CHAPTER - 8

ABSOLUTE CONVERGENCE OF ORTHOGONAL SERIES

Let $\{\phi_n(x)\}$ be any orthonormal system defined in the interval $[a, b]$ and let

$$(8.1.1) \qquad f(x) \sim \sum_{k=0}^{\infty} C_k \phi_k(x)$$

be the Fourier expansion of $f \in L^2 [a, b]$, where

$$C_n = \int_a^b f(x)\, \phi_n(x)\, dx$$

We shall denote by $E_n^{(2)}(f)$ the best approximation to $f(x)$ in L^2 and by $w^{(2)}(\delta, f)$ the qua atic modulus of continuity for $f(x)$

i.e.

$$E_n^{(2)}(f) = \min_T \left\{ \int_a^b [f(x) - T(x)]^2\, dx \right\}^{\frac{1}{2}},$$

where $T(x)$ is the partial sum of any order less than or equal to n of an orthogonal series $\Sigma C_n \phi_n (x)$ with arbitrary coefficients C_n, and

$$w^2(\delta, f) = \sup_{0 \leq h \leq \delta} \left\{ \int_a^b [f(x+h) - f(x)]^2 \, dx \right\}^{\frac{1}{2}}$$

Let $\mathfrak{R}$ denote a class of functions and $\{\phi_n(x)\}$ a given orthonormal system in $[a, b]$. Let us form (for a fixed natural number n) linear combinations of the functions $\phi_o(x), \phi_1(x), \ldots \ldots . \phi_m(x)$ of the form

$$S_n(x) = \sum_{k=o}^{n} a_{nk} \phi_k(x)$$

with real a_{nk}.

Let $d_n(f)$ denote the lower bound of the differences

$$\sup_{a \leq x \leq b} | f(x) - S_n(x) |$$

formed with every possible linear combinations $S_n(x)$. The non-negative number

$$\rho_n(\mathfrak{R}) = \sup_{f \in \mathfrak{R}} d_n(f)$$

is said to be the best degree of approximation attainable for the entire class $\mathfrak{R}$ with arbitrary $S_n(x)$.

Let $d_n^{(p)}(f)$ denote the lower bound of all the numbers

$$\left\{ \int_a^b | f(x) - S_n(x) |^p \, d\mu(x) \right\}^{\frac{1}{p}},$$

formed with linear combinations $S_n(x) = \sum_{k=0}^{n} a_{nk}\, \phi_k(x)$,

then

$$\rho_n(\mathfrak{K}, p) = \sup_{f \in \mathfrak{K}} d_n^{(p)}(f)$$

is said to be best degree of approximation for the class $\mathfrak{K}$ in the sense of L_μ^p - approximation.

Absolute convergence of Fourier series has been investigated by several authors such as Bernstein[1], Szasz[2], Stetchkin S.B.[3], Zygmund[4] etc. While the absolute convergence of orthogonal series has been studied in great details by Stetchkin S.B.[3] Zinovev[5] & Tandori K[6].

Concerning the absolute convergence of Fourier series Stetchkin S.B.[7] has proved the following theorem :

<u>Theorem **A**</u> :-

Let $f \in L^2 [o, 2\pi]$ and let

(8.1.2) $$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

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1) Bernstein [13]
2) Szasz [123]
3) Stetchkin [117]
4) Zygmund [151]
5) Zinovev [146]
6) Tandori K [134]
7) Stetchkin S.B [116]

be its Fourier series. If we fix an increasing sequence of numbers n_k, then

$$\sum_{k=1}^{\infty} | a_{n_k} + b_{n_k} | < \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \quad w^{(2)}\left(\frac{1}{n_k}, f\right).$$

Szasz [1] has proved the following theorem for Fourier series with the assumption that f(x) is of bounded variation.

Theorem B :

If f(x) is of bounded variation and

$$\sum_{n=1}^{\infty} \frac{\sqrt{w\left(\frac{1}{n}, f\right)}}{n} < \infty$$

then the Fourier series converges absolutely.

In this chapter we have generalized the above theorems for orthogonal expansions of f(x). We prove the following theorem.

Theorem 1 :- Let $f \in L^2[a, b]$ and (8.1.1) be its orthonormal expansion. Then

$$(8.1.3) \qquad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \, w^{(2)}\left(\frac{1}{k}, f\right) < \infty$$

1) Szasz [11]

implies the absolute convergence of (8.1.1).

Theorem 2 :- *If f(x) is of bounded variation, then*

$$(8.1.4) \qquad \sum_{k=1}^{\infty} \frac{\sqrt{w(\frac{1}{k}, f)}}{k} < \infty$$

implies the absolute convergence of (8.1.1).

Szasz has proved the following theorem D (refer Alexits[1]) for absolute convergence of orthogonal expansion of a function of a certain class. Bernstein[2] has proved the theorem C for general orthogonal polynomial expansions.

Theorem C :

Let $\{p_n(x)\}$ denote the orthonormal polynomial system, belonging to the distribution $d\mu(x)$ and $\{C_n\}$ the sequence of the expansion coefficients of a function f(x) which satisfies a liptchiz condition of order α with $\alpha > \frac{1}{2}$. Then we have the relation.

$$\sum_{n=0}^{\infty} | C_n | < \infty .$$

Thus, the $p_n(x)$ expansion of f(x) is absolutely convergent in any interval of boundedness of $p_n(x)$.

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1) Alexits G([5] , p. 334) 2) Bernstein [14]

Theorem D :

Let $\{\phi_n(x)\}$ denote an orthonormal system and $\mathfrak{K}$ a class of the functions such that the best L^2 approximation of the functions $f \in \mathfrak{K}$ by linear forms

$$\sum_{k=o}^{n} a_{nk} \phi_k (x)$$

has the degree of approximation

$$\rho_n(\mathfrak{K}, 2) = O\left(\frac{1}{n^\alpha} \right) \quad (\alpha > o)$$

If $2 > \beta > \frac{2}{2\alpha + 1}$, then for the expansion coefficients C_n of arbitrary functions $f \in \mathfrak{K}$, we have the relation

$$\sum_{n=1}^{\infty} | C_n |^\beta < \infty$$

We extend Theorem D, Theorem C directly follows from our theorem excluding one condition.

Theorem : 3 :- Let $\{\phi_n(x)\}$ denote an orthonormal system and $\mathfrak{K}$ a class of functions such that the best L^2 - approximation of the functions $f \in \mathfrak{K}$ by linear forms

$$\sum_{k=o}^{n} a_{nk} \phi_k (x)$$

has the degree of approximation

$$\rho_n (\mathfrak{R}, 2) = O(\frac{1}{n^{\alpha}}), \qquad \alpha > o .$$

If $1 > \beta(\alpha + \frac{1}{2}) > \varepsilon+1$, $\varepsilon > o$ then the expansion coefficients C_n of an arbitrary functions $f \in \mathfrak{R}$, we have the relation,

$$\sum_{n=1}^{\infty} n^{\varepsilon} | C_n |^{\beta} < \infty .$$

Theorem 4 :- Let $\{p_n (x)\}$ denote the orthnormal polynomial system, belonging to the distribution $d\mu(x)$ and $\{C_n\}$ the sequence of the expansion coefficients of a function $f(x)$ which satisfies a Liptchiz condition of order α with $\alpha > o$. Then we have the relation

$$\sum_{n=o}^{\infty} | C_n | < \infty .$$

Thus the $\{p_n (x)\}$ expansion of $f(x)$ is absolutely convergent.

Concerning the functions of the class Lip α, Lorentz has proved the following theorem for Fourier series.

Theorem E[1) :

If $f \in$ Lip α, $\alpha > \frac{1}{p} - \frac{1}{2}$ ($o < p \leq 2$), then

1) Bary N. K. ([11], p.215)

$$\left(\sum_{k=n}^{\infty} \left(| a_k |^p + | b_k |^p \right) \right)^{\frac{1}{p}} \leqslant \frac{C}{n^{\alpha + \frac{1}{2} - \frac{1}{p}}} .$$

We have extended the above theorem for orthogonal expansions of functions of any class.

<u>Theorem 5</u> :- Let $\{\phi_n(x)\}$ denote an orthonormal system and $\mathfrak{K}$ a class of functions such that the best L^2 - approximation of the functions $f \in \mathfrak{K}$ by Linear forms

$$\sum_{k=o}^{\infty} a_{nk} \phi_k (x) \text{ has the degree of approximations}$$

$$\mathcal{E}_n(\mathfrak{K}, 2) = O\left(\frac{1}{n^{\alpha}} \right) .$$

Then, for the expansion coefficients C_n of an arbitrary function $f \in \mathfrak{K}$ we have the relation

$$\left\{ \sum_{k=n}^{\infty} | C_n |^{\beta} \right\}^{\frac{1}{\beta}} \leqslant \frac{C}{n^{\alpha + \frac{1}{2} - \frac{1}{p}}} .$$

For proving these theorems we need the following Lemmas :

<u>Lemma 1</u>[1] : Let $U_n \geqslant o$ $(n = 1, 2, \ldots\ldots)$

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1) Stetchkin [116]

$$\sum_{n=1}^{\infty} U_n^2 < \infty .$$

Let us put

$$r_n = \sum_{k=n}^{\infty} U_k^2$$

Then

$$\sum_{n=1}^{\infty} U_n < \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{\sqrt{r_n}}{n} .$$

Lemma 2[1]: Let $\{p_n(x)\}$ be the orthonormal polynomial system belonging to the distribution $d\mu(x)$ and let $f(x)$ be a function continuous in $[a, b]$ with the continuity modulus $w(f, \delta)$. There exist a sequence $\{S_n(f, x)\}$ of linear forms

$$S_n(f, x) = \sum_{k=o}^{n} a_{nk} p_k(x)$$

for which

$$\sup_{a \leq x \leq b} | f(x) - S_n(f, x) | = O\left[w(f, \frac{1}{n})\right].$$

Proof of Theorem 1: By Bessel's inequality we have

$$\sum_{n=k}^{\infty} C_n^2 \leq \int_a^b | f(x) - S_{k-1}(x) |^2 dx$$

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1) Alexits G. ([5], p. 304)

$$= [E_k^{(2)} (f, \emptyset)]^2$$ (Refer Alexits G.[1],p.23).

Using Lemma 1 we get,

$$\sum_{k=1}^{\infty} | C_k | \leqslant \frac{2}{\sqrt{3}} \sum_{k=1}^{\infty} \sqrt{\frac{1}{k} \sum_{n=k}^{\infty} C_n^2}$$

$$\leqslant \frac{2}{\sqrt{3}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} E_k^{(2)}(f, \emptyset).$$

But by Jackson's inequality for approximations in the metric of the space L^2, we have

$$E_n^{(2)}(f) < C_1 w^{(2)} (\frac{1}{n} , f)$$

Therefore,

$$\sum_{k=1}^{\infty} | C_k | < C_2 \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} w^{(2)} (\frac{1}{k}, f)$$

But by (8.1.3) we have

$$\sum_{k=1}^{\infty} | C_k | < \infty .$$

By Schwarz inequality we have,

(8.1.5) $$\sum_{k=0}^{\infty} \int_a^b | \widehat{C_k \emptyset_k(x)}|dx= O(1) \sum_{k=0}^{\infty} | C_k | \sqrt{\int_a^b \emptyset_k^2(x)dx}$$

$$= O(1) \sum_{k=0}^{\infty} | C_k |$$

$$< \infty .$$

Using B.Levy's theorem, we have

$$\sum_{k=o}^{\infty} | c_k \phi_k(x) | < \infty .$$

Hence the proof.

<u>Proof of Theorem 2</u> : - We have in fact

$$w^{(2)}(\delta, f) < 2 \sqrt{w^{(1)}(\delta, f) \, w(\delta, f)} \quad ^{1)}$$

Now f is a function of bounded variation, then

$$w^{(1)}(\delta, f) = O(\delta) \quad ^{2)}$$

Hence by above theorem,

$$\sum_{k=1}^{\infty} | c_k | < \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} w^{(2)}\left(\frac{1}{k}, f\right)$$

$$< C_3 \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sqrt{\frac{1}{k} w\left(\frac{1}{k}, f\right)}$$

$$= C_3 \sum_{k=1}^{\infty} \frac{\sqrt{w\left(\frac{1}{k}, f\right)}}{k}$$

$$< \infty ,$$

by (8.1.4).

But by (8.1.5) and then applying B.Levy's theorem,

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1) Stechkin [116]   2) Bary [11]

we have,

$$\sum_{k=0}^{\infty} | c_k \phi_k(x) | < \infty$$

Hence the proof.

Proof of Theorem 3 :- By Hölders inequality we have,

$$\sum_{k=2^n+1}^{2^{n+1}} k^{\epsilon} | c_k |^{\beta} \leq \left\{ \sum_{k=2^n+1}^{2^{n+1}} c_k^2 \, k^{\frac{2\epsilon}{\beta}} \right\}^{\frac{\beta}{2}} \left\{ \sum_{k=2^n+1}^{2^{n+1}} 1 \right\}^{\frac{2-\beta}{2}}$$

$$= O\left(2^{\left(\frac{2-\beta}{2}\right)^n} \left\{ \sum_{k=2^n+1}^{2^{n+1}} k^{\frac{2\epsilon}{\beta}} c_k^2 \right\}^{\frac{\beta}{2}} \right)$$

But we have,

$$\left\{ \sum_{k=2^n+1}^{2^{n+1}} c_k^2 \, k^{\frac{2\epsilon}{\beta}} \right\}^{\frac{\beta}{2}} \leq \left\{ 2^{(n+1)\frac{2\epsilon}{\beta}} \sum_{k=2^n+1}^{2^{n+1}} c_k^2 \right\}^{\frac{\beta}{2}}$$

By Bessel's inequality we have,

$$\sum_{k=2^n+1}^{2^{n+1}} c_k^2 \leq \sum_{k=2^n+1}^{\infty} c_k^2 = \int_a^b (f(x) - S_{2^n}(x))^2 \, d\mu(x)$$

$$\leq \rho_{2^n}^2 (\mathcal{R}, 2) = O\left(\frac{1}{2^{2\alpha n}} \right)$$

Therefore,

$$\left\{ \sum_{k=2^n+1}^{2^{n+1}} c_k^2 \, k^{\frac{2\epsilon}{\beta}} \right\}^{\frac{\beta}{2}} = O(1) \left\{ 2^{(n+1)\frac{2\epsilon}{\beta}} \frac{1}{2^{2\alpha n}} \right\}^{\frac{\beta}{2}}$$

So we have,

$$\sum_{k=2^n+1}^{2^{n+1}} k^{\varepsilon} \mid c_k \mid^{\beta} = O(1) \quad 2^{(\frac{2-\beta}{2})n + \left\{(n+1)\frac{2}{\beta} - 2\alpha n\right\}\frac{\beta}{2}}$$

Hence,

$$\sum_{n=1}^{\infty} n^{\varepsilon} \mid c_n \mid^{\beta} = \sum_{n=o}^{\infty} \left(2^{(\varepsilon - \alpha\beta + (1 - \frac{\beta}{2})n} \right)$$

$$< \infty$$

Hence the proof.

Proof of theorem 4 :- Let $\mathfrak{R}$ denote the class of all functions $f(x)$ with $w(f,\delta) = O(\delta^{\alpha})$.

According to Lemma 2 for the approximation of these functions by orthogonal polynomial we have

$$\rho_n(\mathfrak{R}) = O\left(\frac{1}{n^{\alpha}}\right)$$

But

$$\rho_n(\mathfrak{R}, 2) \leqslant \rho_n(\mathfrak{R}) \left\{ \int_a^b d\mu(x) \right\}^{\frac{1}{2}}$$

So we have

$$\rho_n(\mathfrak{R}, 2) = O\left(\frac{1}{n^{\alpha}}\right).$$

Because for $\alpha > o$, we have $\alpha + \frac{1}{2} > \frac{1}{2}$, thus theorem (4) follows from Theorem (3) with $\beta = 1$

and $\varepsilon = \frac{1}{2}$

Hence the proof.

Proof of Theorem 5 :- We know that by Bessel's inequality for any n, we have,

$$\sum_{k=n}^{2n-1} C_k^2 \leqslant \sum_{k=n}^{\infty} C_k^2 = \int_a^b (f(x) - S_n(x))^2 dx$$

$$\leqslant \mathcal{E}_n^2 (\mathcal{R}, 2) = O\left(\frac{1}{n^{2\alpha}}\right)$$

$$\therefore \quad \sum_{k=n}^{2n-1} C_k^2 = O\left(\frac{1}{n^{2\alpha}}\right)$$

Applying Hölders inequality we have,

$$\sum_{k=n}^{2n-1} |C_k|^p \leqslant \left\{ \sum_{k=n}^{2n-1} (|C_k|^p)^{\frac{2}{p}} \right\}^{\frac{p}{2}} \left\{ \sum_{k=n}^{2n-1} 1^{\frac{2}{2-p}} \right\}^{1-\frac{p}{2}}$$

$$= \left\{ \sum_{k=n}^{2n-1} C_k^2 \right\}^{\frac{p}{2}} \left\{ n^{1-\frac{p}{2}} \right\}$$

$$= O(1) \left\{ \frac{1}{n^{2\alpha}} \right\}^{\frac{p}{2}} \left\{ n^{1-\frac{p}{2}} \right\}$$

$$= O(1)\,\frac{1}{n^{\alpha p}}\,\frac{1}{n^{\frac{p}{2}-1}}$$

$$= O(1)\,\frac{1}{n^{p(\alpha+\frac{1}{2})-1}}$$

Hence,

$$\left\{\sum_{k=n}^{\infty} |C_k|^p\right\}^{\frac{1}{p}} = \left\{\sum_{j=o}^{\infty}\sum_{k=2^j n}^{2^{j+1}n-1} |C_k|^p\right\}^{\frac{1}{p}}$$

$$= \left\{\sum_{j=o}^{\infty} \frac{1}{(2^j n)^{p(\alpha+\frac{1}{2})-1}}\right\}^{\frac{1}{p}}$$

$$\leqslant \frac{1}{n^{\alpha+\frac{1}{2}-\frac{1}{p}}}\left\{\sum_{j=o}^{\infty}\left(\frac{1}{2^{p(\alpha+\frac{1}{2})-1}}\right)^j\right\}^{\frac{1}{p}}$$

$$\leqslant C\,\frac{1}{n^{\alpha+\frac{1}{2}-\frac{1}{p}}}$$

Hence the proof.

__Corl. 1__ : If $p = 2$ then we obtain

$$\left\{\sum_{k=n}^{\infty} C_k^{\,2}\right\}^{\frac{1}{2}} = O\left(\frac{1}{n^{\alpha}}\right).$$

Corl. 2 : If $\alpha > \frac{1}{2}$, then above theorem for $p = 1$ reduces to

$$\sum_{k=n}^{\infty} | c_k | = O\left(\frac{1}{n^{\alpha - \frac{1}{2}}} \right)$$