CHAPTER - V

ON THE LEBESGUE FUNCTIONS AND SUMMABILITY OF SERIES IN POLYNOMIAL-LIKE ORTHONORMAL SYSTEMS

5.1 Let $\{\phi_n(x)\}$ $(n=0,1,2.....)$ be an orthonormal system (ONS) of $L^2_{\varrho(x)}$- integrable functions defined in the closed interval $[a,b]$, with respect to a positive, bounded and summable weight function $\varrho(x)$. We consider the orthogonal series

$$(5.1.1) \qquad \sum_{n=0}^{\infty} C_n\phi_n(x)$$

with real coefficients C_n's.

The n^{th} Euler mean of order $q>0$ (or the n^{th} (E,q)-mean) of the sequence of partial sums $\{s_n(x)\}$ of the orthogonal series (5.1.1) is given by

$$\tau_n^{(q)}(x)=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k(x),\quad n=0,1,2,....(q>0),$$

where

$$s_n(x)=\sum_{k=0}^{n}C_k\phi_k(x)$$

The series (5.1.1) is said to be Euler summable by means of order q or more precisely (E,q)-summable to $s(x)$, if

$$\lim_{n\to\infty}\tau_n^{(q)}(x)=s(x).$$

The n^{th}-Riesz mean of order $\alpha>0$ (or the n^{th} (R, λ_n, α)-mean) of the orthogonal series (5.1.1) is given by

$$\sigma_n^{\alpha}(\lambda, x)=\sum_{k=0}^{n}\left(1-\frac{\lambda_k}{\lambda_{n+1}}\right)^{\alpha} c_k \phi_k(x)$$

where $\{\lambda_n\}$ is a positive, strictly increasing numerical sequence with $\lambda_0=0$ and $\lambda_n \to \infty$ as $n \to \infty$.

The series (5.1.1) is said to be (R, λ_n, α)-summable to $s(x)$, if

$$\lim_{n\to\infty} \sigma_n^{\alpha}(\lambda, x)= s(x).$$

An ONS $\{\phi_n(x)\}$ is called polynomial-like if its n^{th} kernel

$$K_n(t, x)=\sum_{k=0}^{n} \phi_k(t)\phi_k(x)$$

has the following structure :

$$(5.1.2)\quad K_n(t, x)=\sum_{k=1}^{r} F_k(t, x)\sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(n)}\phi_{n+i}(t)\phi_{n+j}(x),$$

where p and r are natural numbers independent of n and the constants $|\gamma_{i,j,k}^{(n)}|$ have a common bound independent of n, while the measurable functions $F_k(t, x)$ satisfy the condition

$$F_k(t, x) = O\left(\frac{1}{|t-x|}\right)$$

for every $x\in[a, b]$. We assume that ϕ_{n+i} with negative index is considered to be identically equal to zero.

The ONS $\{\phi_n(x)\}$ is called constant-preserving, if $\phi_o(x)$ = constant.

Define

$$K_n^\beta(t,x) = \sum_{k=o}^{n} \frac{A_{n-k}^\beta}{A_n^\beta} \phi_k(t)\phi_k(x)$$

and

$$L_n^\beta(x) = \int_a^b |K_n^\beta(t,x)|\, \varrho(t)dt , \quad \beta > -1 .$$

Then $K_n^\beta(t,x)$ and $L_n^\beta(x)$ are respectively called the $n^{th}(C,\beta)$-kernel and n^{th}-Lebesgue (C,β)-function of the ONS $\{\phi_n(x)\}$. Further define for $q > o$

$$E_n^{(q)}(t,x) = \frac{1}{(1+q)^n} \sum_{\nu=o}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x)$$

$$F_n^{(q)}(x) = \int_a^b |E_n^{(q)}(t,x)|\, \varrho(t)dt$$

and for $\alpha > o$

$$U_n^\alpha(t,x) = \sum_{k=o}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)^\alpha \phi_k(t)\phi_k(x)$$

and

$$V_n^\alpha(x) = \int_a^b |U_n^\alpha(t,x)|\, \varrho(t)dt.$$

Then $E_n^{(q)}(t,x)$ and $U_n^\alpha(t,x)$ are respectively called the $n^{th}(E,q)$-kernel and the n^{th} (R,λ_n,α)-kernel of the ONS $\{\phi_n(x)\}$ whereas $F_n^{(q)}(x)$ and $V_n^\alpha(x)$ are respectively called the n^{th}-Lebesgue (E,q)-

function and the n^{th}-Lebesgue (R,λ_n,α)-function of the ONS $\{\phi_n(x)\}$.

The partial sums $s_n(x)$ of the expansion of an $L_{\rho(x)}$-integrable function $f(x)$ in the functions of an ONS $\{\phi_n(x)\}$ can be represented by

$$I_n(f,x) = \int_a^b f(t)\ \psi_n(t,x)\ \rho(t)dt$$

where $\psi_n(t,x)$ denotes the sum

$$\sum_{k=o}^{n} \phi_k(t)\phi_k(x).$$

The n^{th} sum

$$t_n(x) = \sum_{k=o}^{n} \alpha_{nk}s_k(x)$$

of an expansion summed by a linear summation process has also the same integral form, where $\psi_n(t,x)$ denotes the sum

$$\sum_{k=o}^{n} \alpha_{nk}\phi_k(t)\phi_k(x).$$

The integral $I_n(f,x)$ is said to be singular (with singular point x), if for an arbitrary positive number δ and for an arbitrary subinterval $[\alpha,\beta]$ of $[a,b]$, the following conditions hold :

(5.1.3) $\lim_{n\to\infty}\int_I \psi_n(t,x)\rho(t)dt=1$ and $\lim_{n\to\infty}\int_J \psi_n(t,x)\rho(t)dt=o$

with $I = [a,b] \cap [x-\delta, x+\delta]$, $J = [\alpha, \beta] - [x-\delta, x+\delta]$.

(5.1.4) $$\operatorname*{ess\,lub}_{t \in [a,b] - [x-\delta, x+\delta]} |\psi_n(t,x)| \leq \psi(\delta)$$

where $\psi(\delta)$ is a number depending on δ and x but independent of n.

If $\psi_n(t,x)$ satisfies uniformly the conditions (5.1.3) and (5.1.4) in an x-set E, then the integral $I_n(f,x)$ is said to be uniformly singular on E.

The convergence of orthogonal series depends upon the Lebesgue functions. This dependence was first investigated by Kolmogoroff-Seliverstoff[1] and Plessner[2] for the case of Fourier trigonometric series. Later on it was extended to the convergence and Cesàro summability and summability by first logarithmic means by Kaczmarz,[3] Tandori[4], Meder[5] and Patel and Sapre[6]. The convergence and summability of non-orthogonal functions series is also studied by Alexits and Sharma[7] and Tandori[8].

The behaviour of the Lebesgue functions for polynomial-like ONS is investigated by Ratajski[9] and Alexits[10]. The convergence and summability of orthogonal expansions for polynomial-like system has been studied by Zinovèv[11] and Alexits[12].

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1) Kolmogoroff-Seliverstoff([35], [36])
2) Plessner [66]
3) Kaczmarz [28]
4) Tandori([82], [85], [88])
5) Meder [47]
6) Patel and Sapre [61]
7) Alexits and Sharma [10]
8) Tandori([86], [87], [89])
9) Ratajski([68], [69])
10) Alexits([4], p.206)
11) Zinovèv [94]
12) Alexits([4], p.267)

Alexits[1] has proved the following theorems :

THEOREM A : If the ONS $\{\phi_n(x)\}$ is polynomial-like and the condition

$$\sum_{k=0}^{n} \phi_k^2(x) = O_x(n)$$

is fulfilled in the set E, then the relation

$$L_n^1(x) = O_x(1)$$

holds almost everywhere in E.

THEOREM B : Let $\{\phi_n(x)\}$ be a complete, constant-preserving polynomial-like ONS with respect to the weight function $\varrho(x)$. Suppose that the functions $F_k(t,x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t=x$ and that the two conditions

$$\sum_{k=0}^{n} \phi_k^2(x) = O(n)$$

and

(5.1.5) $$0 < \varrho(x) \leq \text{const.}$$

are also satisfied in the subinterval $[c,d]$ of $[a,b]$. If the $L^2_{\varrho(x)}$-integrable function $f(x)$ is continuous in $[c,d]$, then its expansion

(5.1.6) $$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x)$$

is uniformly (C,1)-summable in every inner subinterval of $[c,d]$, the sum being $f(x)$.

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1) Alexits ([4], p206, 267)

We extend in this chapter the above results to n^{th} Lebesgue (E,q)-function and n^{th} Lebesgue (R,λ_n,α)-function for polynomial-like ONS and to the (E,q)-summability and (R,λ_n,α)-summability of orthogonal expansion for the constant-preserving polynomial-like ONS. Our results are as follows :

THEOREM 1 : If the ONS $\{\phi_n(x)\}$ is polynomial-like and the condition

$$\phi_n(x) = O_x(1) \tag{5.1.7}$$

is fulfilled in the set E, then the relation

$$F_n^{(q)}(x) = O_x(1)$$

holds almost everywhere in E.

THEOREM 2 : Let $\{\phi_n(x)\}$ be a complete constant-preserving polynomial-like ONS with respect to the weight function $\varsigma(x)$. Suppose that the functions $F_k(t,x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t=x$ and that the two conditions

$$\phi_n(x) = O(1) \tag{5.1.8}$$

and (5.1.5) are also satisfied in the subinterval $[c,d]$ of $[a,b]$. If the $L^2_{\varsigma(x)}$-integrable function $f(x)$ is continuous in $[c,d]$, then its expansion (5.1.6) is uniformly (E,q)-summable $(q>0)$ in every inner sub-interval of $[c,d]$, the sum being $f(x)$.

THEOREM 3 : If the ONS $\{\phi_n(x)\}$ is polynomial-like and the condition (5.1.7) is fulfilled in the set E, then the relation

$$V_n^{\alpha}(x) = O_x(1)$$

holds almost everywhere in E.

THEOREM 4 : Let $\{\phi_n(x)\}$ be a complete constant-preserving polynomial-like ONS with respect to the weight function $\varrho(x)$. Suppose that the functions $F_k(t,x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t=x$ and that the two conditions (5.1.5) and (5.1.8) are also satisfied in the sub-interval $[c,d]$ of $[a,b]$. If the $L^2_{\varrho(x)}$-integrable function $f(x)$ is continuous in $[c,d]$, then its expansion (5.1.6) is uniformly (R, λ_n, α)-summable $(\alpha > 0)$ in every inner sub-interval of $[c,d]$, the sum being $f(x)$.

5.2 The following lemmas will be required for the proofs of the theorems.

LEMMA 1[1] : The expansion coefficients C_n of an L^2_{μ}-integrable function converge to zero as n is indefinitely increased.

LEMMA 2[2] : In order that an ONS $\{\phi_n(x)\}$ should be complete, the validity of Parseval's equation

$$\int_a^b f^2(x)\,d\mu(x) = \sum_{n=0}^{\infty} C_n^2$$

1) Alexits ([4],p.7)

2) Alexits ([4], p.15)

for all $f \in L_{\mu}^2$ is necessary and sufficient.

LEMMA 3[1] : If the function $f(t) \in L_{\varrho(t)}$ is uniformly continuous in a subset E of [a,b] and the conditions (5.1.3), (5.1.4) and

$$\int_a^b |\psi_n(t,x)| \; \varrho(t)dt = O(1)$$

are uniformly satisfied for $x \in E$, then the relation

$$I_n(f,x) \longrightarrow f(x)$$

holds uniformly in E.

LEMMA 4[2] : A monotone sequence of continuous functions, whose limit function is continuous, converges uniformly.

LEMMA 5[3] : For any value of $q > o$, the following evaluation is valid.

$$\max_{o \leq k \leq n} \binom{n}{k} q^k \leq A_q \frac{(1+q)^n}{\sqrt{n}}, \qquad n=1,2,3.......$$

where the constant A_q does not depend on n.

5.3 PROOF OF THEOREM 1 : We have

$$E_n^{(q)}(t,x) = \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} K_{\nu}(t,x).$$

Let $P_n(t,x)$ and $N_n(t,x)$ be the characteristic functions of the sets in which

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1) Alexits ([4], p.260)
2) Alexits ([4], p.266)
3) Ziza [95]

$$\sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x) \geq 0 \text{ and } < 0 ,$$

respectively. From the definition of n^{th} Lebesgue (E,q)- function

$$F_n^{(q)}(x) = \int_a^b \left|E_n^{(q)}(t,x)\right| \varrho(t)dt =$$

$$= \frac{1}{(1+q)^n} \int_a^b P_n(t,x) \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x) \varrho(t)dt -$$

$$- \frac{1}{(1+q)^n} \int_a^b N_n(t,x) \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x) \varrho(t)dt ,$$

i.e.

(5.3.1) $$F_n^{(q)}(x) = \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \int_a^b P_n(t,x) K_\nu(t,x) \varrho(t)dt -$$

$$- \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \int_a^b N_n(t,x) K_\nu(t,x) \varrho(t)dt.$$

Our aim is to show that each of the sum on the R.H.S. of (5.3.1) have the order of magnitude $O_x((1+q)^n)$ for every $x \in E \cap (a+\epsilon, b-\epsilon)$ with arbitrary $\epsilon > 0$ and therefore $F_n^{(q)}(x) = O_x(1)$ holds for almost every $x \in E$. We divide the integral

$$\int_a^b P_n(t,x) K_\nu(t,x) \varrho(t)dt$$

for $n \geq n_\epsilon > \frac{1}{\epsilon}$ into two parts :

$$I_{\nu 1} = \int_{x-n^{-3/2}}^{x+n^{-3/2}} , \quad I_{\nu 2} = \int_a^{x-n^{-3/2}} + \int_{x+n^{-3/2}}^{b} .$$

We first estimate $|I_{\nu 1}|$.

Using Schwarz's inequality

$$I_{\nu 1}^2 \leq \int_{x-n^{-3/2}}^{x+n^{-3/2}} P_n^2(t,x)\ \rho(t)dt \int_{x-n^{-3/2}}^{x+n^{-3/2}} K_{\nu}^2(t,x)\rho(t)dt.$$

Now the conditions (5.1.7) and $P_n^2(t,x) \leq 1$ implies that

$$I_{\nu 1}^2 \leq \int_{x-n^{-3/2}}^{x+n^{-3/2}} \rho(t)dt \sum_{k=0}^{\nu} \phi_k^2(x) =$$

$$= O_x(\nu n^{-3/2}).$$

Hence by Cauchy's inequality and Lemma 5

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 1}| \leq \left\{ \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu}^2 q^{2(n-\nu)} \sum_{\nu=n_\epsilon}^{n} I_{\nu 1}^2 \right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{ \max_{0 \leq \nu \leq n} \binom{n}{\nu} q^{n-\nu} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \sum_{\nu=n_\epsilon}^{n} I_{\nu 1}^2 \right\}^{\frac{1}{2}}$$

$$= O_x(1) \left\{ (1+q)^{2n}\ n^{-\frac{1}{2}}\ n^{-3/2} \sum_{\nu=n_\epsilon}^{n} \nu \right\}^{\frac{1}{2}}$$

$$= O_x((1+q)^n n^{-1})n$$

i.e.

$$(5.3.2) \qquad \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 1}| = O_x((1+q)n).$$

Now, we proceed to estimate

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 2}| \ .$$

Let us put

$$g_k(t,x) = \begin{cases} P_n(t,x)F_k(t,x) & \text{for } t \in [a, x-n^{-3/2}] \cup [x+n^{-3/2}, b] \ . \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 2}| = \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \left(\int_a^{x-n^{-3/2}} + \int_{x+n^{-3/2}}^{b} \right) P_n(t,x) \cdot K_\nu(t,x) \varrho(t) dt \right| .$$

Since, the system $\{\phi_n(x)\}$ is polynomial-like and therefore using the definition (5.1.2) of the kernel $K_n(t,x)$, we have

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 2}| =$$

$$= \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} \left| \left(\int_a^{x-n^{-3/2}} + \int_{x+n^{-3/2}}^{b} \right) P_n(t,x) \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \cdot \phi_{\nu+i}(t)\, \phi_{\nu+j}(x) \varrho(t) dt \right|$$

$$\leq \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |\gamma_{i,j,k}^{(\nu)}|\, |\phi_{\nu+j}(x)| \left| \left(\int_a^{x-n^{-3/2}} + \int_{x+n^{-3/2}}^{b} \right) \cdot P_n(t,x) F_k(t,x) \phi_{\nu+i}(t)\, \varrho(t) dt \right| .$$

Using, the definition of the function $g_k(t,x)$, we obtain

$$\sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 2}| =$$

$$=O_x(1)\sum_{k=1}^{r}\sum_{i,j=-p}^{p}\sum_{\nu=n_\epsilon}^{n}\binom{n}{\nu}q^{n-\nu}\left|\int_a^b g_k(t,x)\phi_{\nu+i}(t)\,\varrho(t)dt\right|$$

$$=O_x(1)\sum_{k=1}^{r}\sum_{i=-p}^{p}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}\left|\int_a^b g_k(t,x)\,\phi_{\nu+i}(t)\,\varrho(t)dt\right|.$$

i.e.

$$(5.3.3)\qquad \sum_{\nu=n_\epsilon}^{n}\binom{n}{\nu}q^{n-\nu}|I_{\nu 2}|=O_x(1)\sum_{k=1}^{r}\sum_{i=-p}^{p}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}\left|\int_a^b g_k(t,x)\cdot \phi_{\nu+i}(t)\,\varrho(t)dt\right|.$$

Now $\quad F_k(t,x)=O(|t-x|^{-1})$ and $|t-x|\geq n^{-3/2}$ imply that

$$|g_k(t,x)|\leq P_n(t,x)\,|F_k(t,x)|=O(n^{3/2}).$$

i.e. $\quad g_k(t,x)$ is bounded for a fixed n.

i.e. $\quad g_k(t,x)$ is integrable, which means that the integrals on the R.H.S. of the above relation are the expansion coefficients of $L^2_{\varrho(t)}$-integrable function (with fixed i) which tends to zero as $\nu\to\infty$ due to Lemma 1.

Now, let us choose n large enough and fix it.

Since

$$\int_a^b g_k(t,x)\phi_{\nu+i}(t)\varrho(t)dt\longrightarrow 0\quad\text{as }\nu\longrightarrow\infty$$

for given $\epsilon>0$, $\exists\, n_0>0$ such that $\nu\geq n_0$ implies

$$\left|\int_a^b g_k(t,x)\phi_{\nu+i}(t)\,\rho(t)\,dt\right| < \epsilon .$$

Let

$$M = \max\left\{ \left|\int_a^b g_k(t,x)\,\phi_{\nu+i}(t)\,\rho(t)\,dt\right| ,\ \text{where } \nu = 0,1,2,\ldots,n_\epsilon - 1 \right\}.$$

Then

$$\sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \left|\int_a^b g_k(t,x)\phi_{\nu+i}(t)\,\rho(t)\,dt\right|$$

$$\leq M \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu}$$

$$= M(1+q)^n .$$

Consequently, from (5.3.3) we get

$$(5.3.4) \qquad \sum_{\nu=n_\epsilon}^{n} \binom{n}{\nu} q^{n-\nu} |I_{\nu 2}| = O_x((1+q)^n).$$

Hence, it follows from (5.3.2) and (5.3.4) that

$$\sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \int_a^b P_n(t,x)\, K_\nu(t,x)\rho(t)\,dt = O_x((1+q)^n)$$

is true for almost every $x \in E \cap [a+\epsilon,\ b-\epsilon]$ and in similar way we obtain that the estimate

$$\sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \int_a^b N_n(t,x) K_\nu(t,x)\rho(t)\,dt = O_x((1+q)^n)$$

holds for almost every $x \in E \cap [a+\epsilon,\ b-\epsilon]$ whence due to (5.3.1) we obtain that the estimate

$$F_n^{(q)}(x) = O_x(1)$$

holds almost everywhere in E.

This completes the proof of our theorem.

5.4 PROOF OF THEOREM 2 : For $x \in [c+\delta, d-\delta]$

$$\left|\left(\int_a^{x-\delta} + \int_{x+\delta}^b\right) f(t) E_n^{(q)}(t,x) \rho(t)\,dt\right|$$

$$= \left|\left(\int_a^{x-\delta} + \int_{x+\delta}^b\right) f(t) \cdot \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x) \rho(t)\,dt\right|$$

$$= \left|\frac{1}{(1+q)^n} \left(\int_a^{x-\delta} + \int_{x+\delta}^b\right) f(t) \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \cdot \phi_{\nu+i}(t) \phi_{\nu+j}(x) \rho(t)\,dt\right|$$

$$= \left|\frac{1}{(1+q)^n} \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \gamma_{i,j,k}^{(\nu)} \phi_{\nu+j}(x) \left(\int_a^{x-\delta} + \int_{x+\delta}^b\right) f(t) F_k(t,x) \cdot \phi_{\nu+i}(t) \rho(t)\,dt\right|.$$

Now, let us put

$$h_k(t,x) = \begin{cases} f(t) F_k(t,x), & t \in [a, x-\delta] \cup [x+\delta, b] \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\left|\left(\int_a^{x-\delta} + \int_{x+\delta}^b\right) f(t) E_n^{(q)}(t,x) \rho(t)\,dt\right| \leq$$

$$\leq \frac{1}{(1+q)^n} \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} |\gamma^{(\nu)}_{i,j,k}| \; |\phi_{\nu+j}(x)| \; \Big| \int_a^b h_k(t,x) \cdot$$

$$\cdot \phi_{\nu+i}(t) \varrho(t)dt \Big|$$

$$= O(1) \sum_{k=1}^{r} \sum_{i=-p}^{p} \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \Big| \int_a^b h_k(t,x) \phi_{\nu+i}(t) \; \varrho(t)dt \Big| .$$

Further by Cauchy's inequality

$$\Big| \Big(\int_a^{x-\delta} + \int_{x+\delta}^{b} \Big) f(t) \; E_n^{(q)} \; (t,x) \; \varrho \; (t)dt \Big|$$

$$= O(1) \sum_{k=1}^{r} \sum_{i=-p}^{p} \frac{1}{(1+q)^n} \Big[\sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \Big\{ \int_a^b h_k(t,x) \cdot$$

$$\cdot \phi_{\nu+i}(t) \; \varrho \; (t)dt \Big\}^2 \Big]^{\frac{1}{2}}$$

i.e.

$$(5.4.1) \qquad \Big| \Big(\int_a^{x-\delta} + \int_{x+\delta}^{b} \Big) f(t) E_n^{(q)}(t,x) \varrho(t)dt \Big| =$$

$$= O(1) \sum_{k=1}^{r} \sum_{i=-p}^{p} \Big[\frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \Big\{ \int_a^b h_k(t,x) \phi_{\nu+i}(t) \; \varrho \; (t)dt \Big\}^2 \Big]^{\frac{1}{2}}$$

The integrals on the right hand side of the above relation are the $(\nu+i)^{th}$ expansion coefficients $C_{\nu+i}(x)$ of the function $h_k(t,x)$. Also, the system $\{\phi_n(x)\}$ is complete and hence (with fixed i) according to Lemma 2.

$$(5.4.2) \qquad \sum_{\nu=-i}^{\infty} C^2_{\nu+i}(x) = \int_a^b h_k^2(t,x) \varrho(t)dt \; .$$

Now, we proceed to prove that the function

$$G_k(x) = \int_a^b h_k^2(t,x)\varsigma(t)dt$$

is continuous on $[c+\delta, d-\delta]$.

It is given that $f(t)$ is continuous on $[c,d]$.

Define

$$f(t)=o, \quad t\notin[c,d].$$

Since $F_k(t,x)$ is continuous in the square $a\leq t\leq b$, $a\leq x\leq b$ except for the diagonal points $t=x$, for each t and every $x\in[a,b]$, $x\neq t$, $F_k(t,x)$ is continuous as a function of x only. Hence given $\epsilon>o, \exists\ \delta_{1x}>o$, such that $o<|h|<\delta_{1x}<\delta<\frac{\epsilon}{4K}$ implies

$$\left|F_k^2(t,x+h)-F_k^2(t,x)\right| < \frac{\epsilon}{2M_1^2.(d-c)},$$

where M_1 denotes the bound for f. (Since the δ chosen above is arbitrary, we may take $\delta<\frac{\epsilon}{4K}$, where $K>o$ denote the bound for the functions $f^2(t)F_k^2(t,x)$ and $f^2(t)F_k^2(t,x+h)$ in the intervals $[x-\delta, x-\delta+h]$ and $[x+\delta, x+\delta+h]$ respectively. This is possible as the function $f(t)$ is continuous in $[c,d]$ and $F_k(t,x)$ are continuous in the intervals $[x-\delta, x-\delta+h]$ and $[x+\delta, x+\delta+h]$ as the functions of t.

Now, for $x\in[c+\delta, d-\delta]$ and $o<|h|<\delta_{1x}<\delta<\frac{\epsilon}{4K}$

$$\left|G_k(x+h)-G_k(x)\right| = \left|\int_a^b h_k^2(t,x+h)\varsigma(t)dt - \int_a^b h_k^2(t,x)\varsigma(t)dt\right|$$

$$= \left| \left(\int_a^{x+h-\delta} + \int_{x+h+\delta}^{b} \right) f^2(t) F_k^2(t,x+h)\, \varrho(t)\, dt - \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f^2(t) F_k^2(t,x)\, \varrho(t)\, dt \right| .$$

Let us put

$$E = [a, x-\delta+h] \cup [x+\delta, b] \cap [c,d] .$$

Then, the continuity of $F_k(t,x)$ is true for any $t \in E$ and all x and therefore

$$|G_k(x+h) - G_k(x)| =$$

$$= \left| \left(\int_E f^2(t) F_k^2(t,x+h) \varrho(t) dt - \int_{x+\delta}^{x+\delta+h} f^2(t) F_k^2(t,x+h) \varrho(t) dt \right) - \right.$$

$$\left. - \left(\int_E f^2(t) F_k^2(t,x) \varrho(t) dt - \int_{x-\delta}^{x-\delta+h} f^2(t) F_k^2(t,x) \varrho(t) dt \right) \right|$$

$$\leq \left| \int_E f^2(t) F_k^2(t,x+h) \varrho(t) dt - \int_E f^2(t) F_k^2(t,x) \varrho(t) dt \right| +$$

$$+ \left| \int_{x-\delta}^{x-\delta+h} f^2(t) F_k^2(t,x) \varrho(t) dt - \int_{x+\delta}^{x+\delta+h} f^2(t) F_k^2(t,x+h) \varrho(t) dt \right|$$

$$\leq \int_E f^2(t) \left| F_k^2(t,x+h) - F_k^2(t,x) \right| \varrho(t) dt + \int_{x-\delta}^{x-\delta+h} f^2(t) F_k^2(t,x) \varrho(t) dt +$$

$$+ \int_{x+\delta}^{x+\delta+h} f^2(t) F_k^2(t,x+h) \varrho(t) dt$$

$$< M_1^2 \frac{\epsilon}{2M_1^2(d-c)} |E| + 2K|h|$$

$$< \frac{\epsilon}{2} + 2K \frac{\epsilon}{4K} = \epsilon$$

Hence, it follows that $G_k(x)$ is uniformly continuous.

Now, we proceed to prove that $C_n(x)$ is also uniformly continuous in $[c+\delta, d-\delta]$. As noted above, since for each t and every $x \in [a,b]$, $x \neq t$ $F_k(t,x)$ is continuous as a function of x only for given $\epsilon > 0$, $\exists$ $\delta_{2x} > 0$, $0 < |h| < \delta_{2x} < \delta$ $\ni$

$$|F_k(t,x+h) - F_k(t,x)| < \frac{\epsilon}{M_1 \; |E|^{\frac{1}{2}}} .$$

Now, for $x \in [c+\delta, d-\delta]$ and $|h| < \delta_{2x}$, by Cauchy's inequality

$$|C_n(x+h) - C_n(x)| = \left| \int_a^b h_k(t,x+h)\phi_n(t)\, \rho(t)dt - \int_a^b h_k(t,x)\phi_n(t)\, \rho(t)dt \right|$$

$$\leq \left[\int_a^b \left\{ h_k(t,x+h) - h_k(t,x) \right\}^2 \rho(t)dt \int_a^b \phi_n^2(t)\rho(t)dt \right]^{\frac{1}{2}}$$

$$= \left[\int_E f^2(t)(F_k(t,x+h) - F_k(t,x))^2 \rho(t)dt \right]^{\frac{1}{2}}$$

$$< \left[M_1^2 \quad \frac{\epsilon^2}{M_1^2 \; |E|} \quad |E| \right]^{\frac{1}{2}}$$

$$= \epsilon$$

Hence, it follows that $C_n(x)$ is also uniformly continuous on $[c+\delta, d-\delta]$. Consequently, it follows from (5.4.2) and Lemma 4 that the series

$$\sum_{\nu=1}^{\infty} C_{\nu+i}^2 \; (x)$$

Converges uniformly and therefore, it follows that the sequence $\{C^2_{\nu+i}(x)\}$ converges uniformly to zero as $\nu \longrightarrow \infty$ and this implies that the n^{th} (E,q)-mean of $\{C^2_{\nu+i}(x)\}$ also converge uniformly to zero, i.e.

$$\frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \left\{ \int_a^b h_k(t,x)\phi_{\nu+i}(t) \; \varrho(t)dt \right\}^2 = o(1).$$

Thus, it follows from (5.4.1) that for $x \in [c+\delta, d-\delta]$

$$\left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f(t)E_n^{(q)}(t,x) \; \varrho(t)dt = o(1).$$

Since, this relation is true for any $L^2_{\varrho(t)}$-integrable function f, continuous in $[c,d]$, in particular taking $f(t)=1$, $t \in [a,b]$, we have

$$(5.4.3) \qquad \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) E_n^{(q)}(t,x) \; \varrho(t)dt = o(1).$$

Now

$$E_n^{(q)}(t,x) = \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x) =$$

$$= \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \sum_{m=0}^{\nu} \phi_m(t)\phi_m(x)$$

$$= \frac{1}{(1+q)^n} \sum_{m=0}^{n} \phi_m(t)\phi_m(x) \sum_{\nu=m}^{n} \binom{n}{\nu} q^{n-\nu}.$$

Hence, it follows from the constant-preserving property of the system $\{\phi_n(x)\}$ that

$$\int_a^b E_n^{(q)}(t,x)\varrho(t)dt = \frac{1}{(1+q)^n} \int_a^b \left\{ \sum_{m=0}^{n} \phi_m(t)\phi_m(x) \sum_{\nu=m}^{n} \binom{n}{\nu} q^{n-\nu} \right\} \varrho(t)dt =$$

$$= \frac{1}{(1+q)^n} \sum_{m=0}^{n} \frac{\phi_m(x)}{\phi_0(x)} \sum_{\nu=m}^{n} \binom{n}{\nu} q^{n-\nu} \int_a^b \phi_m(t)\phi_0(t)\varrho(t)dt$$

$$= \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu}$$

$$=1 .$$

Consequently, it follows from (5.4.3) that

$$(5.4.4) \qquad \int_{x-\delta}^{x+\delta} E_n^{(q)}(t,x)\varrho(t)dt = 1+o(1).$$

Thus, it follows from (5.4.3) and (5.4.4) that the relation (5.1.3) is uniformly satisfied in $[c+\delta, d-\delta]$ with $\psi_n(t,x)=E_n^{(q)}(t,x)$. Further, for $x \in [c+\delta, d-\delta]$ and $t \in [a, x-\delta] \cup [x+\delta, b]$

$$|E_n^{(q)}(t,x)| = \left| \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} K_\nu(t,x) \right|$$

$$= \left| \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \cdot \phi_{\nu+i}(t)\phi_{\nu+j}(x) \right|$$

$$\leq \frac{1}{(1+q)^n} \sum_{k=1}^{r} \sum_{i,j=-p}^{p} |F_k(t,x)| \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} |\gamma_{i,j,k}^{(\nu)}| \, |\phi_{\nu+i}(t)| \, |\phi_{\nu+j}(x)|$$

$$= O(1) \frac{1}{(1+q)^n} \sum_{k=1}^{r} \sum_{i,j=-p}^{p} |F_k(t,x)| \; (1+q)^n$$

$$=O(1)\sum_{k=1}^{r}|F_k(t,x)|$$

$$=O\left(\frac{1}{|t-x|}\right)$$

$$=O\left(\frac{1}{\delta}\right)$$

Thus

$$|E_n^{(q)}(t,x)| \leq \psi(\delta).$$

i.e. The relation (5.1.4) is uniformly satisfied in $[c+\delta, d-\delta]$ with $\psi_n(t,x)=E_n^{(q)}(t,x)$. In otherwords, we have proved that the means $\tau_n^{(q)}(x)$ of the expansion (5.1.6) are uniformly singular in the interval $[c+\delta, d-\delta]$. Also from Theorem 1, the validity of the relation

$$F_n^{(q)}(x)=O(1)$$

follows in every subinterval $[c+\delta, d-\delta]$ of (c,d). Consequently, it follows from Lemma 3 that

$$I_n(f,x) = \tau_n^{(q)}(x) = \frac{1}{(1+q)^n}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}s_\nu(x) =$$

$$= \frac{1}{(1+q)^n}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}\sum_{k=0}^{\nu}c_k\phi_k(x)$$

$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}c_k\phi_k(x)\sum_{\nu=k}^{n}\binom{n}{\nu}q^{n-\nu}$$

$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\left(\int_a^b f(t)\phi_k(t)\rho(t)dt\right)\phi_k(x)\sum_{\nu=k}^{n}\binom{n}{\nu}q^{n-\nu}$$

$$= \int_a^b f(t)\left\{\frac{1}{(1+q)^n}\sum_{k=o}^{n} \phi_k(t)\phi_k(x)\sum_{\nu=k}^{n}\binom{n}{\nu}q^{n-\nu}\right\}\varsigma(t)dt$$

$$= \int_a^b f(t)\left\{\frac{1}{(1+q)^n}\sum_{\nu=o}^{n}\binom{n}{\nu}q^{n-\nu}\sum_{k=o}^{\nu}\phi_k(t)\phi_k(x)\right\}\varsigma(t)dt$$

$$= \int_a^b f(t)\cdot\frac{1}{(1+q)^n}\sum_{\nu=o}^{n}\binom{n}{\nu}q^{n-\nu}K_\nu(t,x)\,\varsigma(t)dt.$$

$$= \int_a^b f(t)E_n^{(q)}(t,x)\varsigma(t)dt$$

converges to $f(x)$ uniformly in $[c+\delta, d-\delta]$.

With this the theorem is proved.

5.5 PROOF OF THEOREM 3 : We have

$$U_n^\alpha(t,x)=\sum_{k=o}^{n}\left(1-\frac{\lambda_k}{\lambda_{n+1}}\right)^\alpha\phi_k(t)\phi_k(x).$$

Effecting Abel's transformation, we obtain

$$U_n^\alpha(t,x)=\sum_{\nu=o}^{n}\left\{\left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha-\left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha\right\}K_\nu(t,x).$$

Let $P_n(t,x)$ and $N_n(t,x)$ denote the characteristic functions of the sets in which

$$\sum_{\nu=o}^{n}\left\{\left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha-\left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha\right\}K_\nu(t,x)\geq o \quad\text{and} < o$$

respectively.

From the definition of n^{th} Lebesgue (R, λ_n, α)-function, we have

$$(5.5.1) \quad V_n^{\alpha}(x) = \int_a^b |U_n^{\alpha}(t,x)| \ \varrho(t)dt$$

$$= \sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} \int_a^b P_n(t,x)K_\nu(t,x)\varrho(t)dt-$$

$$- \sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} \int_a^b N_n(t,x)K_\nu(t,x)\varrho(t)dt.$$

Our aim is to show that each of the sum on the R.H.S. of (5.5.1) have the order of magnitude $O_x(1)$ for every $x \in E \cap (a+\epsilon,\ b-\epsilon)$ with arbitrary $\epsilon > o$. We divide the integral

$$\int_a^b P_n(t,x)K_\nu(t,x)\varrho(t)dt$$

for $n \geq n_\epsilon > 1/\epsilon$ into two parts :

$$I_{\nu 1} = \int_{x-n^{-2}}^{x+n^{-2}}, \quad I_{\nu 2} = \int_a^{x-n^{-2}} + \int_{x+n^{-2}}^{b}.$$

We first estimate $|I_{\nu 1}|$.

Effecting Schwarz's inequality we have

$$I_{\nu 1}^2 \leq \int_{x-n^{-2}}^{x+n^{-2}} P_n^2(t,x)\varrho(t)dt \int_{x-n^{-2}}^{x+n^{-2}} K_\nu^2(t,x)\varrho(t)dt.$$

We infer from the condition (5.1.7) and $P_n^2(t,x) \leq 1$ that

$$I_{\nu 1}^2 \leq \int_{x-n^{-2}}^{x+n^{-2}} \rho(t)dt \sum_{k=0}^{\nu} \phi_k^2(x)$$

$$= O_x(\nu n^{-2})$$

Consequently, by Cauchy's inequality we have

$$\sum_{\nu=n_\epsilon}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}|I_{\nu 1}|$$

$$\leq \left[\sum_{\nu=n_\epsilon}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}^2 \sum_{\nu=n_\epsilon}^{n} I_{\nu 1}^2\right]^{\frac{1}{2}}$$

$$= \left[\sum_{\nu=n_\epsilon}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}^2 \sum_{\nu=n_\epsilon}^{n} O_x(\nu n^{-2})\right]^{\frac{1}{2}}$$

$$= O_x(n^{-1})\left\{n(n-n_\epsilon+1)\right\}^{\frac{1}{2}}\left[\sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}^2\right]^{\frac{1}{2}}$$

$$= O_x(1)\left[\left[\sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\right]^2\right]^{\frac{1}{2}}$$

$$= O_x(1)\sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}$$

Thus

$$(5.5.2) \quad \sum_{\nu=n_\epsilon}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}|I_{\nu 1}| = O_x(1).$$

Now, we proceed to estimate

$$\sum_{\nu=n_\epsilon}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}|I_{\nu 2}| .$$

Let us put

$$g_k(t,x)=\begin{cases} P_n(t,x)F_k(t,x) & \text{for } t\in [a,x-n^{-2}]\cup[x+n^{-2},b] \\ 0 & \text{otherwise.}\end{cases}$$

Now

$$\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\}|I_{\nu 2}| =$$

$$=\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\}\left|\left(\int_a^{x-n^{-2}}+\int_{x+n^{-2}}^{b}\right)P_n(t,x)K_\nu(t,x)\rho(t)dt\right|.$$

Since, the system $\{\phi_n(x)\}$ is polynomial-like and therefore, by definition (5.1.2) of the kernel $K_n(t,x)$ we have

$$\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\}|I_{\nu 2}|=$$

$$=\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\}\left|\left(\int_a^{x-n^{-2}}+\int_{x+n^{-2}}^{b}\right)P_n(t,x)\sum_{k=1}^{r}F_k(t,x)\cdot\right.$$

$$\left.\cdot\sum_{i,j=-p}^{p}\gamma_{i,j,k}^{(\nu)}\phi_{\nu+i}(t)\phi_{\nu+j}(x)\,\rho(t)dt\right|$$

$$\leq \sum_{k=1}^{r}\sum_{i,j=-p}^{p}\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\}|\gamma_{i,j,k}^{(\nu)}|\,|\phi_{\nu+j}(x)|\cdot$$

$$\cdot\left|\left(\int_a^{x-n^{-2}}+\int_{x+n^{-2}}^{b}\right)P_n(t,x)F_k(t,x)\phi_{\nu+i}(t)\rho(t)dt\right|.$$

Using the definition of the function $g_k(t,x)$ and the condition (5.1.7), we obtain

$$\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}|I_{\nu 2}| =$$

$$= O_x(1)\sum_{k=1}^{r}\sum_{i=-p}^{p}\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\left|\int_a^b g_k(t,x)\phi_{\nu+i}(t)\cdot \rho(t)dt\right|.$$

Applying Cauchy's inequality, we get

$$\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}|I_{\nu 2}|$$

$$= O_x(1)\sum_{k=1}^{r}\sum_{i=-p}^{p}\left[\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\cdot \sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\left\{\int_a^b g_k(t,x)\phi_{\nu+i}(t)\rho(t)dt\right\}^2\right]^{\frac{1}{2}}$$

$$= O_x(1)\sum_{k=1}^{r}\sum_{i=-p}^{p}\left[\sum_{\nu=o}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\cdot \sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\left\{\int_a^b g_k(t,x)\phi_{\nu+i}(t)\rho(t)dt\right\}^2\right]^{\frac{1}{2}}.$$

i.e.

$$(5.5.3)\quad \sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}|I_{\nu 2}| =$$

$$= O_x(1)\sum_{k=1}^{r}\sum_{i=-p}^{p}\left[\sum_{\nu=n_\epsilon}^{n}\left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\cdot \left\{\int_a^b g_k(t,x)\phi_{\nu+i}(t)\rho(t)dt\right\}^2\right]^{\frac{1}{2}}$$

Now, $F_k(t,x) = O(|t-x|^{-1})$ and $|t-x| \geq n^{-2}$ imply that

$$|g_k(t,x)| \leq P_n(t,x)\ |F_k(t,x)| = O(n^2).$$

i.e. $g_k(t,x)$ is bounded for a fixed n and hence it is integrable. Therefore, it means that the integrals on the R.H.S. of the above relation are the expansion coefficients of $L^2_{\varsigma(t)}$-integrable function (with fixed i) each of which tends to zero as $\nu \longrightarrow \infty$ due to Lemma 1.

Now, let us choose n large enough and fix it.

Since

$$\int_a^b g_k(t,x)\phi_{\nu+i}(t)\ \varsigma(t)dt \longrightarrow 0 \text{ as } \nu \longrightarrow \infty$$

for given $\epsilon > 0$, $\exists\ n_0 > 0$ such that $\nu \geq n_0$ implies

$$\left|\int_a^b g_k(t,x)\phi_{\nu+i}(t)\varsigma(t)dt\right| < \epsilon .$$

Let $M_2 = \max\left\{ \left|\int_a^b g_k(t,x)\phi_{\nu+i}(t)\varsigma(t)dt\right| ,\ \text{where } \nu = 0,1,2.\ldots \ldots n_0-1, \epsilon\right\}.$

Then

$$\sum_{\nu=n_\epsilon}^{n} \left\{\left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha - \left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha\right\}\left\{\int_a^b g_k(t,x)\phi_{\nu+i}(t)\varsigma(t)dt\right\}^2$$

$$\leq M_2^2 \sum_{\nu=0}^{n} \left\{\left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha - \left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha\right\}$$

$$= M_2^2 .$$

Hence, it follows from (5.5.3) that

$$(5.5.4)\ \sum_{\nu=n_\epsilon}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} |I_{\nu 2}| = O_x(1).$$

Hence, it follows from (5.5.2) and (5.5.4) that

$$\sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} \int_a^b P_n(t,x)K_\nu(t,x)\varrho(t)dt = O_x(1)$$

is true for almost every $x \in E \cap [a+\epsilon, b-\epsilon]$ and in similar way, we obtain that the estimate

$$\sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} \int_a^b N_n(t,x)K_\nu(t,x)\varrho(t)dt = O_x(1)$$

holds for almost every $x \in E \cap [a+\epsilon, b-\epsilon]$, whence (5.5.1) implies that the estimate

$$V_n^{\alpha}(x) = O_x(1)$$

is valid almost everywhere in E.

This completes the proof of the theorem.

5.6 <u>PROOF OF THEOREM 4</u> :- For $x \in [c+\delta, d-\delta]$

$$\left| \left(\int_a^{x-\delta} + \int_{x+\delta}^{b}\right) f(t) U_n^{\alpha}(t,x)\varrho(t)dt \right|$$

$$= \left| \left(\int_a^{x-\delta} + \int_{x+\delta}^{b}\right) f(t) \sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} K_\nu(t,x)\varrho(t)dt \right|$$

$$= \left| \left(\int_a^{x-\delta} + \int_{x+\delta}^{b}\right) f(t) \left[\sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} \sum_{k=1}^{r} F_k(t,x) \cdot \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(\nu)} \phi_{\nu+i}(t)\phi_{\nu+j}(x)\right] \varrho(t)dt \right|$$

$$= \left| \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=o}^{n} \left\{ \left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha - \left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha \right\} \gamma^{(\nu)}_{i,j,k} \, \phi_{\nu+j}(x) \cdot \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f(t) F_k(t,x) \phi_{\nu+i}(t) \rho(t)\, dt \right|.$$

Now, applying Cauchy's inequality we obtain

$$\left| \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f(t) U_n^\alpha(t,x) \rho(t)\, dt \right| =$$

$$= O(1) \sum_{k=1}^{r} \sum_{i=-p}^{p} \left[\sum_{\nu=o}^{n} \left\{ \left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha - \left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha \right\} \cdot \sum_{\nu=o}^{n} \left\{ \left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha - \left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha \right\} \left\{ \int_a^b h_k(t,x) \phi_{\nu+i}(t) \rho(t)\, dt \right\}^2 \right]^{\frac{1}{2}}$$

where

$$h_k(t,x) = \begin{cases} f(t) F_k(t,x), & t \in [a, x-\delta] \cup [x+\delta, b] \\ o & \text{otherwise.} \end{cases}$$

i.e.

$$(5.6.1) \quad \left| \left(\int_a^{x-\delta} + \int_{x+\delta}^{b} \right) f(t) U_n^\alpha(t,x) \rho(t)\, dt \right| =$$

$$= O(1) \sum_{k=1}^{r} \sum_{i=-p}^{p} \left[\sum_{\nu=o}^{n} \left\{ \left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)^\alpha - \left(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}}\right)^\alpha \right\} \cdot \left\{ \int_a^b h_k(t,x) \phi_{\nu+i}(t) \rho(t)\, dt \right\}^2 \right]^{\frac{1}{2}}.$$

The integrals on the R.H.S. of the above relation are the $(\nu+i)^{th}$ expansion coefficients $C_{\nu+i}(x)$ of the function $h_k(t,x)$.

Also, the system $\{\phi_n(x)\}$ is complete and hence (with fixed i) according to Lemma 2

$$(5.6.2)\quad \sum_{\nu=-i}^{\infty} C^2_{\nu+i}(x) = \int_a^b h_k^2(t,x)\varrho(t)dt.$$

The continuity of the functions

$$G_k(x)=\int_a^b h_k^2(t,x)\varrho(t)dt$$

and $C_n(x)$ in the interval $[c+\delta, d-\delta]$ follows on the same line as proved in Theorem 2. Consequently, it follows from (5.6.2) and Lemma 4 that the series

$$\sum_{\nu=-i}^{\infty} C^2_{\nu+i}(x)$$

converges uniformly and therefore, it follows that the sequence $\{C^2_{\nu+i}(x)\}$ converges uniformly to zero as $\nu \longrightarrow \infty$ and this implies that the $n^{th}(R, \lambda_n, \alpha)$-mean of $\{C^2_{\nu+i}(x)\}$ also converge uniformly to zero, i.e.,

$$\sum_{\nu=0}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha\right\}\left\{\int_a^b h_k(t,x)\phi_{\nu+i}(t)\,\varrho(t)dt\right\}^2 =o(1).$$

Thus, it follows from (5.6.1) that for $x\in[c+\delta, d-\delta]$

$$\left(\int_a^{x-\delta} + \int_{x+\delta}^{b}\right)f(t)U_n^\alpha(t,x)\varrho(t)dt=o(1).$$

Since, this relation is true for any $L^2_{\varrho(t)}$-integrable function f continuous in $[c,d]$, in particular taking $f(t)=1$, $t\in[a,b]$, we have

$$(5.6.3)\qquad \left(\int_a^{x-\delta} + \int_{x+\delta}^{b}\right)U_n^\alpha(t,x)\varrho(t)dt=o(1).$$

Now $$U_n^{\alpha}(t,x) = \sum_{k=o}^{n} (1-\frac{\lambda_k}{\lambda_{n+1}})^{\alpha} \phi_k(t)\phi_k(x)$$

and therefore, it follows from the constant-preserving property of the system $\{\phi_n(x)\}$ that

$$\int_a^b U_n^{\alpha}(t,x)\varsigma(t)dt = \int_a^b \sum_{k=o}^{n} (1-\frac{\lambda_k}{\lambda_{n+1}})^{\alpha} \phi_k(t)\phi_k(x)\varsigma(t)dt$$

$$= \sum_{k=o}^{n} (1-\frac{\lambda_k}{\lambda_{n+1}})^{\alpha} \phi_k(x) \int_a^b \phi_k(t)\varsigma(t)dt$$

$$= \sum_{k=o}^{n} (1-\frac{\lambda_k}{\lambda_{n+1}})^{\alpha} \frac{\phi_k(x)}{\phi_o(x)} \int_a^b \phi_k(t)\phi_o(t)\,\varsigma(t)dt$$

$$=(1-\frac{\lambda_o}{\lambda_{n+1}})^{\alpha}$$

$$= 1.$$

Consequently, it follows from (5.6.3) that

$$(5.6.4) \qquad \int_{x-\delta}^{x+\delta} U_n^{\alpha}(t,x)\varsigma(t)dt = 1+o(1).$$

Thus, it follows from (5.6.3) and (5.6.4) that the relation (5.1.3) is uniformly satisfied in $[c+\delta, d-\delta]$ with $\Psi_n(t,x)=U_n^{\alpha}(t,x)$. Further, for $x \in [c+\delta, d-\delta]$ and $t \in [a, x-\delta] \cup [x+\delta, b]$

$$|U_n^{\alpha}(t,x)| = \left|\sum_{\nu=o}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} K_\nu(t,x)\right|$$

$$= \left|\sum_{\nu=o}^{n} \left\{(1-\frac{\lambda_\nu}{\lambda_{n+1}})^{\alpha}-(1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^{\alpha}\right\} \sum_{k=1}^{r} F_k(t,x) \sum_{i,j=-p}^{p} \cdot\right.$$

$$\left. \cdot\, \gamma_{i,j,k}^{(\nu)}\, \phi_{\nu+i}(t)\phi_{\nu+j}(x)\right|$$

$$= \left| \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=o}^{n} \left\{ (1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha - (1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha \right\} \gamma_{i,j,k}^{(\nu)} F_k(t,x) \cdot \cdot \emptyset_{\nu+j}(x)\emptyset_{\nu+i}(t) \right|$$

$$\leqslant \sum_{k=1}^{r} \sum_{i,j=-p}^{p} \sum_{\nu=o}^{n} \left\{ (1-\frac{\lambda_\nu}{\lambda_{n+1}})^\alpha - (1-\frac{\lambda_{\nu+1}}{\lambda_{n+1}})^\alpha \right\} |\gamma_{i,j,k}^{(\nu)}| \, |F_k(t,x)| \cdot \cdot |\emptyset_{\nu+j}(x)| \; |\emptyset_{\nu+i}(t)|$$

$$= O(1) \sum_{k=1}^{r} |F_k(t,x)|$$

$$= O\left(\frac{1}{|t-x|} \right)$$

$$= O\left(\frac{1}{\delta} \right)$$

Thus

$$|U_n^\alpha(t,x)| \leqslant \psi(\delta).$$

i.e. The relation (5.1.4) is uniformly satisfied in $[c+\delta, d-\delta]$ with $\gamma_n(t,x)=U_n^\alpha(t,x)$. In otherwords, we have proved that the means $\sigma_n^\alpha(\lambda,x)$ of the expansion (5.1.6) are uniformly singular in the interval $[c+\delta, d-\delta]$. Also from Theorem 3, the validity of the relation

$$V_n^\alpha(x)=O(1)$$

follows in every subinterval $[c+\delta, d-\delta]$ of $[c,d]$. Consequently, it follows from Lemma 3 that

$$I_n(f,x)=\sigma_n^\alpha(\lambda,x)= \sum_{k=o}^{n} (1-\frac{\lambda_k}{\lambda_{n+1}})^\alpha C_k\emptyset_k(x)=$$

$$= \sum_{k=o}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)^{\alpha} \int_a^b f(t)\phi_k(t)\phi_k(x)\varrho(t)dt$$

$$= \int_a^b f(t)\left\{ \sum_{k=o}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)^{\alpha} \phi_k(t)\phi_k(x)\right\} \varrho(t)dt$$

$$= \int_a^b f(t)U_n^{\alpha}(t,x)\varrho(t)dt.$$

converges to $f(x)$ uniformly in $[c+\delta, d-\delta]$.

With this the theorem is proved.