infty} 2^{2n\beta} \int_a^b (s_{2^n}(x) - \tau_{2^n}(x))^2 dx$$

$$\leqslant C \sum_{n=1}^{\infty} \frac{2^{2n\beta}}{2^{2n}} \sum_{k=1}^{2^n} k^2 c_k^2$$

$$= C \sum_{n=1}^{\infty} \frac{2^{2n\beta}}{2^{2n}} \sum_{m=0}^{n-1} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2$$

$$= C \sum_{m=0}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2 \sum_{n=m+1}^{\infty} \frac{2^{2n\beta}}{2^{2n}}$$

$$= O(1) \sum_{m=0}^{\infty} \frac{2^{2m\beta}}{2^{2m}} \sum_{k=2^m+1}^{2^{m+1}} k^2 c_k^2$$

$$= O(1) \sum_{k=0}^{\infty} c_k^2 \, k^{2\beta} < \infty .$$

Therefore, by B. Levy's theorem, it follows that the series

$$\sum_{n=1}^{\infty} 2^{2n\beta} (s_{2^n}(x) - \tau_{2^n}(x))^2 < \infty$$

almost everywhere in (a,b).

Consequently, the relation

$$s_{2^n}(x) - \tau_{2^n}(x) = o_x(2^{-n\beta}) \tag{3.4.2}$$

holds almost everywhere in (a,b).

Now, it remains to prove that the relation

$$(3.4.3) \qquad \sum_{k=2^m+1}^{2^{m+1}} k^{2\beta+1} \left(\tau_k(x) - \tau_{k-1}(x) \right)^2 = o_x(1)$$

holds almost everywhere in (a, b).

We have

$$\tau_n(x) - \tau_{n-1}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k(x) - \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} s_k(x)$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k(x) - \frac{1}{2^{n-1}} \sum_{k=0}^{n} \binom{n-1}{k} s_k(x)$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \left\{ \binom{n}{k} - 2\binom{n-1}{k} \right\} s_k(x).$$

Now $\qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$

Therefore

$$\tau_n(x) - \tau_{n-1}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \left\{ \binom{n-1}{k-1} - \binom{n-1}{k} \right\} s_k(x)$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \left\{ \binom{n-1}{k-1} - \binom{n-1}{k} \right\} \sum_{r=0}^{k} c_r \phi_r(x)$$

$$= \frac{1}{2^n} \sum_{r=0}^{n} c_r \phi_r(x) \sum_{k=r}^{n} \left\{ \binom{n-1}{k-1} - \binom{n-1}{k} \right\}$$

$$= \frac{1}{2^n} \sum_{r=0}^{n} \binom{n}{r} \frac{r}{n} c_r \phi_r(x).$$

Consequently,

$$\sum_{n=2}^{\infty} n^{2\beta+1}\int_a^b (\tau_n(x)-\tau_{n-1}(x))^2\,dx$$

$$= \sum_{n=2}^{\infty} n^{2\beta+1}\frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{n}{k}^2\frac{k^2}{n^2}c_k^2$$

$$\leq \sum_{n=2}^{\infty}\frac{n^{2\beta-1}}{2^{2n}}\left\{\max_{0\leq k\leq n}\binom{n}{k}\right\}^2\sum_{k=1}^{n}k^2c_k^2 .$$

By virtue of Lemma 5, we obtain

$$\sum_{n=2}^{\infty} n^{2\beta+1}\int_a^b (\tau_n(x)-\tau_{n-1}(x))^2dx$$

$$\leq B_1\sum_{n=2}^{\infty} n^{2\beta-2}\sum_{k=1}^{n}k^2c_k^2$$

$$\leq B_1\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} n^{2\beta-2}\sum_{\nu=0}^{m}\sum_{k=2^\nu+1}^{2^{\nu+1}}k^2c_k^2$$

$$=O(1)\sum_{m=0}^{\infty}2^{m(2\beta-1)}\sum_{\nu=0}^{m}2^{2\nu}\sum_{k=2^\nu+1}^{2^{\nu+1}}c_k^2$$

$$=O(1)\sum_{\nu=0}^{\infty}2^{2\nu}\sum_{k=2^\nu+1}^{2^{\nu+1}}c_k^2\sum_{m=\nu}^{\infty}2^{m(2\beta-1)}$$

$$=O(1)\sum_{\nu=0}^{\infty}2^{2\nu}\sum_{k=2^\nu+1}^{2^{\nu+1}}c_k^2\,2^{\nu(2\beta-1)}$$

$$= O(1) \sum_{\nu=0}^{\infty} \sum_{k=2^{\nu}+1}^{2^{\nu+1}} c_k^2 2^{\nu(2\beta+1)}$$

$$= O(1) \sum_{\nu=0}^{\infty} \sum_{k=2^{\nu}+1}^{2^{\nu+1}} c_k^2 k^{2\beta+1} < \infty .$$

Hence, by B . Levy's theorem, it follows that the series

$$\sum_{n=2}^{\infty} n^{2\beta+1} (\tau_n(x) - \tau_{n-1}(x))^2$$

converges almost everywhere in (a,b) and therefore, the relation (3.4.3) holds almost everywhere in (a,b).

Consequently, for $2^m < n < 2^{m+1}$

$$|\tau_n(x) - \tau_{2^m}(x)| = \left|\sum_{k=2^m+1}^{n} (\tau_k(x) - \tau_{k-1}(x))\right|$$

$$= \left| \sum_{k=2^m+1}^{n} k^{\beta+\frac{1}{2}} (\tau_k(x) - \tau_{k-1}(x)) \frac{1}{k^{\beta+\frac{1}{2}}} \right|$$

$$\leq \left\{ \sum_{k=2^m+1}^{2^{m+1}} k^{2\beta+1} (\tau_k(x) - \tau_{k-1}(x))^2 \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k^{2\beta+1}} \right\}^{\frac{1}{2}}$$

$$= o_x(2^{-m\beta})$$

$$= o_x(n^{-\beta})$$

holds almost everywhere in (a,b).

i.e. The relation

$$(3.4.4) \qquad |\tau_n(x) - \tau_{2^m}(x)| = o_x(n^{-\beta})$$

holds for $2^m < n < 2^{m+1}$ almost everywhere in (a,b).

Therefore, by (3.4.1), (3.4.2) and (3.4.4), it follows that the relation

$$|\tau_n(x) - f(x)| = o_x(n^{-\beta})$$

holds almost everywhere in (a, b).

This proves the theorem.