CHAPTER - IV

ON THE ORDER OF CERTAIN SUMMABILITY MEANS

4.1 Let $\{\phi_n(x)\}$ $(n=0,1,2,\ldots\ldots)$ be an orthonormal system (ONS) of L^2-integrable functions defined in the closed interval $[a,b]$. We consider the orthogonal series

$$(4.1.1) \qquad \sum_{n=0}^{\infty} C_n\phi_n(x)$$

with real coefficients C_n's.

The notations $s_n(x)$, $t_n(x)$, $\{p_n\} \in M^{\alpha}$ and $\{p_n\} \in BVM^{\alpha}$ mean the same as referred in Chapter II.

Define $$K_n(t,x) = \sum_{k=0}^{n} \phi_k(t)\phi_k(x)$$

and

$$L_n(x) = \int_a^b |K_n(t,x)| \, dt$$

Here $K_n(t,x)$ and $L_n(x)$ are respectively called the n^{th} kernel and n^{th} Lebesgue function of the ONS $\{\phi_n(x)\}$.

The behaviour of Lebesgue functions effects upon the convergence of orthogonal series. The connection between the behaviour of Lebesgue functions and convergence was first discovered by Kolmogoroff-Seliverstoff[1)] and Plessner[2)] for

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1) Kolmogoroff and Seliverstoff ([35], [36])

2) Plessner [66]

the case of Fourier trigonometric series. It was then extended to orthogonal series for the convergence and Cesàro summability by Kaczmarz[1] and Tandori[2] and for summability by first Logarithmic means by Meder.[3] Sunouchi[4] has also worked in this dierection for Riesz means. While discussing the influence of Lebesgue functions on the Cesàro summability of orthogonal series Alexits[5] has proved the following theorem.

<u>THEOREM A</u> : If the Lebesgue functions

$$L_{2^n}(x) = \int_a^b \left| \sum_{k=0}^{2^n} \phi_k(t)\phi_k(x) \right| dt \qquad (4.1.2)$$

of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E \subset [a,b]$, then the condition

$$\sum_{n=0}^{\infty} c_n^2 < \infty \qquad (4.1.3)$$

implies the $(C,\alpha>0)$-summability of the orthogonal series (4.1.1) almost everywhere on E.

In this chapter we extend the above result to Riesz and Nörlund summability as follows :

<u>THEOREM 1</u> : <u>If the Lebesgue functions (4.1.2) of an ONS</u> $\{\phi_n(x)\}$ <u>are uniformly bounded on the set</u> $E \subset [a,b]$, <u>then the relation</u> (4.1.3) <u>implies that the estimate</u>

$$\sigma_n(\lambda,x) = o_x(n)$$

1) Kaczmarz [28]

2) Tandori [82]

3) Meder [47]

4) Sunouchi [77]

5) Alexits ([4], p.185)

holds almost everywhere on E.

THEOREM 2 : If

(4.1.4) $\{p_n\} \in BVM^{\alpha}, \ \alpha > \frac{1}{2}$

and the Lebesgue functions (4.1.2) of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E \subset [a,b]$, then the orthogonal series (4.1.1) is (N, p_n)- summable almost everywhere under the condition (4.1.3).

Further, while discussing the Cesàro summability of the orthogonal series (4.1.1) Alexits[1)] has made the following assertion :

THEOREM B : If the coefficients of the orthogonal series (4.1.1) satisfy the condition (4.1.3) and the series (4.1.1) is A-summable almost everywhere, then the relation

(4.1.5) $\sigma_n(x) = o_x(\log\log n)$

holds almost everywhere.

An attempt has been made here to prove the above theorem without the A-summability condition. In what follows we prove the following theorem :

THEOREM 3[2)]: If the coefficients of the orthogonal series (4.1.1) satisfy the condition (4.1.3), then the relation (4.1.5) holds almost everywhere.

1) Alexits ([4], p.129)

2) Kantawala [30]

4.2 In order to prove the above theorems, we need the following lemmas :

LEMMA 1[1] : Under the condition (4.1.3) the relation

$$s_{\nu_n}(x) - \sigma_{\nu_n}(x) = o_x(1)$$

is valid almost everywhere for every index sequence $\{\nu_n\}$ with

$$\frac{\nu_{n+1}}{\nu_n} \geq q > 1.$$

LEMMA 2[2] : If the coefficients of the orthogonal series (4.1.1) satisfy the condition

$$\sum_{n=1}^{\infty} c_n^2 (\log\log n)^2 < \infty ,$$

then the series (4.1.1) is (C,α)- summable for every $\alpha > 0$.

LEMMA 3[3] : If $\{\lambda_n\}$ is a positive, non-decreasing number sequence for which the relation

$$L_{\nu_n}(x) = O(\lambda_{\nu_n}) \qquad (\nu_1 < \nu_2 < \ldots\ldots\ldots)$$

holds in a set $E \subset [a,b]$, then for the partial sums $\{s_{\nu_n}(x)\}$ of the orthogonal series (4.1.1) under the condition (4.1.3), the estimate

$$s_{\nu_n}(x) = O_x(\lambda_{\nu_n}^{\frac{1}{2}})$$

holds almost everywhere on E.

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1) Alexits ([4], p.118)
2) Alexits ([4], p.125)
3) Alexits ([4], p.172)

LEMMA 4[1) : Let $s_n(x)$ be the n^{th} partial sum of the orthogonal series (4.1.1) with coefficients C_n satisfying the condition (4.1.3) and $\{n_k\}$ be an arbitrary increasing sequence of indices satisfying the condition

$$1 < q \leq \frac{n_{k+1}}{n_k} \leq r \quad \text{for } k=0,1,2,\dots$$

where r and q are constants. If

$$\{p_n\} \in BVM^{\alpha}, \quad \alpha > \tfrac{1}{2}$$

then the orthogonal series (4.1.1) is (N, p_n)- summable almost everywhere if and only if the sequence $\{s_{n_k}(x)\}$ is convergent almost everywhere.

LEMMA 5 : Let

$$\sum_{n=0}^{\infty} U_n$$

be a given series with $\{s_n\}$ and $\{\sigma_n\}$ as its sequences of partial sums and n^{th} (C,1)- means respectively. If $\{m_n\}$ is a convex null sequence and

(4.2.1) $$\sigma_n = O(1)$$

then

$$\sigma_n(\lambda, m) = o(n)$$

where $\sigma_n(\lambda, m)$ denotes the $(R, \lambda_n, 1)$- mean of the series

$$\sum_{n=0}^{\infty} U_n m_n .$$

PROOF : Using Abel's transformation, we obtain

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1) Meder [48]

$$\sigma_n(\lambda, m) = \sum_{k=o}^{n} \left(1-\frac{\lambda_k}{\lambda_{n+1}}\right) U_k m_k =$$

$$= \sum_{k=o}^{n} \delta_k U_k, \quad \text{where} \quad \delta_k = \left(1-\frac{\lambda_k}{\lambda_{n+1}}\right) m_k$$

$$= \sum_{k=o}^{m} (\delta_k - \delta_{k+1}) s_k + \delta_n s_n$$

$$= \sum_{k=o}^{n} \delta'_k s_k + \delta_n s_n, \quad \text{where} \quad \delta'_k = \delta_k - \delta_{k+1} .$$

Hence, it follows on account of $\delta_n = \delta'_n$ by Abel's transform that

$$\sigma_n(\lambda, m) = \sum_{k=o}^{n} \delta'_k s_k =$$

$$= \sum_{k=o}^{n-1} \delta''_k (k+1)\sigma_k + \delta'_n (n+1)\sigma_n,$$

where $\delta''_k = \delta'_k - \delta'_{k+1}$

The second difference δ''_k may be represented as follows :

$$\delta''_k = \delta'_k - \delta'_{k+1} =$$

$$= \delta_k - 2\delta_{k+1} + \delta_{k+2}$$

$$= \left(1-\frac{\lambda_k}{\lambda_{n+1}}\right)\Delta m_k - \left(1-\frac{\lambda_{k+1}}{\lambda_{n+1}}\right)\Delta m_{k+1} +$$

$$+ \left(\frac{\lambda_{k+1}-\lambda_k}{\lambda_{n+1}}\right) m_{k+1} + \left(\frac{\lambda_{k+1}-\lambda_{k+2}}{\lambda_{n+1}}\right) m_{k+2},$$

where $\Delta m_k = m_k - m_{k+1}$.

Therefore,

$$\sigma_n(\lambda, m) = \sum_{k=o}^{n-1} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)(k+1)\sigma_k \Delta m_k -$$

$$- \sum_{k=o}^{n-1} \left(1 - \frac{\lambda_{k+1}}{\lambda_{n+1}}\right)(k+1)\sigma_k \Delta m_{k+1} +$$

$$+ \sum_{k=o}^{n-1} \left(\frac{\lambda_{k+1} - \lambda_k}{\lambda_{n+1}}\right)(k+1)\sigma_k m_{k+1} +$$

$$+ \sum_{k=o}^{n-1} \left(\frac{\lambda_{k+1} - \lambda_{k+2}}{\lambda_{n+1}}\right)(k+1)\sigma_k m_{k+2} +$$

$$+ \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(n+1)\sigma_n m_n .$$

From condition (4.2.1), we get

$$\sigma_n(\lambda, m) = O(1) \sum_{k=o}^{n-1} (k+1)\Delta m_k + O(1) \sum_{k=o}^{n-1} (k+1)\Delta m_{k+1} +$$

$$+ O(1) \sum_{k=o}^{n-1} \left(\frac{\lambda_{k+1} - \lambda_k}{\lambda_{n+1}}\right)(k+1) m_{k+1} +$$

$$+ O(1) \sum_{k=o}^{n-1} \left(\frac{\lambda_{k+1} - \lambda_{k+2}}{\lambda_{n+1}}\right)(k+1) m_{k+2} +$$

$$+ O(1)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(n+1) m_n .$$

Since $\{m_n\}$ is a convex null sequence

$$n\Delta m_n = o(1)$$

and consequently

$$\sigma_n(\lambda, m)= o(n)+ o(n) +O(1) \sum_{k=o}^{n-1} \left(\frac{\lambda_{k+1}-\lambda_k}{\lambda_{n+1}} \right)(k+1)m_{k+1} +$$

$$+O(1) \sum_{k=o}^{n-1} \left(\frac{\lambda_{k+1}-\lambda_k}{\lambda_{n+1}} \right)(k+1)m_{k+2}+o(n).$$

$$=o(n)+O(1) \sum_{k=o}^{n-1} \left(\frac{\lambda_{k+1}-\lambda_k}{\lambda_{n+1}}\right)(k+1)\Delta m_{k+1}$$

$$= o(n)+o(1)$$

$$= o(n).$$

With this the lemma is proved.

LEMMA 6[1) : If $\{\lambda_n\}$ is positive, concave, monotone increasing and tending so slowly to infinity that $\Delta\lambda_n = O(n^{-1})$, then (C,1)-summability of the series

$$\sum_{n=o}^{\infty} U_n$$

implies the estimate

$$\sigma_n(\lambda)= o(\lambda_n)$$

where $\sigma_n(\lambda)$ denotes the n^{th} (C,1)- mean of the series

$$\sum_{n=o}^{\infty} U_n \lambda_n .$$

1) Alexits ([4] , p.74)

LEMMA 7 : The sequence $\{\log\log n\}$ is concave and $\Delta \log\log n = O((n\log n)^{-1})$, where for any sequence $\{\lambda_n\}$ we mean $\Delta\lambda_k=\lambda_k-\lambda_{k+1}$ and $\Delta^2\lambda_k=\Delta\lambda_k-\Delta\lambda_{k+1}$.

PROOF : We observe that

$$\Delta^2\log\log n = \log\log n + \log\log(n+2) - \log\log^2(n+1).$$

Thus in order to prove the sequence $\{\log\log n\}$ to be concave, we must have

$$\log\left(\frac{\log n.\log(n+2)}{\log^2(n+1)} \right) \leqslant 0$$

or

$$(4.2.2)\quad \frac{\log n.\log(n+2)}{\log^2(n+1)} \leqslant 1 .$$

Now

$$\log n=\log(n+1-1)$$

$$= \log(n+1) + \log(1-\frac{1}{n+1}) .$$

i.e.

$$(4.2.3)\quad \log n= \log(n+1)-\frac{1}{(n+1)} - \frac{1}{2(n+1)^2} - \cdots\cdots$$

and

$$\log(n+2) = \log(n+1+1).$$

i.e.,

(4.2.4) $$\log(n+2) = \log(n+1) + \frac{1}{(n+1)} - \frac{1}{2(n+1)^2} + \cdots\cdots$$

Hence, from (4.2.3) and (4.2.4)

$$\log n . \log(n+2) \leqslant \log^2(n+1) + \log(n+1)\left\{\frac{1}{(n+1)} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} - \cdots\cdots - \frac{1}{(n+1)} - \frac{1}{2(n+1)^2} - \cdots\cdots\right\}$$

$$= \log^2(n+1) - \log(n+1)\left\{\frac{1}{(n+1)^2} + \frac{1}{2(n+1)^4} + \cdots\cdots\right\}$$

$$< \log^2(n+1).$$

This proves (4.2.2).

Now, we proceed to prove that

$$\Delta(\log\log n) = O((n\log n)^{-1}).$$

We have

$$|\Delta \log\log n| = |\log\log n - \log\log(n+1)|$$

$$= \log\log(n+1) - \log\log n$$

$$= \int_{\log n}^{\log(n+1)} \frac{dx}{x}$$

$$\leqslant \frac{\log(n+1) - \log n}{\log n}$$

Since $\Delta \log n = O(n^{-1})$, we have

$$|\Delta \log\log n| = \frac{1}{\log n} O(n^{-1}) = O((n\log n)^{-1}).$$

This completes the proof of the lemma.

4.3 PROOF OF THEOREM 1 : Since

$$\sum_{n=o}^{\infty} c_n^2 < \infty$$

there exists a monotone number sequence $\{\mu_n^2\}$, such that $\mu_n \longrightarrow \infty$ and

$$\sum_{n=o}^{\infty} c_n^2/\mu_n^2 < \infty .$$

It is also easy to construct (for instance geometrically) a strictly increasing concave sequence $\{m_n\}$ with $m_n^2 \leqslant \mu_n^2$, $m_n^2 \longrightarrow \infty$ and

$$\sum_{n=o}^{\infty} c_n^2 m_n^2 < \infty .$$

Since $\{m_n\}$ is concave and tending to infinity, $\{1/m_n\}$ is a convex null sequence. Let $s_n(m,x)$ and $\sigma_n(m,x)$ denote the n^{th} partial sum and the $n^{th}(C,1)$- mean of the orthogonal series

$$\sum_{n=o}^{\infty} c_n m_n \phi_n(x)$$

respectively.

From our assumption about the uniform boundedness of the Lebesgue functions (4.1.2), it follows by Lemma 3 that the relation

$$s_{2^p}(m,x) = O_x(1)$$

holds almost everywhere on E.

Hence, the relation

$$\sigma_{2^p}(m,x) = O_x(1)$$

is valid almost everywhere on E due to Lemma 1.

Also

$$\sigma_n(m,x) - \sigma_{2^p}(m,x) \longrightarrow 0 \quad ^{1)}$$

almost everywhere with $2^p < n < 2^{p+1}$.

Therefore, the relation

$$\sigma_n(m,x) = O_x(1)$$

is valid almost everywhere on E.

Since the series

$$\sum_{n=0}^{\infty} c_n \phi_n(x)$$

arises from the series

$$(4.3.1) \qquad \sum_{n=0}^{\infty} c_n m_n \phi_n(x)$$

by multiplying by turns the terms of the series (4.3.1) by the terms of the convex null sequence $\{1/m_n\}$, it follows by

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1) Alexits ([4], p.119)

Lemma 5 that the estimate

$$\sigma_n(\lambda, x) = o_x(n)$$

holds almost everywhere on E.

This proves the theorem completely,

4.4 PROOF OF THEOREM 2 : From the given conditions, we can conclude by Theorem A that the orthogonal series (4.1.1) is $(C, \alpha > o)$-summable almost everywhere on E. Therefore, Lemma 1 implies the convergence of the sequence $\{s_{2^n}(x)\}$ of the partial sums of the series (4.1.1).

Hence by Lemma 4, it follow that the orthogonal series (4.1.1) is (N, p_n)- summable almost everywhere on E.

4.5 PROOF OF THEOREM 3 : We have

$$\sum_{n=o}^{\infty} c_n^2 < \infty ,$$

i.e.

$$\sum_{n=3}^{\infty} \frac{c_n^2}{(\text{loglog}n)^2} (\text{loglog}n)^2 < \infty ,$$

Consequently, it follows from Lemma 2 that the orthogonal series

$$\sum \frac{c_n}{\text{loglog}n} \phi_n(x)$$

is (C, α)-summable for every $\alpha > o$.

Moreover, due to Lemma 7, it follows that the sequence $\{\log\log n\}$ is positive, concave, monotone increasing and tending so slowly to infinity and $\Delta(\log\log n) = O(n^{-1})$, hence Lemma 6 implies that

$$\sigma_n(x) = o_x(\log\log n)$$

is valid almost everywhere.

With this the theorem is proved.