

CHAPTER - 1

I N T R O D U C T I O N

1.1 The theory of orthogonal series was originated during the discussion of the problems of vibrating string more than two hundred years ago, considered by Euler in 1753 in connection with the work of Daniel Bernoulli. During their discussion they had advanced the theory of vibrating strings to the stage where the partial differential equation $y_{tt} = a^2 y_{xx}$ was known and the solution of the boundary value problem had been found from the general solution of that equation. Thus they have led to the possibility of representing an arbitrary function by a trigonometrical series. The problem of what functions can be represented by trigonometric series arise again later during the researches by French mathematical Physicist J. B. J. Fourier.

The last several years have been the period of intensive development in the theory of Fourier series with respect to trigonometric orthogonal systems. Less attention has been paid to the theory of general orthogonal series. The present work is concerning the theory of general orthogonal

series. During the first half of the present century, some of the leading mathematicians like Fejér, Hobson, Hardy, Hilbert, Lebesgue, F.Riesz, M.Riesz, Alexits, Wiener, Weyl, Kaczmarz, Menchoff, Lorentz, Tandori and Meder were mainly working in the field of convergence, summation and approximation problems of general and special orthogonal expansions. We would like to discuss some of the problems connected with the convergence and summability of general orthogonal series. We shall start with a number of definitions and concepts relevant to our work.

1.2 Throughout the thesis we shall make use of either Stieltjes - Lebesgue integral or the Lebesgue integral. The notion of orthogonality is introduced by means of the Stieltjes Lebesgue integral. Let $\mu(x)$ be a positive bounded and monotone increasing function in the closed interval $[a, b]$. Such a function is called the distribution function¹⁾.

A real function $f(x)$ is called L_μ - integrable, if it is μ - measurable and

$$(1.2.1) \quad \int_a^b |f(x)| d\mu(x) < \infty$$

1) Freud [37]

If $\mu(x)$ is absolutely continuous and $g(x) = \mu'(x)$, then for any L_μ - integrable function $f(x)$, the relation

$$(1.2.2) \quad \int_a^b f(x) d\mu(x) = \int_a^b f(x) g(x) dx$$

is valid. In this case we shall say that $f(x)$ is $L_{g(x)}$ - integrable function and we call $g(x)$ a covering function or weight function. If, in particular, $g(x) = 1$ then we shall say in accordance with the usual terminology that $f(x)$ is L - integrable.

A function $f(x)$ is called L_μ^2 or $L_{g(x)}^2$ - integrable, if it is L_μ or $L_{g(x)}$ - integrable respectively and if, furthermore,

$$\int_a^b f^2(x) d\mu(x) < \infty \quad \text{or} \quad \int_a^b f^2(x) g(x) dx < \infty$$

holds. We shall talk about an L^2 - integrable function, if $g(x) = 1$.

ORTHOGONALITY :- A finite or denumerably infinite system $\{\phi_n(x)\}$ of L_μ^2 - integrable functions is said to be orthogonal with respect to the distribution $d\mu(x)$ in the interval $[a, b]$, if

$$(1.2.3) \quad \int_a^b \phi_m(x) \phi_n(x) d\mu(x) = 0, \quad m \neq n.$$

holds and none of the functions $\phi_n(x)$ vanishes almost everywhere.

A system $\{\phi_n(x)\}$ is said to be orthonormal, if in addition to the condition (1.2.3) the condition

$$\int_a^b \phi_n^2(x) d\mu(x) = 1, \quad n = 0, 1, 2, \dots$$

is also satisfied. Every orthogonal system $\{\psi_n(x)\}$ can be converted into an ONS by means of multiplying every one of its members by a suitably chosen constant factor. For, since none of the functions $\psi_n(x)$ can vanish almost everywhere, the functions

$$\phi_n(x) = \frac{\psi_n(x)}{\left\{ \int_a^b \psi_n^2(x) d\mu(x) \right\}^{1/2}}$$

exist and it is immediately evident that they constitute an ONS with respect to $d\mu(x)$. If, in particular $\mu(x) = x$ i.e. $\mu'(x) = g(x) = 1$, then $\{\phi_n(x)\}$ is simply an ONS in the ordinary sense.

ORTHOGONALIZATION :- A system of functions $\{f_n(x)\}$ is called linearly independent in $[a, b]$, if the validity of the relation of the form

$$\sum_{k=0}^n a_k f_k(x) = 0,$$

for μ - almost every $x \in [a, b]$ necessarily implies the relation

$$a_0 = a_1 = \dots = a_n = 0, \text{ for all } n \in \mathbb{N}.$$

Every orthogonal system $\{\phi_n(x)\}$ is linearly independent.¹⁾ Conversely, any linearly independent system of functions $\{f_n(x)\}$ can be converted in to an ONS $\{\phi_n(x)\}$ such that for each n , $\phi_n(x)$ is a linear combinations of functions from $\{f_n(x)\}$. The process of constructing an orthonormal system from a linearly independent system is known as Schmidt's²⁾ general process of Orthogonalization.

ORTHOGONAL SERIES AND ORTHOGONAL EXPANSION :-

Let $\{\psi_n(x)\}$ be an orthonormal system. A series of the form

$$(1.2.4) \quad \sum_{n=0}^{\infty} C_n \psi_n(x)$$

where C_0, C_1, C_2, \dots etc. are arbitrary real numbers,

¹⁾ Alexits ([5], p.4) ²⁾ Schmidt [108]

is called an orthogonal series. However, if the coefficients C_n in the series (1.2.4) are representable in the form

$$C_n = \frac{1}{\int_a^b \psi_n^2(x) d\mu(x)} \int_a^b f(x) \psi_n(x) d\mu(x),$$

$$n = 0, 1, 2, \dots$$

for a certain function f ,

according to Fourier's manner, then we shall say that the series (1.2.4) is the orthogonal expansion of the function $f(x)$ and we shall express this relation by

$$f(x) \sim \sum_{n=0}^{\infty} C_n \psi_n(x).$$

In this case we shall call the numbers C_0, C_1, C_2, \dots the expansion coefficients of the function $f(x)$.

The orthogonal expansion and orthogonal series differ from each other due to the following minimum property established by Gram¹⁾.

Let $f(x) \in L_{\mu}^2[a, b]$ and $\{\phi_n(x)\}$ be an arbitrary ONS. Among all expressions of the form

$$S_n(x) = \sum_{k=0}^n a_k \phi_k(x).$$

1) Gram [40]

the integral

$$I(S_n) = \int_a^b [f(x) - S_n(x)]^2 d\mu(x)$$

attains the least value for $S_n(x) = s_n(x)$, where

$$s_n(x) = \sum_{k=0}^n C_k \phi_k(x), \quad C_k = \int_a^b f(t) \phi_k(t) d\mu(t).$$

An immediate consequence of Gram's theorem is the Bessel's inequality¹⁾.

$$\sum_{n=0}^{\infty} C_n^2 \leq \int_a^b f^2(x) d\mu(x).$$

Bessel's inequality implies that the expansion coefficients C_n of an L^2_{μ} - integrable function converge to zero as $n \rightarrow \infty$.

The most fundamental theorem in the theory of orthogonal series is the Riesz - Fischer theorem proved nearly simultaneously and independently by Riesz²⁾ and Fischer³⁾. The above theorem was later on generalized by Fomin⁴⁾ as follows.

Let $\{\phi_k(x)\}$ be an ONS on the interval

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- 1) Tricomí [140]
2) Riesz [103]

- 3) Fischer [33]
4) Fomin [35]

$[a, b]$, $\phi_k \in L^q[a, b]$, $k = 0, 1, 2, \dots$, $1 < q < \infty$
 and let p be given by the relation $\frac{1}{p} + \frac{1}{q} = 1$, if
 $q < \infty$, and $p = 1$ if $q = \infty$. If there exist an
 increasing sequence $\{V_k\}$ of natural numbers, such that,
 $V_k \rightarrow \infty$ and

$$\sum_{k=0}^{\infty} \left(\frac{1}{V_k} - \frac{1}{V_{k+1}} \right) \int_a^b \left| \sum_{m=0}^k a_m V_m \phi_m(x) \right|^p dx < \infty$$

with a_k real, then there exists a function $f \in L^p[a, b]$,
 such that

$$a_k = \int_a^b f(x) \phi_k(x) dx \quad \text{for } k = 0, 1, 2, \dots$$

ORTHOGONAL POLYNOMIALS :

The system of function $\{x^n\}$, for integral n ,
 is a linearly independent system. An orthogonalization¹⁾ of
 this system by the Schmidt's process gives a polynomial of
 degree exactly equal to n and it may be observed that
 we can choose the sign of the highest degree term in x in
 the polynomial to be positive. This polynomial of degree n
 will be denoted by $p_n(x)$.

Supposing that the orthogonalization of the

1) Freud [37]

system $\{x^n\}$ has been done under the distribution $d\mu(x) = \rho(x) dx$, we could assign special forms to $\rho(x)$ in order to obtain some well-known systems of polynomials. For example,

(i) Jacobi polynomials for :

$$\rho(x) = (b-x)^\alpha (x-a)^\beta, \quad \alpha > -1, \beta > -1.$$

Particular cases of Jacobi polynomials are :

(A) Ultraspherical polynomials for,

$$\alpha = \beta, \quad \alpha > -1,$$

$$a = -1, \quad b = 1,$$

(B) Legendre polynomials for,

$$\alpha = \beta = 0, \quad a = -1, \quad b = 1,$$

(C) Chebyscheff polynomials for,

$$\alpha = \beta = -\frac{1}{2}, \quad a = -1, \quad b = 1.$$

(ii) Laguerre polynomials :

$$a = 0, \quad b = +\infty, \quad \rho(x) = e^{-x} x^\alpha, \quad \alpha > -1$$

(except for a constant factor).

(iii) Hermite polynomials :

$$a = -\infty, \quad b = +\infty, \quad \rho(x) = e^{-x^2}$$

(except for a constant factor).

$\lambda(n)$ - LACUNARY ORTHOGONAL SERIES :

Let $\lambda(x)$ denote a positive function concave from below, defined for $x \geq 1$, such that $\lambda(x) \leq x$, and is increasing monotonely to infinity. We shall call the orthogonal series

$$\sum_{n=0}^{\infty} C_n \rho_n(x)$$

$\lambda(n)$ - lacunary, if the number of its non-vanishing coefficients C_k with $n < k \leq 2n$ does not exceed $\lambda(n)$.

Furthermore, we shall say that the coefficients have the positive number sequence $\{q_n\}$ as a majorant, if the relation

$$C_n = O(q_n)$$

holds.

1.3 We would like to define various summability methods which would be used in the body work of the thesis. In each of the following definition we take

$$(1.3.1) \quad \sum_{n=0}^{\infty} U_n$$

as the infinite series and S_n to be its n^{th} -partial sum.

CESÀRO SUMMABILITY :

For $\alpha > -1$,

let $A_n^\alpha = \binom{n+\alpha}{n}$ denote the n^{th} coefficient of the binomial series

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = \frac{1}{(1-x)^{1+\alpha}} \quad (\alpha \neq -1, -2, \dots)$$

We write

$$\sigma_n^0 = S_n^0 = S_n = U_0 + U_1 + \dots + U_n$$

and

$$S_n^\alpha = \sum_{k=0}^n A_{n-k}^{\alpha-1} S_k = \sum_{k=0}^n A_{n-k}^\alpha U_k.$$

Then the quotient

$$\sigma_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha}$$

is called the n^{th} Cesàro mean of order α of the sequence $\{S_n\}$ or simply (C, α) mean. The series (1.3.1) is said to be (C, α) -summable to $S^1)$ if $\sigma_n^\alpha \rightarrow S$ as $n \rightarrow \infty$.

The series (1.3.1) is said to be strongly (C, α) -summable with index k to the sum S , if

$$\frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} |S_v - S|^k \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For $\alpha = 1$, this gives the definition of strong summability $(H, k)^2)$.

1) Cesàro [25], Chapman [26], Knopp ([56] [57])

2) Zygmund ([149], p.180), Bary ([11], p.2), Moricz [54]

Riesz Summability :

Let $\{\lambda_n\}$ be a positive, strictly increasing sequence of real numbers with $\lambda_0 = 0$ and $\lambda_n \rightarrow \infty$, as $n \rightarrow \infty$.

The series (1.3.1) is said to be summable by Riesz means of order α or summable (R, λ_n, α) ($\alpha > 0$) to the sum S if

$$\sigma_n^\alpha(\lambda) = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)^\alpha U_k \rightarrow S \quad \text{as } n \rightarrow \infty.$$

Here $\sigma_n^\alpha(\lambda)$ is called the n^{th} $(R, \lambda_n, \alpha)^{1)}$ - mean of the given series.

In particular for $\alpha = 1$

$$\sigma_n^1(\lambda) = \sigma_n(\lambda) = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right) U_k = \frac{1}{\lambda_{n+1}} \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) s_k$$

defines the n^{th} $(R, \lambda_n, 1)$ - mean of the series (1.3.1).

Obviously, the Riesz method of summation is a generalization of $(C, 1)$ method, which is obtained by putting $\lambda_n = n$. In case $\lambda_n = \log n$, the Riesz summability is known as Riesz logarithmic summability.

1) Das [30], Lorentz [71]

de-la Valle'e - Poussin's summability :

Define

$$V_n = \frac{S_n + S_{n+1} + \dots + S_{2n-1}}{n}$$

called the n^{th} de-la Valle'e - Poussin's mean¹⁾ of the series (1.3.1). If

$$V_n \rightarrow S \text{ as } n \rightarrow \infty$$

then the series (1.3.1) is said to be summable by de-la Valle'e Poussin's method to the sum S .

Euler Summability :

A sequence to sequence transformation given by the equation

$$T_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} S_k,$$

$$(n = 0, 1, 2, \dots)$$

defines the sequence $\{T_n^q\}$ of (E, q) means ($q > 0$), known as Euler means²⁾.

If

$$T_n^{(q)} \rightarrow S \text{ as } n \rightarrow \infty$$

then the series (1.3.1) is said to be (E, q) summable to S .

1) Tandori [133]

2) Hardy [42]

In particular for $q = 1$

$$T_n^{(1)} = T_n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} S_k$$

defines the $(E, 1)$ - mean of the series (1.3.1).

Nörlund Summability :

Let $\{p_n\}$ be a sequence of nonnegative real numbers. A sequence to sequence transformation given by the equation

$$t_n = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} S_k$$

where $P_n = p_0 + p_1 + \dots + p_n$, $p_0 > 0$, $p_n \geq 0$ defines the sequence $\{t_n\}$ of Nörlund means¹⁾ of the sequence $\{S_n\}$. The series (1.3.1) is said to be summable (N, p_n) to the sum S if

$$\lim_{n \rightarrow \infty} t_n = S.$$

It is well known that the method (N, p_n) is regular, if and only if,

$$\lim_{n \rightarrow \infty} \frac{p_n}{P_n} = 0.$$

1) Nörlund [86], Woroni [145]

The sequence $\{p_n\}$ will be said to belong to class $M^{\alpha 1}$, for a certain real $\alpha \geq 0$, if,

$$(i) \quad 0 < p_n < p_{n+1} \quad \text{for } n = 0, 1, 2, \dots$$

$$\text{or } 0 < p_{n+1} < p_n \quad \text{for } n = 0, 1, 2, \dots$$

$$(ii) \quad p_0 + p_1 + \dots + p_n = p_n \uparrow \infty$$

$$(iii) \quad \lim_{n \rightarrow \infty} \frac{np_n}{p_n} = \alpha$$

Obviously, if $\{p_n\} \in M^{\alpha}$, then the method (N, p_n) is regular.

If for some sequence $\{p_n\}$, conditions (i) and (ii) are satisfied and moreover, if

$$\lim_{n \rightarrow \infty} \frac{n \Delta p_{n-1}}{p_n} = 1 - \alpha, \quad \text{where } \alpha \geq 0,$$

$\Delta p_{n-1} = p_{n-1} - p_n$, then we shall say that the sequence $\{p_n\}$ belongs to the class \bar{M}^{α} .²⁾

(\bar{N}, p_n) Summability :-

The series (1.3.1) is to be (\bar{N}, p_n) summable to the value S , if $p_n > 0$, $p_0 > 0$, $P_n = p_0 + p_1 + \dots + p_n$ and

$$\bar{I}_n = \frac{1}{P_n} \sum_{k=0}^n p_k S_k \longrightarrow S \quad \text{as } n \rightarrow \infty.$$

1) Meder [78]

2) Meder [80]

The expressions \bar{T}_n will be called the (\bar{N}, p_n) means of the series ¹⁾ (1.3.1).

The series (1.3.1) is said to be strongly (\bar{N}, p_n) summable to the sum S if

$$\frac{1}{P_n} \sum_{k=0}^n p_k (S_k - S)^2 \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Logarithmic Summability :- The series (1.3.1) is to be summable by first logarithmic means or summable by $(R, 1)$, if

$$L_n = \frac{1}{\log n} \left(\frac{S_1}{1} + \frac{S_2}{2} + \dots + \frac{S_n}{n} \right) \longrightarrow S \quad \text{as } n \rightarrow \infty$$

The expression L_n will be called n^{th} - logarithmic means of sequence $\{S_n\}$. It is known that if a series is summable to S by the method $(C, 1)$, then it is also summable to S by the method $(R, 1)^{2)}$.

An increasing sequence of natural numbers

$$n_1 < n_2 < \dots < n_k < \dots$$

1) Hardy [42]

2) Meder [77]

is said to satisfy the condition (L) if the series

$$\sum \frac{1}{n_k}$$

satisfies the condition (L) ; i.e.

$$\sum_{k=m}^{\infty} \frac{1}{n_k} = O\left(\frac{1}{n_m}\right) \quad 1)$$

1.4 Lebesgue functions :

The concept of Lebesgue function was introduced by Lebesgue²⁾. He investigated the influence of these function on the divergence of Fourier series. Lebesgue functions $L_n(x)$ are constants for the trigonometric system and are therefore called the Lebesgue constants.

Summability Kernels and Lebesgue functions :-

The sums

$$k_n(t, x) = \sum_{k=0}^n \phi_k(t) \phi_k(x) \quad \text{and} \quad k_n^\alpha(t, x) = \sum_{k=0}^n \frac{A_{n-k}^\alpha}{A_n^\alpha} \phi_k(t) \phi_k(x)$$

$(\alpha > -1)$ are called the n^{th} -Kernel and $n^{\text{th}}(C, \alpha)$ Kernel respectively of the ONS $\{\phi_n(x)\}$, whereas

1) Bary [12].

2) Lebesgue [61]

$$L_n(x) = \int_a^b |k_n(t, x)| d\mu(t) \quad \text{and} \quad L_n^\alpha(x) = \int_a^b |k_n^\alpha(t, x)| d\mu(t)$$

are called its n^{th} Lebesgue function and n^{th} Lebesgue (C, α) function, respectively.

The n^{th} - (N, p_n) Kernel $N_n(t, x)$ and n^{th} Lebesgue (N, p_n) - function $\bar{Z}_n(x)$ of the ONS $\{\phi_n(x)\}$ are defined by

$$N_n(t, x) = \frac{1}{p_n} \sum_{v=0}^n p_{n-v} k_v(t, x)$$

and

$$\bar{Z}_n(x) = \int_a^b |N_n(t, x)| d\mu(t).$$

$$\bar{N}_n(t, x) = \frac{1}{p_n} \sum_{v=0}^n p_v k_v(t, x)$$

and

$$\bar{Q}_n(x) = \int_a^b |\bar{N}_n(t, x)| d\mu(t)$$

respectively are defined as the n^{th} - (\bar{N}, p_n) Kernel and (\bar{N}, p_n) Lebesgue function for the ONS $\{\phi_n(x)\}$.

Riesz Kernel and its Lebesgue function as well as Euler Kernel and its Lebesgue function were defined by Kantawala P.S.¹⁾ in a similar way.

1.5 Singular integrals :- This concept was introduced by Lebesgue¹⁾ and it has important convergence properties.

The partial sums $S_n(x)$ of the expansion of an $L_p(x)$ - integrable function in the functions of an ONS $\{\phi_n(x)\}$ are of the form :

$$(1.5.1) \quad I_n(f, x) = \int_a^b f(t) \psi_n(t, x) \phi(t) dt$$

where $\psi_n(t, x)$ denotes the sum

$$\sum_{k=0}^n \phi_k(t) \phi_k(x) .$$

The n^{th} sums

$$t_n(x) = \sum_{k=0}^n \alpha_{nk} S_k(x)$$

of an expansion summed by a linear summation process are also representable by the integral (1.5.1) where $\psi_n(t, x)$

1) Kantawala P. S. [50]

1) Lebesgue [61]

denotes the sum

$$\psi_n(t, x) = \sum_{k=0}^n a_{nk} \phi_k(t) \phi_k(x).$$

The integral $I_n(f, x)$ is said to be singular (with singular point x) if, for an arbitrary number $\delta > 0$ and for an arbitrary subinterval $[\alpha, B]$ of $[a, b]$,

$$(1.5.2) \quad \lim_{n \rightarrow \infty} \int_I \psi_n(t, x) \varphi(t) dt = 1 \quad \text{and}$$

$$\lim_{n \rightarrow \infty} \int_J \psi_n(t, x) \varphi(t) dt = 0,$$

where $I = [a, b] \cap [x - \delta, x + \delta]$, $J = [\alpha, B] - [x - \delta, x + \delta]$ and

$$(1.5.3) \quad \text{ess. lqb.} \quad |\psi_n(t, x)| \leq \psi(\delta)$$

$$t \in [a, b] - [x - \delta, x + \delta]$$

Should holds.

Where $\psi(\delta)$ is a number depending on δ and x but independent of n .

If $\psi_n(t, x)$ satisfies uniformly the conditions (1.5.2) and (1.5.3) in an x -set E , then the integral $I_n(f, x)$ is said to be uniformly singular on E .

1.6 Constant preserving and polynomial like Orthonormal Systems :-

The idea of the polynomial - like orthogonal system was introduced by Alexits¹⁾.

An ONS $\{\phi_n(x)\}$ is called polynomial like, if its n^{th} Kernel $k_n(t, x)$ has the following structure :

$$k_n(t, x) = \sum_{k=1}^r F_k(t, x) \sum_{i, j=-p}^p Y_{i, j, k}^{(n)} \phi_{n+i}(t) \phi_{n+j}(x)$$

Where p and r are natural numbers, independent of n and the constants $|Y_{i, j, k}^{(n)}|$ have a common bound independent of n , while the measurable functions $F_k(t, x)$ satisfy the condition

$$F_k(t, x) = O\left(\frac{1}{|t-x|}\right)$$

for every $x \in [a, b]$. Here the function ϕ_{n+i} with negative index is considered to be identically equal to zero.

The orthonormal system $\{\phi_n(x)\}$ is called constant preserving if $\phi_0(x) = \text{constant}$. In this case besides C_0 all the expansion coefficients of the constant

1) Alexits ([5] P.11)

function $f(x) = C$ vanish and therefore, we have for the n^{th} partial sum $S_n(x)$ of its expansion

$$S_n(x) = C \int_a^b \phi_0(t) \phi_0(x) d\mu(t) = C.$$

i.e. a representation, preserving constancy.

The system of orthogonal polynomials $\{P_n(x)\}$ and the trigonometrical systems are polynomial like.

1.7 Degree of Approximation of a Class of functions :-

Let \mathcal{R} denote a class of functions and $\{\phi_n(x)\}$ a given orthonormal system in $[a, b]$. Let us form (for a fixed natural number n) linear combinations of the functions $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$ of the form

$$S_n(x) = \sum_{k=0}^n a_{nk} \phi_k(x)$$

with real a_{nk} . Let $d_n(f)$ denote the lower bound of

$$\sup_{a \leq x \leq b} |f(x) - S_n(x)|$$

formed with every possible linear combinations $S_n(x)$. The non-negative number

$$\rho_n(\mathcal{R}) = \sup_{f \in \mathcal{R}} d_n(f).$$

is said to be the best degree of approximation attainable for the entire class \mathcal{R} with arbitrary $S_n(x)$.

If $d_n^{(p)}(f)$ denotes the lower bound of all the numbers

$$\left\{ \int_a^b |f(x) - S_n(x)|^p d\mu(x) \right\}^{1/p}$$

formed with linear combinations $S_n(x) = \sum_{k=0}^n a_{nk} \phi_k(x)$

then, $\mathcal{E}_n(\mathcal{R}, p)$ can be considered as best degree of approximation for the class \mathcal{R} in the sense of the L_μ^p -approximation.

1.8 Convergence and Summability of Orthogonal series :- (a brief history):-

The question of convergence of orthogonal series was originally started by Jerosch and Weyl¹⁾ who pointed out that the condition

$$C_n = O\left(n^{-\frac{3}{4}-\xi}\right), \quad \xi > 0.$$

is sufficient for the convergence of the series

$$(1.8.1) \quad \sum_{n=0}^{\infty} C_n \phi_n(x).$$

1) Jerosch and Weyl [49]

Further, Weyl¹⁾ has improved this condition by showing that the condition,

$$\sum_{n=1}^{\infty} C_n^2 \sqrt{n} < \infty$$

is sufficient for the convergence of the series (1.8.1).
 Later on Hobson²⁾ modified the above condition to the form

$$\sum_{n=1}^{\infty} C_n^2 n^{\xi} < \infty, \quad \xi > 0$$

and Plancherel³⁾ has solved the same problem with the condition

$$\sum_{n=2}^{\infty} C_n^2 \log^3 n < \infty.$$

The Chain of ideas in this direction continued and finally a masterpiece work regarding the convergence of the orthogonal series (1.8.1) was carried out nearly, simultaneously and independently of one another by Rademacher⁴⁾ and Menchoff⁵⁾. They have shown that the series (1.8.1) is convergent almost everywhere in the interval of orthogonality if the condition

$$\sum_{n=1}^{\infty} C_n^2 \log^2 n < \infty,$$

is satisfied. Further generalizations of this theorem were

1) Weyl [144]
 2) Hobson [44]
 3) Plancherel [98]

4) Rademacher [100]
 5) Menchoff [73]

given by Gaposkin¹⁾, Salem²⁾, Talalyan³⁾ and Walfisz⁴⁾.

The theorem of Rademacher and Menchoff is the best of its kind is obvious from the following fundamental theorem of convergence theory given by Menchoff.

If $w(n)$ is an arbitrary positive monotone increasing sequence of numbers with $w(n) = o(\log n)$, then there exist an everywhere divergent orthogonal series

$$\sum_{n=0}^{\infty} C_n \psi_n(x)$$

whose coefficients satisfy the condition

$$\sum_{n=1}^{\infty} C_n^2 w^2(n) < \infty.$$

Another result which needs to be mentioned in this direction is due to Tandori⁵⁾, who proved that if $\{C_n\}$ is a positive monotone decreasing sequence of numbers for which

$$\sum_{n=1}^{\infty} C_n^2 \log n = \infty$$

holds, then there exist in $[a, b]$ an ONS $\psi_n(x)$

1) Gaposkin [39]
2) Salem [104]
3) Talalyan [126]

4) Walfisz [143]
5) Tandori [136]

dependent on C_n such that the orthogonal series

$$\sum_{n=0}^{\infty} C_n \psi_n(x)$$

is divergent everywhere in $[a, b]$.

For many special orthogonal series the condition for convergence has still better form. Kolmogoroff - Seliverstov¹⁾ and Plessner²⁾ showed that under the condition

$$(1.8.2) \quad \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \log n < \infty,$$

the Fourier series,

$$(1.8.3) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is convergent almost everywhere.

Further, it was proved by Plessner that the condition (1.8.2) is equivalent to

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(x+t) - f(x-t)|^2}{t} dt dx < \infty$$

where $f(x)$ is the function whose Fourier series is (1.8.3).

1) Kolmogoroff - Seliverstov [59]

2) Plessner [97]

Besides the question of convergence, Menchoff and Kaczmarz have discussed the Cesàro summability of (1.8.1). The Fundamental theorem concerning the Cesàro summability of orthogonal series was at first proved by Menchoff¹⁾ and independently also by Kaczmarz²⁾.

They have shown that if $\{w(n)\}$ denotes a positive monotone increasing sequence of numbers whose terms are of order of magnitude $w(n) = o(\log \log n)$ then there exist an orthogonal series

$$\sum_{n=1}^{\infty} a_n \psi_n(x)$$

which is nowhere A - summable although its coefficients satisfy the condition

$$\sum_{n=1}^{\infty} a_n^2 w^2(n) < \infty.$$

Lebesgue functions on the convergence and summability of series :

The question of convergence of orthogonal expansion is also smoothed by means of Lebesgue functions introduced by Lebesgue³⁾, who investigated the influence of these functions on the divergence of Fourier series. The effect of Lebesgue functions on the convergence of Fourier

1) Menchoff ([74][75])

2) Kaczmarz [52]

3) Lebesgue [61]

series was investigated by Kolmogoroff-Seliverstoff¹⁾ and Plessner²⁾.

The idea of Lebesgue functions in the convergence and summability theory of orthogonal series was generalized by Kaczmarz³⁾, Tandori⁴⁾, Meder⁵⁾, Zinovev⁶⁾ Alexits⁷⁾ and Osilenker⁸⁾.

Kaczmarz⁹⁾ has shown that if

$$L_n(x) = O(\lambda_n)$$

where $\lambda(n) \leq \lambda(n+1)$

and $\sum_{n=1}^{\infty} c_n^2 \lambda^2(n) < +\infty$,

then the series (1.8.1) converges almost everywhere. The analogous result for $(C, \alpha > 0)$ summability was also introduced by him.

The order of Lebesgue functions which plays an important role in the convergence theory of orthogonal series was estimated by Moricz¹⁰⁾, Olevskii¹¹⁾, Ratajski¹²⁾ and Alexits¹³⁾.

-
- | | |
|----------------------------------|-----------------------------|
| 1) Kolmogoroff-Seliverstoff [59] | 8) Osilenker [89] |
| 2) Plessner [97] | 9) Kaczmarz [51] |
| 3) Kaczmarz [51] | 10) Moricz [84] |
| 4) Tandori ([127], [135], [137]) | 11) Olevskii ([87], [88]) |
| 5) Meder [77] | 12) Ratajski ([101], [102]) |
| 6) Zinovev [147] | 13) Alexits [5] |
| 7) Alexits [5] | |

The chain of ideas was extended in this field of functional series also by Alexits and Sharma¹⁾, Tandori²⁾ and Moricz³⁾. Alexits and Sharma have proved that, if

$$\sum_{k=0}^{\infty} a_k^2 < \infty$$

and the Lebesgue functions

$$L_n^1(x) = \int_E |k_n^1(t, x)| d\mu(t) \text{ where } k_n^1(t, x) = \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \cdot f_k(t) f_k(x)$$

of the sequence of μ -integrable functions $\{f_n(x)\}$ on μ -measurable set $E \subset x$, which is measurable with a positive measure μ , satisfy the condition $L_n^1(x) = O(\lambda_n)$ uniformly on the measurable set E of finite measure, then the sums

$$\sigma_n(x) = \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) a_k f_k(x)$$

have the order of magnitude $O_x(\lambda_n^{1/2})$ on E almost everywhere. Moreover, they have proved that, if the Lebesgue functions $L_n^1(x)$ are uniformly bounded on the measurable set E of finite measure and

$$\sum a_n^2 < \infty$$

-
- 1) Alexits and Sharma [10]
 - 2) Tandori ([139], [138])
 - 3) Moricz [83]

then the series

$\sum a_n f_n(x)$ is $(C, 1)$ - summable almost everywhere.

Moricz has generalized these theorems of Alexits and Sharma by estimating the order of Lebesgue function corresponding to general summation process.

1.9 Sunouchi¹⁾ has found the order of approximation of

$$(1.9.1) \quad \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(x)|^k}{n}, \quad k > 1$$

under the restriction of boundedness of the function $\phi_n(x)$.

In Chapter II of the thesis we have established the convergence of

$$(1.9.2) \quad \sum_{n=1}^{\infty} \frac{(S_n(x) - \bar{T}_n(x))^2}{n}$$

and

$$\sum_{n=1}^{\infty} \frac{[S_n(x) - \sigma_n^{\alpha}(x)]^2}{n}$$

and also generalize this result as follows.

1) Sunouchi [118]

Theorem: If $p_0 > 0$, $p_n \geq 0$, $np_n = O(p_n)$ and $|\phi_n(x)| \leq k$, then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \bar{T}_n(x)|^q}{n} dx = O(1) \sum_{n=1}^{\infty} |C_n|^q n^{q-2}, q \geq 2.$$

Theorem : If $|\phi_n(x)| \leq k$, $n = 0, 1, 2, \dots$

then

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n^\alpha(x)|^q}{n} dx \leq A \sum_{n=1}^{\infty} n^{q-2} |C_n|^q, q \geq 2.$$

Moreover, in this chapter we have also discussed the convergence of (1.9.2) with logarithmic means. Further we have also discussed the convergence of ,

$$\sum_{n=1}^{\infty} \frac{[S_n(x) - \sigma_n(\lambda, x)]^2}{n^p} < \infty, \quad p > 1.$$

and

$$\sum_{n=1}^{\infty} \frac{[S_n(x) - V_n(x)]^2}{n^p} < \infty, \quad p > 1.$$

in this chapter.

The order of approximation of the type (1.9.1) with Euler and Riesz means was carried out by

Patel¹⁾ and for Nörlund means, it was established by
 Kantawala²⁾.

The approximation of summability means to their
 generating function for the orthogonal series

$$(1.9.3) \quad \sum_{n=0}^{\infty} C_n \phi_n(x)$$

has been studied by Alexits and Kralik³⁾, Leindler⁴⁾ and
 Bolgov and Efimov⁵⁾. Leindler⁴⁾ has proved the
 following theorem:

Theorem A :- If

$$(1.9.4) \quad \sum_{n=1}^{\infty} C_n^2 n^{2\beta} < \infty, \quad (0 < \beta < 1)$$

then

$$\sigma_n(x) - f(x) = O_x(n^{-\beta})$$

holds almost everywhere in (a, b) .

The same result in this direction for Nörlund means and
 Euler means was proved by Kantawala²⁾.

- 1) Patel [94]
- 2) Kantawala [50]
- 3) Alexits and Kralik [7]
- 4) Leindler [66]
- 5) Bolgov and Efimov [23]

In Chapter III we have generalized the above result to (\bar{N}, p_n) means as follows :

Theorem : If $p_n \in \bar{M}^\alpha$, $\alpha > \frac{1}{2}$, then under the condition (1.9.4) the relation

$$\bar{T}_n(x) - f(x) = o_x(n^{-\beta}) \quad (0 < \beta < \frac{1}{2})$$

holds almost everywhere in (a, b) .

1)
Sunouchi has discussed the strong
2)
summability of (1.9.1). Maddox has generalized the
Sunouchi's result by considering the weaker hypothesis.
This result was extended by Kantawala³⁾ for strong Nörlund
summability.

In this chapter, we extend the results of Sunouchi and Maddox to (\bar{N}, p_n) means and Euler means.

In chapter IV we have estimated the order of certain summability means. Alexits⁴⁾ has proved the following theorem :

Theorem B :- If the Lebesgue functions

$$(1.9.5) \quad L_{2^n}(x) = \int_a^b \left| \sum_{k=0}^{2^n} \phi_k(t) \phi_k(x) \right| dt$$

1) Sunouchi [120]

2) Maddox [72]

3) Alexits [5]

4) Kantawala [50]

of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E \subset [a, b]$, then the condition

$$(1.9.6) \quad \sum_{n=0}^{\infty} C_n^2 < \infty$$

implies the $(C, \alpha > 0)$ summability of the orthogonal series (1.9.1) almost everywhere on E .

Kantawala¹⁾ has extended the above result to Riesz and Nörlund summabilities. Here, we extend the result of Kantawala and Alexits for Euler and (\bar{N}, p_n) means. Our theorem are as follows.

Theorem :

If the Lebesgue functions (1.9.5) of ONS $\{\phi_n(x)\}$ are uniformly bounded in the set $E \subset [a, b]$ then the relation (1.9.6) implies the estimate

$$T_n(x) = O_x(n).$$

Theorem :

If $\{p_n\} \in M^\alpha$ and the Lebesgue functions of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E \subset [a, b]$, then the orthogonal series (1.9.1) is (\bar{N}, p_n) summable almost everywhere under the condition (1.9.6).

1) Kantawala [50]

Moreover in this chapter we have also discussed the order of magnitude of Nörlund Lebesgue function and (\bar{N}, p_n) Lebesgue function in the direction of Alexits¹⁾.

Chapter 5 is devoted to estimate the order of Lebesgue functions for polynomial - like ONS, corresponding to Nörlund and (\bar{N}, p_n) means. We have also discussed in this chapter the Nörlund and (\bar{N}, p_n) summability of orthogonal series. These results are extensions of the following results proved by Alexits.

Theorem C :

If the ONS $\{\phi_n(x)\}$ is polynomial - like and the condition

$$\sum_{k=0}^n \phi_k^2(x) = O_x(n)$$

is fulfilled in the set E , then the relation

$$L_n^1(x) = O_x(1)$$

holds almost everywhere in E .

Theorem D :

Let $\{\phi_n(x)\}$ be a complete constant preserving polynomial like ONS with respect to the weight function

1) Alexits ([4], p.206,207)

$\varrho(x)$. Suppose that the function $F_k(t, x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t = x$ and that the two conditions

$$\sum_{k=0}^{\infty} \varrho_k^2(x) = O(n)$$

and

$$(1.9.7) \quad 0 < \varrho(x) \leq \text{constant}$$

are also satisfied in the subinterval $[c, d]$ of $[a, b]$. If the $L^2_{\varrho}(x)$ - integrable function $f(x)$ is continuous in $[c, d]$, then its expansion

$$(1.9.8) \quad f(x) \sim \sum_{n=0}^{\infty} C_n \varphi_n(x)$$

is uniformly $(C, 1)$ - summable in every inner subinterval of $[c, d]$, the sum being $f(x)$.

We are stating below two of the theorems proved by us.

Theorem :

If the ONS $\{\varphi_n(x)\}$ is polynomial like and the condition

$$\varphi_n(x) = O_x(1)$$

is fulfilled in the set E , then the relation

$$\bar{Q}_n(x) = O_x(1)$$

holds almost everywhere on E .

Theorem :

Let $\{\phi_n(x)\}$ be a complete constant preserving polynomial - like ONS with respect to the weight function $g(x)$. Suppose that the functions $F_k(t, x)$ are continuous in the square $a \leq t \leq b$, $a \leq x \leq b$ with eventual exception of the diagonal $t = x$ and that the two conditions.

$$\phi_n(x) = O(1)$$

and (1.9.7) are also satisfied in the subinterval $[C, d]$ of $[a, b]$. If the $L^2_{g(x)}$ - integrable function $f(x)$ is continuous in $[C, d]$, then its expansion (1.9.8) is uniformly (\bar{N}, p_n) summable in every inner subinterval of $[C, d]$, the sum being $f(x)$.

1.10 Absolute Summabilities of Orthogonal Series :

Absolute summability of Fourier - trigonometric series by Cesàro, Nörlund and Riesz means have been engaging the attention of a large number of workers in this line. A systematic account of the available literature on absolute summability of a Fourier trigonometric series has been given by Prasad¹⁾.

1) Prasad [99].

In case of Fourier orthogonal expansion the earliest result on $|C, \alpha|$ - summability are due to Tsuchikura¹⁾ and Tandori²⁾. Tandori³⁾ has proved that the condition

$$(1.10.1) \quad \sum_{m=0}^{\infty} \left(\sum_{k=2^m+1}^{2^{m+1}} c_k^2 \right)^{\frac{1}{2}} < \infty$$

is necessary and sufficient for $|C, 1|$ - summability of (1.9.3).

The necessity was later on extended by Billard⁴⁾, Leindler⁵⁾, Grepcevskaja⁶⁾ and Patel⁷⁾ have extended Tandori's theorem to $|C, \alpha|$ - summability. Szalay⁸⁾ has generalized these theorems for generalized absolute Cesàro summability.

Absolute Euler summability of orthogonal series has been studied by Patel⁹⁾ and Bhatnagar¹⁰⁾. Absolute Riesz summability of orthogonal series was discussed by Alexits and Kralik¹¹⁾, Moricz¹²⁾ and P. Srivastava¹³⁾, while considering the absolute Nörlund summability, Meder¹⁴⁾ has proved the following theorem.

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|---------------------------|----------------------------|
| 1) Tsuchikura [141] | 8) Szalay [124] |
| 2) Tandori ([131], [133]) | 9) Patel [94] |
| 3) Tandori [131] | 10) Bhatnagar [117] |
| 4) Billard [20] | 11) Alexits and Kralik [6] |
| 5) Leindler [62] | 12) Moricz [84] |
| 6) Grepcevskaja [41] | 13) P. Srivastava [113] |
| 7) Patel [94] | 14) Meder [81] |

Theorem E :

Let $\{p_n\} \in \bar{M}^\alpha$, $\alpha > \frac{1}{2}$ then (1.10.1) is the necessary and sufficient condition for the series (1.9.3) to $|N, p_n|$ summable in the interval $[0, 1]$.

In chapter VI we prove the same result of Meder under the weaker condition for Nörlund means, (\bar{N}, p_n) means and $|V, \lambda|$ means. Moreover in this chapter we have also discussed absolute harmonic summability. We prove,

Theorem :

If $np_n = O(p_n)$, then

$$(1.10.2) \quad \sum_{m=0}^{\infty} \left\{ \sum_{v=0}^m c_v^2 \right\}^{\frac{1}{2}} < \infty$$

implies the $|N, p_n|$ summability of (1.9.3).

Theorem :

If $np_n = O(p_n)$ then (1.10.2) implies the $|\bar{N}, p_n|$ summability of (1.9.3).

Theorem :

Let $\lambda = \{\lambda_n\}$ be a monotonic nondecreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$ and $\lambda_1 = 1$, then

$$\sum_{n=1}^{\infty} \left\{ \sum_{k=n-\lambda_n}^{n+1} c_k^2 \right\}^{\frac{1}{2}} < \infty$$

implies the $|V, \lambda|$ summability of (1.9.3).

1.11 Convergence and summability of orthogonal polynomial

Series :

The series of orthogonal polynomials i.e. the series

$$(1.11.1) \quad \sum_{n=0}^{\infty} C_n p_n(x)$$

carry their own importance. The convergence and Cesàro summability was first discussed in great details by

1) Jackson¹⁾. Subsequent papers in this line are due to Chen²⁾ and Freud³⁾. Regarding the $[C, 1]$ summability of the series (1.11.1) Jackson⁴⁾ showed that if the weight function $\varphi(x)$ is bounded and $\varphi(t) \varphi^2(t)$ is summable in the interval $[-1, 1]$ then the series (1.11.1) is summable $[C, 1]$ to the generating function $f(x)$, $\varphi(t)$ being given by

$$\frac{f(t) - f(x)}{t - x}$$

In chapter VII of our thesis we have generalized this result to the case of Nörlund and (\bar{N}, p_n) means. Patel⁴⁾ has proved the same result of Jackson for Riesz and Euler means.

1) Jackson [47]

2) Chen [27]

3) Freud [36]

4) Patel [94]

Jackson¹⁾ and Alexits²⁾ have estimated the degree of approximation with the help of trigonometrical system. Moreover in this chapter we have also discussed the degree of approximation for orthogonal polynomial system in the direction of Alexits³⁾, he has proved the theorem for orthogonal system.

Some of the theorems were proved by different authors for convergence and summability of general orthogonal series, e.g. Alexits and Kralik⁴⁾, Tandori⁵⁾ and Leindler⁶⁾. Further in this chapter we extend the above results of Tandori, Alexits and Kralik for orthogonal polynomial system under weaker condition.

1.12 Absolute convergence of orthogonal series :

Absolute convergence of Fourier series has been studied in great details by Bernstein⁷⁾, Bary⁸⁾, Zygmund⁹⁾ & Stetchkin S.B.¹⁰⁾. Absolute convergence of orthogonal series has been discussed by Stetchkin¹¹⁾, Alexits¹²⁾, Bochkarev¹³⁾, Zinovev¹⁴⁾, Tandori¹⁵⁾.

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|-----------------------|-----------------------|--------------------|-------|
| 1) Jackson | [46] | 9) Zygmund | [150] |
| 2) Alexits | [5] | 10) Stetchkin S.B. | [117] |
| 3) Alexits | [5] | 11) Stetchkin | [114] |
| 4) Alexits and Kralik | [7] | 12) Alexits | [5] |
| 5) Tandori | ([128], [129], [130]) | 13) Bochkarev | [22] |
| 6) Leindler | ([66], [67], [68]) | 14) Zinovev | [146] |
| 7) Bernstein | [13] | 15) Tandori | [134] |
| 8) Bary | [12] | | |

In chapter VIII we have discussed the absolute convergence of orthogonal series. We extend the results of Szasz's¹⁾ for Fourier series to orthogonal expansion by proving the following theorems:

Theorem :

Let $f \in L^2[a, b]$ and $f(x) \sim \sum c_n \phi_n(x)$

be its orthonormal expansion. Then

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \quad {}^{(2)} w \left(\frac{1}{k}, f \right) < \infty$$

implies the absolute convergence of (1.9.8).

Theorem :

If $f(x)$ is of bounded variation, then

$$\sum_{k=1}^{\infty} \frac{\sqrt{w \left(\frac{1}{k}, f \right)}}{k} < \infty$$

implies the absolute convergence of (1.9.8).

Moreover in this chapter we have also discussed some results on the absolute convergence of orthogonal expansions of the function of certain class and orthogonal polynomial expansion in the direction of Alexits²⁾.

1.13 In chapter IX of our thesis, we have discussed the summability and convergence of $\lambda(n)$ - lacunary

1) Szasz [123]

2) Alexits [5]

orthogonal series. Alexits¹⁾ has proved the following theorem for Cesàro summability of $\lambda(n)$ - lacunary orthogonal series :

Theorem :

If the coefficients of $\lambda(n)$ - lacunary orthogonal series (1.9.3) have as a majorant a positive monotone decreasing number sequence $\{q_n\}$, satisfying the condition.

$$(1.12.1) \quad \sum_{n=1}^{\infty} \frac{\sqrt{\lambda(n)} q_n}{n} < \infty$$

then the condition

$$\sum_{n=0}^{\infty} c_n^2 < \infty$$

implies the (C, α) summability almost everywhere of the orthogonal series (1.9.3). The same result was extended by Bhatnagar²⁾ and Kantawala³⁾ for Euler means and Nörlund means. In this chapter we proved the analogous result for Riesz summability and (\bar{N}, p_n) summability of $\lambda(n)$ - lacunary orthogonal series.

We have also discussed convergence of $\lambda(n)$ - lacunary orthogonal series in the direction of Alexits⁴⁾.

- 1) Alexits [5]
- 2) Bhatnagar [17]
- 3) Kantawala [50]
- 4) Alexits [5]