CHAPTER - 6

ABSOLUTE SUMMABILITY OF ORTHOGONAL SERIES

Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be an orthonormal system (ONS) of L^2 - integrable functions defined in the closed interval $[a, b]$. We consider the orthogonal series

$$(6.1.1) \qquad \sum_{n=0}^{\infty} c_n \phi_n(x)$$

with real coefficients c_n's.

The (N, p_n) means and $(\bar{N}, p_n)$ means of the orthogonal series (6.1.1) are given by,

$$t_n(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \, S_k(x)$$

and

$$\bar{T}_n(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_k(x),$$

respectively.

The series (6.1.1) is said to be (N, p_n) and $(\bar{N}, p_n)$ summable to $S(x)$ if

$$\lim_{n \to \infty} t_n(x) = S(x)$$

and

$$\lim_{n \to \infty} \bar{T}_n(x) = S(x), \text{ respectively.}$$

Let $\lambda = \{\lambda_n\}$ be a monotonic non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leqslant 1$, and $\lambda_1 = 1$. The transformation

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{V=n-\lambda_n+1}^{n} S_V$$

defines the sequence $\{V_n(\lambda)\}$ of generalized [1] de-la Valle'e Poussin means of (6.1.1) generated by λ.

The series (6.1.1) is said to be absolute Nörlund summable, absolute $(\bar{N}, p_n)$ summable and $|V, \lambda|$ summable if,

$$\sum_{n=1}^{\infty} | t_n - t_{n-1} |,$$

$$\sum_{n=1}^{\infty} | \bar{T}_n - \bar{T}_{n-1} |$$

and

$$\sum_{n=1}^{\infty} | V_{n+1}(\lambda) - V_n(\lambda) |$$

are convergent respectively.

Billiard[2] and Tandori[3] have studied the $|(C, 1)|$

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1) Sharma P.L. and Jain R.K. [111)
2) Billiard P. [20]
3) Tandori K. ([131], [132])

summability of (6.1.1). Tandori[1] has proved the following theorem.

<u>Theorem A</u> : The condition

$$(6.1.2) \qquad \sum_{m=o}^{\infty} \left\{ \sum_{k=2^m+1}^{2^{m+1}} C_k^{\,2} \right\}^{\frac{1}{2}} < \infty$$

is the necessary and sufficient condition for the series (6.1.1) to be $|(C, 1)|$ - summable almost everywhere.

If for some sequence $\{p_n\}$ the conditions

(1) $\quad 0 < p_n < p_{n+1} \quad$ for $\quad n = o, 1, 2, \ldots.$

or $\quad 0 < p_{n+1} < p_n \quad$ for $\quad n = o, 1, 2, \ldots.$

(2) $\quad p_o + p_1 + \ldots.+ p_n = P_n \uparrow \infty$

(3) $\quad \lim\limits_{n \to \infty} \dfrac{n \, \Delta \, p_{n-1}}{P_n} = 1 - \alpha, \quad$ where $\quad \alpha \geqslant o,$

$\Delta p_{n-1} = p_{n-1} - p_n$ are satisfied, then we shall say that the sequence $\{p_n\}$ belongs to the class $\overline{M}^{\alpha}$ [2].

In the special case when $p_n = \dfrac{1}{n+1}$, $P_n \sim \log n$, and the Nörlund means t_n reduces to harmonic means[3].

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1) Tandori K. ([131], [132])
2) Meder J. [80]
3) Bhatt S.N. [16]

In the same direction the same result was generalized by different authors for different summabilties. Tandori K.[1] and P. Srivastava[2] have proved the result for absolute Riesz summability. Meder J.[3] has proved the following theorem for absolute Nörlund summability:

Theorem B :-

Let $\{p_n\} \in \bar{M}^{\alpha}$, $\alpha > \frac{1}{2}$, then

$$\sum_{m=o}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} a_k^2 < \infty,$$

is the necessary and sufficient condition for the series (6.1.1) to be $|N, p_n|$ summable in interval [o, 1].

In this chapter we extend the above result for $|\bar{N}, p_n|$ and $|V, \lambda|$ summability. We also generalize the Theorem B of Meder. Our theorems are as follows.

Theorem 1 :-

If $np_n = O(P_n)$, then

$$\sum_{m=o}^{\infty} \left\{ \sum_{v=o}^{m} C_v^2 \right\}^{\frac{1}{2}} < \infty$$

implies the $|N, p_n|$ summability of (6.1.1).

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1) Tandori K. ([131], [132])
2) P.Srivastava [113]
3) Meder J. [77]

Theorem 2 :-

If $np_n = O(P_n)$ then

$$\sum_{n=o}^{\infty} \left\{ \sum_{v=o}^{n} C_v^2 \right\}^{\frac{1}{2}} < \infty$$

implies the $|\bar{N}, p_n|$ summability of (6.1.1).

Theorem 3 :-

Let $\lambda = \{\lambda_n\}$ be a monotonic nondecreasing sequence of natural numbers with

$\lambda_{n+1} - \lambda_n \leqslant 1$ and $\lambda_1 = 1$, then

$$\sum_{n=1}^{\infty} \left\{ \sum_{k=n-\lambda_n+2}^{n+1} C_k^2 \right\}^{\frac{1}{2}} < \infty$$

implies the $|V, \lambda|$ summability of orthogonal series (6.1.1).

In order to prove above theorems, we need the following Lemma :

Lemma[1] :- If $\{p_n\} \in M^{\alpha}$, $\alpha > \frac{1}{2}$, then

$$\lim_{n \to \infty} \frac{n}{P_n^2} \sum_{k=o}^{n} \frac{P_k^2}{(k+1)^2} = \frac{1}{2\alpha - 1} .$$

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1) Meder J. [78]

Proof of Theorem 1 :-

We have,

$$t_n(x) - t_{n-1}(x) = \frac{1}{P_n} \sum_{k=o}^{n} p_{n-k} \, S_k(x) - \frac{1}{P_{n-1}} \sum_{k=o}^{n} p_{n-k-1} \, S_k(x)$$

$$= \frac{1}{P_n} \sum_{r=o}^{n} C_r \phi_r(x) \sum_{k=r}^{n} p_{n-k} - \frac{1}{P_{n-1}} \sum_{r=o}^{n} C_r \phi_r(x) \sum_{k=r}^{n} p_{n-k-1}$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} P_{n-k} \, C_k \phi_k(x) - \frac{1}{P_{n-1}} \sum_{k=o}^{n} P_{n-k-1} \, C_k \phi_k(x)$$

$$= \frac{1}{P_n P_{n-1}} \sum_{k=o}^{n} (p_{n-k} \, P_n - p_n P_{n-k}) \, C_k \phi_k(x)$$

$$= \frac{1}{P_{n-1}} \sum_{k=o}^{n} p_{n-k} C_k \phi_k(x) - \frac{n}{P_n n P_{n-1}} \sum_{k=o}^{n} p_n P_{n-k} \, C_k \phi_k(x)$$

$$= O(1) \left\{ \frac{1}{P_{n-1}} \sum_{k=o}^{n} p_{n-k} \, C_k \phi_k(x) - \frac{1}{nP_{n-1}} \sum_{k=o}^{n} p_{n-k} \, C_k \phi_k(x) \right\}.$$

$$= O(1)\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n} p_v C_{n-v}\,\phi_{n-v}(x) - \frac{1}{nP_{n-1}}\right.$$

$$\left.\sum_{v=o}^{n} p_v C_{n-v}\phi_{n-v}(x) + p_o C_n \phi_n(x)\right\}$$

$$= O(1)\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n}\frac{P_v}{v}\, C_{n-v}\,\phi_{n-v}(x) - \frac{1}{nP_{n-1}}\right.$$

$$\left.\sum_{v=o}^{n} p_v C_{n-v}\,\phi_{n-v}(x) + p_o C_n \phi_n(x)\right\}$$

$$= O(1)\left\{\frac{P_n}{P_{n-1}}\sum_{v=1}^{n} C_{n-v}\,\phi_{n-v}(x) - \frac{P_n}{nP_{n-1}}\sum_{v=o}^{n}\right.$$

$$\left. C_{n-v}\,\phi_{n-v}(x) + p_o C_n \phi_n(x)\right\}$$

$$= O(1)\left\{\frac{P_n}{P_{n-1}}\sum_{v=1}^{n} C_{n-v}\,\phi_{n-v}(x) - \frac{P_n}{P_{n-1}}\sum_{v=o}^{n}\right.$$

$$\left. C_{n-v}\,\phi_{n-v}(x) + p_o C_n \phi_n(x)\right\}$$

Therefore,

$$|t_n - t_{n-1}| = O(1)\left\{\left|\frac{P_n}{P_{n-1}}\sum_{v=1}^{n} C_{n-v}\phi_{n-v}(x) - \frac{P_n}{P_{n-1}}\sum_{v=1}^{n}\right.\right.$$

$$\left.\left. C_{n-v}\phi_{n-v}(x) + p_o C_n \phi_n(x)\right|\right\}$$

$$= O(1)\left\{\left|\frac{P_n}{P_{n-1}}\sum_{v=1}^{n} C_{n-v}\phi_{n-v}(x)\right| + \left|\frac{P_n}{P_{n-1}}\sum_{v=o}^{n}\right.\right.$$

$$\left.\left. C_{n-v}\phi_{n-v}(x)\right| + |p_o C_n \phi_n(x)|\right\}$$

$$= O(1) \left\{ \left| \frac{P_n}{P_{n-1}} \sum_{v=1}^{n} C_{n-v} \phi_{n-v}(x) \right| + |p_o C_n \phi_n(x)| \right\}$$

Therefore, by Schwar'z inequality,

$$\sum_{n=1}^{\infty} \int_a^b |t_n(x) - t_{n-1}(x)| \, dx = O(1) \sum_{n=1}^{\infty} \left\{ \int_a^b (t_n(x) - t_{n-1}(x))^2 dx \right\}^{\frac{1}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \left\{ \frac{P_n^2}{P_{n-1}^2} \sum_{v=o}^{n} C_v^2 \right\}^{\frac{1}{2}}$$

$$+ \sum_{n=1}^{\infty} \left\{ C_n^2 \right\}^{\frac{1}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \left\{ \sum_{v=o}^{n} C_v^2 \right\}^{\frac{1}{2}}$$

$$< \infty .$$

Hence, by B. Levy's theorem, we have

$$\sum_{n=1}^{\infty} | t_n(x) - t_{n-1}(x) | < \infty .$$

Hence the proof.

<u>Remark</u> :- Under the same condition as of Theorem 1, the given series is absolutely harmonic summable.

<u>Proof of Theorem 2</u> :-

$$| \bar{T}_n(x) - \bar{T}_{n-1}(x)| = \frac{1}{P_n} \sum_{k=o}^{n} p_k S_k(x) - \frac{1}{P_{n-1}} \sum_{k=o}^{n-1} p_k S_k(x)$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} p_k S_k(x) - \frac{1}{P_{n-1}} \sum_{k=o}^{n-1} p_k S_k(x)$$

$$+ \frac{p_n}{P_n} S_n(x)$$

$$= \left(\frac{1}{P_n} - \frac{1}{P_{n-1}} \right) \sum_{k=o}^{n-1} p_k S_k(x) + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{-p_n}{P_n P_{n-1}} \sum_{k=o}^{n-1} p_k \sum_{v=o}^{k} C_v \emptyset_v(x) + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{-p_n}{P_n P_{n-1}} \sum_{k=o}^{n-1} p_k \sum_{v=o}^{k} C_v \emptyset_v(x) + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{-p_n}{P_n P_{n-1}} \sum_{v=o}^{n-1} C_v \emptyset_v(x) \sum_{k=o}^{n-1} p_k + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{-p_n}{P_n P_{n-1}} \sum_{v=o}^{n-1} C_v \emptyset_v(x) \sum_{k=o}^{n-1} p_k + \frac{p_n}{P_n P_{n-1}}$$

$$\sum_{v=o}^{n-1} C_v \emptyset_v(x) \sum_{k=o}^{v-1} p_k + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{p_n}{P_n} (S_n(x) - S_{n-1}(x)) + \frac{p_n}{P_n P_{n-1}}$$

$$\sum_{v=o}^{n-1} C_v \emptyset_v(x) P_{v-1}$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=o}^{n} C_v \emptyset_v(x) P_{v-1} .$$

By Schwarz inequality,

$$\sum_{n=1}^{\infty} \int_a^b |\bar{T}_n(x) - \bar{T}_{n-1}(x)| \, dx = O(1) \sum_{n=1}^{\infty} \left\{ \int_a^b (\bar{T}_n(x) - \bar{T}_{n-1}(x))^2 \, dx \right\}^{\frac{1}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \left\{ \frac{p_n^2}{P_n^2 \, P_{n-1}^2} \sum_{v=o}^{n} C_v^2 \left(\sum_{i=o}^{n} p_i \right)^2 \right\}^{\frac{1}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \left\{ \frac{p_n^2}{P_n^2 \, P_{n-1}^2} \sum_{v=o}^{n} C_v^2 \left(\frac{v}{P_v^2} \sum_{i=o}^{v} p_i^2 \right) \frac{P_v^2}{v} \right\}^{\frac{1}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \left\{ \frac{1}{P_n^2 P_{n-1}^2} \sum_{v=o}^{n} C_v^2 P_v^2 \right\}^{\frac{1}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sum_{v=o}^{n} C_v^2 \right\}^{\frac{1}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \left\{ \sum_{v=o}^{n} C_v^2 \right\}^{\frac{1}{2}}$$

$$< \infty .$$

Hence the proof.

Proof of Theorem 3 :-

We have,

$$|v_{n+1}(\lambda) - v_n(\lambda)| = \frac{1}{\lambda_n \lambda_{n+1}} \left\{ \sum_{k=n-\lambda_n+2}^{n+1} \left\{ (\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n \right\} C_k \phi_k(x) \right\}$$

$$= \left| \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right) \sum_{k=n-\lambda_n+2}^{n+1} (k-n-1) C_k \phi_k(x) + \frac{1}{\lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} C_k \phi_k(x) \right|$$

Using Schwarz inequality and by B.Levy's theorem, We have,

$$\sum_{n=1}^{\infty} \int_a^b |v_{n+1}(\lambda) - v_n(\lambda)| \, dx = O(1) \sum_{n=1}^{\infty} \left[\left\{ \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right)^2 \sum_{k=n-\lambda_n+2}^{n+1} (k-n-1)^2 C_k^2 + \frac{1}{\lambda_{n+1}^2} \sum_{k=n-\lambda_n+2}^{n+1} C_k^2 \right\}^{\frac{1}{2}} \right.$$

$$= O(1) \sum_{n=1}^{\infty} \left\{ \frac{1}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n+1} (k-n-1)^2 C_k^2 + \frac{1}{\lambda_{n+1}^2} \sum_{k=n-\lambda_n+2}^{n+1} (k-n-1)^2 C_k^2 + \frac{1}{\lambda_{n+1}^2} \sum_{k=n-\lambda_n+2}^{n+1} C_k^2 \right\}^{\frac{1}{2}}$$

$$= O(1)\left\{\sum_{n=1}^{\infty}\left\{\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n+1}(k-n-1)^2 C_k^2\right.\right.$$

$$+\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n+1}(k-n-1)^2 C_k^2$$

$$\left.\left.+\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n+1}(k-n-1)^2 C_k^2\right\}^{\frac{1}{2}}\right\}$$

$$= O(1)\sum_{n=1}^{\infty}\left\{\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n+1}(k-n-1)^2 C_k^2\right\}^{\frac{1}{2}}$$

$$= O(1)\sum_{n=1}^{\infty}\left\{\frac{1}{\lambda_n^2}\lambda_n^2\sum_{k=n-\lambda_n+2}^{n+1}C_k^2\right\}^{\frac{1}{2}}$$

$$= O(1)\sum_{n=1}^{\infty}\left\{\sum_{k=n-\lambda_n+2}^{n+1}C_k^2\right\}^{\frac{1}{2}}$$

$$< \infty .$$

Hence the proof.