CHAPTER - 7

CONVERGENCE AND SUMMABILITY OF ORTHOGONAL POLYNOMIAL SERIES

Let $p_o(x)$, $p_1(x)$, $p_2(x)$, ..., $p_n(x)$ be the system of normalized orthogonal polynomials in the interval $(-1, 1)$, corresponding to a positive bounded and summable weight function $w(x)$. The Fourier expansion corresponding to a function $f(x) \in L^2[-1, 1]$ in this system is given by

$$(7.1.1) \qquad f(x) \sim \sum_{n=o}^{\infty} C_n p_n(x).$$

Let

$$S_n(x) = \sum_{v=o}^{n} C_v p_v(x)$$

$$t_n(x) = \frac{1}{P_n} \sum_{v=o}^{n} p_{n-v} S_v(x)$$

$$\sigma_n^{\alpha}(x) = \frac{1}{A_n^{\alpha}} \sum_{v=o}^{n} A_{n-v}^{\alpha-1} S_v(x)$$

be sequence of partial sums, (N, p_n) means and (C, α) means respectively, of the series (7.1.1).

Let $\{p_n(x)\}$ be a system of orthonormal polynomial system belonging to the distribution $d\mu(x)$. Let $\mathfrak{K}$ denote a class of functions and

$$S_n(x) = \sum_{k=o}^{n} a_{nk} p_k(x)$$

be a linear combination of $p_o(x), p_1(x), \ldots\ldots\ldots, p_n(x)$, with a_{nk}. Let $d_n^{(2)}(f)$ denote the lower bound of all the numbers,

$$\left[\int_a^b [f(x) - S_n(x)]^2 \, d\mu(x) \right]^{\frac{1}{2}},$$

formed with arbitrary $S_n(x)$.

Then,

$$p_n(\mathfrak{K}, 2) = \sup_{f \in \mathfrak{K}} d_n^{(2)}(f)$$

is called the best degree of approximation for the class $\mathfrak{K}$ in the sense of L^2 - approximation.

If for some sequence $\{p_n\}$ the condition

(i) $0 < p_n < p_{n+1}$ for $n = o, 1, 2, \ldots\ldots$

or

$0 < p_{n+1} < p_n$ for $n = o, 1, 2, \ldots\ldots$

(ii) $p_o + p_1 + \dots\dots\dots + p_n = P_n \uparrow \infty$

(iii) $$\lim_{n \to \infty} \frac{n \, \Delta p_n}{P_n} = \alpha$$

where $\alpha \geqslant 0$,

are satisfied, then we shall say that the sequence $\{p_n\}$ belongs to the class M^{α} 1).

Let $w(f, \delta, a, b)$ denote the continuity modulus of the function $f(x)$ in the interval $[a, b]$ i.e.

$$w(f, \delta, a, b) = \sup_{\substack{|t - x| \leqslant \delta \\ t, x \in [a, b]}} |f(t) - f(x)|.$$

$w(f, \delta, a, b)$ can be represent as $w(f, \delta)$. We wenote by $w(\delta)$ a majorant function of $w(f, \delta, a, b)$ i.e. a function satisfying the condition

$$w(\delta) \geqslant w(f, \delta, a, b).$$

Cesàro, Riesz, Euler and Nörlund summability of series of orthogonal functions (not necessarily polynomials) have been studied by Alexits G.[1], Kaczmarz[2], Menchoff[3], Meder[4], Tandori[5], Lorentz[6], Zygmund[7], Patel C. M.[8], Bhatnagar P. C.[9] and Leindler L.[10] Jackson D.[11] applied the Cesàro summability to series of orthogonal polynomials for the first time. He proved the following theorem for Cesàro mean of order 1.

Theorem A :-

If the weight function $\rho(t)$ is a bounded function and if $\rho(t)\,\phi^2(t)$ is summable in the interval (-1, 1), then the series (7.1.1) is summable (C, 1) to a function f(x) in (-1, 1). Where,

$$\phi(t) = \frac{f(t) - f(x)}{t - x}$$

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1) Alexits G. [5]
2) Kaczmarz [52]
3) Menchoff [74]
4) Meder J. ([79], [80])
5) Lorentz [71]
6) Tandori ([127], [128])
7) Zygmund [152]
8) Patel C.M. [92]
9) Bhatnagar [15]
10) Leindler [64]
11) Jackson [48]

Similar results were proved by Patel[1], Bhatnagar[2] for Riesz and Euler means of order 1. In this chapter we prove the following theorems for Nörlund and Cesàro summability.

Theorem 1 :

If $\rho(t)\,\phi^2(t)$ is summable in $(-1, 1)$ then the series (7.1.1) is (N, p_n) summable to a function $f(x)$ in $(-1, 1)$ if $p_n \leftarrow M^{\alpha}$.

Theorem 2 :

If $\rho(t)\,\phi^2(t)$ is summable in $(-1, 1)$ then the series (7.1.1) is (C, α) summable in $(-1, 1)$.

Jackson D.[3] has proved several theorems while discussing the degree of approximation with the help of trigonometrical system. One of the theorems,

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1) Patel [91]
2) Bhatnagar [15]
3) Jackson [46]

for the degree of approximation in the sense of L^2 approximation, proved by Jackson[1] is as follows.

Theorem B :

Let $\rho_n(\hat{R}, 2)$ denote the best approximation in the L^2 space of the class $\hat{R}$ of all the 2π periodic functions $f(x)$ possessing an L^2 continuity modulus $w_2(f, \delta) < w_2(\delta)$ where $w_2(\delta)$ is a majorant of $w_2(f, \delta)$, Then we have

$$\rho_n(\hat{R}, 2) = O\left[w_2\left(\frac{1}{n}\right) \right].$$

In this chapter we extend the above result for orthogonal polynomial system under a weaker condition.

Theorem 3 :

Let $\{p_n(x)\}$ be an orthonormal polynomial

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1) Jackson [46]

system in the interval $[-1, 1]$ belonging to the weight function $\rho(x)$.

$$0 < \rho = O\left(\frac{1}{\sqrt{1 - x^2}}\right) .$$

Let $\rho_n(\mathfrak{K}, 2)$ denote the best degree of approximation, by linear forms of the system, in the L^2 - space of the class $\mathfrak{K}$ of all functions which possess continuity modulus $w(f, \delta) < w(\delta)$, then we have

$$\rho_n(\mathfrak{K}, 2) = O\, w\left(f, \frac{1}{n}\right) .$$

Moreover, while proving the converging almost everywhere of general orthogonal series in the intervals of continuity, Alexits and Kralik[1] has proved the following theorem.

Theorem C :

Let $\{\phi_n(x)\}$ denote a constant preserving polynomial like orthonormal system with respect to (x) whose functions $\phi_n(x)$ are uniformly bounded. The condition,

$$\sum_{k=0}^{n} \phi_k^2(x) = O(n)$$

1) Alexits ([5], p 312).

is satisfied in the interval of orthogonality while the condition (5.1.7)
satisfied uniformly in the subinterval

$$[C, d] \text{ of } [a, b].$$

If $f(x)$ is an L^2 - integrable function, continuous in $[C, d]$ with the continuity modulus

$$w(f, \delta, C, d) = O\left(\frac{1}{\sqrt{\phi(1/\delta)}}\right)$$

where $\phi(x) > 0$, is an arbitrary function, monotone increasing for $x > 1$ and satisfying the condition

$$\int_2^\infty \frac{dx}{x\phi(x-1)} < \infty$$

then the expansions

$$(7.1.2) \qquad f(x) \sim \sum_{n=0}^{\infty} C_n \phi_n(x)$$

converges to $f(x)$ almost everywhere in $[C, d]$.

In this chapter we extend the above theorem for orthogonal polynomial system under weaker condition for certain class of functions.

<u>Theorem 4 :</u>

If $\{p_n(x)\}$ *is any orthonormal polynomial* system belonging to the distribution $d\mu(x)$ and $f(x)$ is a function with the continuity modulus

(7.1.3) $$w(f, \delta) = O\left(\sqrt{\frac{1}{\phi(1/\delta)}}\right)$$

then

(7.1.4) $$\int_1^{\infty} \frac{dx}{x\phi(x)} < \infty$$

Implies the expansion (7.1.1) converges almost everywhere in $[a, b]$ for any order of its term.

In order to prove the above theorems we need the following Lemmas :

Lemma 1[1)]:

If $\{p_n\} \in M^{\alpha}$, $\alpha > \frac{1}{2}$,

then

$$\lim_{n \to \infty} \frac{n}{P_n^2} \sum_{k=0}^{n} \frac{P_k^2}{(k+1)^2} = \frac{1}{2\alpha - 1} .$$

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1) Meder [78]

Lemma 2[1] :

If $\phi(x)$ satisfies the condition (7.1.4) and if $\mathfrak{R}_{\phi}^{\mu}(\{\phi_n(x)\})$ be the class of all functions f(x) for which

$E_n^2 = O(\frac{1}{\phi(n)})$ then the expansion (7.1.2) converges almost everywhere in [a, b] for any order of its term. Where E_n^2 denote the best degree of approximation of f(x) by linear forms in the space L^2.

Proof of Theorem 1 :

$$t_n(x) - f(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(x) - f(x)$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(x) - \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} f(x)$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} (S_k(x) - f(x))$$

(7.1.7) $$P_n \{ t_n(x) - f(x) \} = \sum_{k=0}^{n} p_{n-k} (S_k(x) - f(x))$$

We have,

$$S_k(x) = C_o p_o(x) + C_1 p_1(x) + \text{------} + C_k p_k(x)$$

1) Leindler L. [63]

$$= \int_{-1}^{1} f(t)\, \varrho(t) p_o(t)\ p_o(x)\ dt + \int_{-1}^{1} f(t)\ \varrho(t) p_1(t) p_1(x) dt$$

$$+ \int_{-1}^{1} f(t)\ \varrho(t)\ p_2(t)\ p_2(x)\ dt + - - - - +$$

$$\int_{-1}^{1} f(t)\, \varrho(t)\ p_k(t)\ p_k(x)\ dt$$

$$= f(t) \int_{-1}^{1} \{ p_o(x)\ p_o(t) + p_1(t)\ p_1(x) + - - - -$$

$$+ p_k(t)\ p_k(x) \}\ \varrho(t)\ dt$$

(7.1.8) $$S_k(x) = \int_{-1}^{1} f(t)\, K_k(x, t)\ \varrho(t)\ dt$$

where

$$K_k(x,t) = p_o(x)\ p_o(t) + p_1(t)\ p_1(x) + - - - - +$$

$$p_k(t)\ p_k(x)$$

$$= \sum_{v=0}^{k} p_v(x)\ p_v(t)$$

Let α_k denotes the positive coefficients of x^k in $p_k(x)$.

Since 1 is a polynomial of degree 0 ,

$$\int_{-1}^{1} \varrho(t)\, P_k(t)\; dt = \int_{-1}^{1} \varrho(t)\, P_k(t) \cdot 1\; dt$$

$$= 0 \qquad \text{for } k \geq 1.$$

and

$$p_o(x) = p_o(t) = \alpha_o$$

Here $\int_{-1}^{1} \varrho(t)\, p_o(x)\, p_o(t)\; dt = \int_{-1}^{1} \varrho(t)\, p_o(t) dt = 1$

by the orthonormality property.

Now,

$$\varrho(t)\, p_o(x)\, p_o(t) + \varrho(t)\, p_1(t)\, p_1(x) + ------ + \varrho(t) \cdot$$

$$p_k(t)\, p_k(x) = \varrho(t)\, K_k(x,t).$$

Therefore,

(7.1.9) $$\int_{-1}^{1} \varrho(t)\, K_k(x,t)\; dt = 1$$

as a special case of (7.1.8) with $f(t) = 1$.

As $f(x)$ is constant with respect to the variable of integration , multiplying (7.1.9) by $f(x)$ we get

(7.1.10) $$f(x) = \int_{-1}^{1} \varrho(t)\; f(x)\, K_k(x,t)\; dt$$

From (7.1.8) and (7.1.10) we have

$$S_k(x) - f(x) = \int_{-1}^{1} \varrho(t) \left\{ f(t)-f(x) \right\} K_k(x,t)\, dt$$

Hence, (7.1.7) gives

$$P_n \left\{ t_n(x) - f(x) \right\} = \int_{-1}^{1} \sum_{k=0}^{n} p_{n-k}\, \varrho(t) \left\{ f(t)-f(x) \right\} K_k(x,t)\, dt$$

$$= \int_{-1}^{1} \sum_{k=0}^{n} p_{n-k}\, \varrho(t) \frac{\left\{ f(t)-f(x) \right\}}{(t-x)} (t-x) K_k(x,t)\, dt$$

By applying Christoffel-Darboux formula[1)] we get,

$$(7.1.11) \quad = \int_{-1}^{1} \sum_{k=0}^{n} p_{n-k} \phi(t) \frac{\alpha_k}{\alpha_{k+1}} \left\{ p_{k+1}(t)\, p_k(x) - p_{k+1}(x)\, p_k(t) \right\} \varrho(t)\, dt.$$

Putting $H_n(\lambda, x, t) = \sum_{k=0}^{n} p_{n-k} \frac{\alpha_k}{\alpha_{k+1}} p_{k+1}(t)\, p_k(x)$.

R.H.S. of (7.1.11) is

$$(7.1.12) \quad \int_{-1}^{1} \varrho(t)\, \phi(t)\, H_n(\lambda,x,t)\, dt - \int_{-1}^{1} \varrho(t) \phi(t)\, H_n(\lambda,x,t) dt$$

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1) Alexits G. ([5], p. 26)

Using Schwarz inequality we obtain,

(7.1.13) $$\left\{ \int_{-1}^{1} \varrho(t)\, H_n(\lambda, x, t)\, dt \right\}^2$$

$$\leq \int_{-1}^{1} \varrho(t)\, \phi^2(t)\, dt \int_{-1}^{1} \varrho(t)\, [H_n(\lambda, x, t)]^2\, dt$$

$$= O(1) \int_{-1}^{1} \left\{ H_n(\lambda, x, t) \right\}^2 dt\, \varrho(t)$$

$$= O(1) \sum_{k=o}^{n} p_{n-k}^2\, p_k^2(x)$$

As $\frac{\alpha_k}{\alpha_{k+1}}$ is bounded for large k.[1]

$$= O(n) \sum_{k=o}^{n} \frac{p_{n-k}^2}{(n-k)^2} \frac{(n-k)^2}{P_{n-k}^2} P_{n-k}^2$$

$$= O(n) \sum_{k=o}^{n} \frac{P_{n-k}^2}{(n-k)^2}$$

$$= O(1) \frac{n}{P_n^2} \sum_{v=o}^{n} \frac{P_v^2}{(v+1)^2} P_v^2$$

$$= O(P_n^2)$$

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1) Alexits G. [5], p. 28

this implies that

$$\lim_{n \to \infty} t_n(x) = f(x)$$

Hence the proof.

<u>Proof of Theorem 2</u> :-

$$\sigma_n^{\alpha}(x) - f(x) = \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} A_{n-k}^{\alpha-1} S_k(x) - \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} A_{n-k}^{\alpha-1} f(x)$$

$$= \frac{1}{A_n^{\alpha}} \sum_{k=o}^{n} A_{n-k}^{\alpha-1} (S_k(x) - f(x)).$$

(7.1.14) $$A_n^{\alpha} \left\{ \sigma_n^{\alpha}(x) - f(x) \right\} = \sum_{k=o}^{n} A_{n-k}^{\alpha-1} (S_k(x) - f(x)$$

From theorem 1

$$S_k(x) - f(x) = \int_{-1}^{1} \rho(t) \left\{ f(t) - f(x) \right\} k_k(x, t) dt$$

Hence (7.1.14) gives,

$$A_n^{\alpha} (\sigma_n^{\alpha}(x) - f(x) = \int_{-1}^{1} \sum_{k=o}^{n} A_{n-k}^{\alpha-1} \rho(t) \left\{ f(t) - f(x) \right\} k_k(x, t) \, dt .$$

By applying Christoffel – Darboux formula[1],

$$(7.1.15) \qquad = \sum_{k=o}^{n} A_{n-k}^{\alpha-1} \; \phi(t) \frac{\alpha_k}{\alpha_{k+1}} \Big\{ p_{k+1}(t)\, p_k(x) - p_{k+1}(x)\, p_k(t) \Big\} \varrho(t)\, dt .$$

Putting $H_n(\lambda, x, t) = \sum_{k=o}^{n} A_{n-k}^{\alpha-1} \frac{\alpha_k}{\alpha_{k+1}} p_{k+1}(t)\, p_k(x)$.

R.H.S. of (7.1.15) is

$$\int_{-1}^{1} \varrho(t)\, \phi(t)\, H_n(\lambda, x, t)\, dt - \int_{-1}^{1} \varrho(t)\, \phi(t)\, H_n(\lambda, x, t) dt$$

Using Schwarz inequality we have,

$$\left\{ \int_{-1}^{1} \varrho(t)\, \phi(t)\, H_n(\lambda, x, t)\, dt \right\}^2 \leqslant \int_{-1}^{1} \varrho(t)\, \phi^2(t)\, dt \; [H_n(\lambda, x, t)]^2 dt$$

$$= O(1) \int_{-1}^{1} [H_n(\lambda, x, t)]^2 \varrho(t)\, dt$$

$$= O(1) \sum_{k=o}^{n} (A_{n-k}^{\alpha-1})^2\, p_k^2(x).$$

as $\frac{\alpha k}{\alpha_{k+1}}$ is bounded for large k.

$$= O(1)\, n^{2\alpha-1}$$

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1) Alexits G. ([5], p. 26)

$$= O(1)\, n^{2\alpha}$$

$$= O(\, A_n^{\alpha}\,)^2 \quad .$$

This implies that

$$\lim_{n \to \infty} \sigma_n^{\alpha}(x) = f(x)$$

Hence the proof.

Proof of Theorem 3 :-

Let $S_n(x)$ be the n^{th} partial sums of the orthonormal expansion of $f(x)$.

Then,

$$\int_{-1}^{1} [\, f(x) - S_n(x)\,]^2 \rho(x)\, dx$$

$$= O(1) \int_{-1}^{1} [f(x) - p_n(x)\,]^2 \rho(x)\, dx \quad 1)$$

$$= O(1) \int_{-1}^{1} [f(x) - p_n(x)\,]^2 \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= O(1) \int_{0}^{\pi} [\, f(\cos\theta\,) - p_n(\cos\theta)]^2\, d\theta$$

$$= O(1) \int_{0}^{\pi} [\, f(\cos\theta) - p_n(\cos\theta)]^2\, d\theta\ .$$

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1) Alexits G.([5], p. 6)

The function $G(\Theta) = f(\cos\Theta)$ is defined in $[o, \pi]$ and possesses a continuity modulus $w(G,\delta)$. Also we have,

$$w(G(\Theta),\delta) = w(f(\cos\Theta),\delta)$$

$$= \sup_{|\phi_1-\phi_2|\leq\delta} |G(\Theta_1) - G(\Theta_2)|$$

$$< \sup_{|x_1-x_2|\leq\delta} |f(x_1) - f(x_2)|$$

$$= w(f,\delta).$$

Therefore, by theorem B and using condition $w_2(f,\delta) \leq w(f,\delta)$.

$$\int_{-1}^{1}[f(x) - S_n(x)]^2 \varrho(x)dx] = O(1) [w_2^2 (f, \frac{1}{n})]$$

Hence,

$$\int_{-1}^{1}[f(x) - S_n(x)]^2 \varrho(x)\, dx = O(1) [w(f, \frac{1}{n})]$$

Hence,

$$\rho_n(\mathfrak{R}, 2) = O[w(f, \frac{1}{n})].$$

<u>Proof of Theorem 4 :</u>

From theorem 3 we evidently have the relation

$$E_n^2 = O[w^2 (f, \frac{1}{n})]$$

From this condition and by (7.1.9) we have,

$$f \left(\Re_{\phi}^{(\mu)} \left\{ p_n(x) \right\} \right.$$

Hence the proof directly follows by applying Lemma 2.