CHAPTER - 3

STRONG APPROXIMATION OF ORTHOGONAL SERIES

Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be an orthonormal system (ONS) of L^2 - integrable functions defined in the closed interval $[a, b]$. We consider the orthogonal series.

(3.1.1) $$\sum_{n=0}^{\infty} C_n \phi_n(x)$$

with real coefficients C_n's.

Let us denote the partial sums, $(\bar{N}, p_n)$ means, and Euler means of the series (3.1.1) by

$$S_n(x) = \sum_{v=0}^{n} C_v \phi_v(x)$$

$$\bar{T}_n(x) = \frac{1}{P_n} \sum_{v=0}^{n} p_v S_v(x)$$

$$T_n(x) = \frac{1}{2^n} \sum_{v=0}^{n} \binom{n}{v} S_v(x),$$

respectively, where $P_n = p_0 + p_1 + p_2 + \ldots + p_n$, $p_0 > 0$, $p_n \geqslant 0$.

The series (3.1.1) is said to be $(\bar{N}, p_n)$ summable to S if

$$\lim_{n \to \infty} \bar{T}_n(x) = S.$$

The series (3.1.1) is said to be (E, 1) summable to S if

$$\lim_{n \to \infty} T_n(x) = S.$$

The sequence $\{p_n\}$ will be said to belong to the class M^α for a certain real $\alpha \geq 0$, if

(i) $0 < p_n < p_{n+1}$ for $n = 0, 1, 2, \ldots$

or $0 < p_{n+1} < p_n$ for $n = 0, 1, 2, \ldots$

(ii) $p_0 + p_1 + \ldots + p_n = P_n \uparrow \infty$

(iii) $\lim_{n \to \infty} \frac{np_n}{P_n} = \alpha$.

The n^{th} (C, 1) - means of the orthogonal series (3.1.1) have been approximated by Tandori[1], Meder[2], Alexits and Kralik[3] and Leindler[4]. Leindler[5] approximated the de-la Valle'e Poussion mean of the orthogonal series (3.1.1). The Riesz means were

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1) Tandori [133]
2) Meder J. [82]
3) Alexits and Kralik [6]
4) Leindler ([65], [66])
5) Leindler ([64], [65], [66])

approximated by Leindler[1]. Later on the above results were generalized to strong approximation of $(C, \alpha > o)$-means by Sunouchi[2] and Leindler[3]. Bolgov and Efimov[4] have generalized the above results to the means generated by triangular matrices.

Leindler[5] has proved the following theorem.

<u>Theorem A</u> :- If

$$(3.1.2) \qquad \sum_{n=1}^{\infty} C_n^{\,2}\, n^{2\beta} < \infty , \quad (o < \beta < 1),$$

then

$$\sigma_n(x) - f(x) = o_x\,(n^{-\beta})$$

holds almost everywhere in (a, b).

Similarly Kantawala P.S.[6] has generalized the above theorem to Nörlund summability. In this chapter we extend the above result of Leindler to $(\bar{N}, p_n)$ means.

<u>Theorem 1</u> :- <u>If</u> $p_n \in \bar{M}^{\alpha}$, $\alpha > \frac{1}{2}$, <u>then under the the condition (3.1.2), the relation</u>.

$$\bar{T}_n(x) - f(x) = o_x\,(n^{-\beta}) \quad (o < \beta < \tfrac{1}{2})$$

<u>holds almost everywhere in (a, b)</u>.

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1) Lindler [64]
2) Sunouchi G.[119]
3) Lindler [67]
4) Bolgov and Efimov [23]
5) Leindler [66]
6) Kantawala P.S. [1]

The n^{th} (C, 1) - means of Fourier series and the Walsh expansion of the function f(x) satisfying the Liptchiz condition were approximated by Bernstein[1] and Fine[2] respectively. The strong (C, 1) summability of Fourier series, conjugate Fourier series and orthogonal series was investigated by Alexits[3], Alexits and Kralik[4], Sun young Sheng[5], Alexits and Leindler[6] and Turan[7].

Strong Cesàro summability of orthogonal series (3.1.1) was discussed by Sunouchi[8]. He has proved the following theorem :

Theorem A :- If the orthogonal series (3.1.1) with

$$(3.1.3) \qquad \sum_{n=o}^{\infty} c_n^2 < \infty$$

is (C, 1) - summable to f(x) almost everywhere in [a, b], then

$$\lim_{n \to \infty} \frac{1}{A_n^{\beta}} \sum_{v=o}^{n} A_{n-v}^{\beta-1} \left| S_v(x) - f(x) \right|^k = o$$

almost everywhere in [a, b] for any $\beta > o$ and $k > o$.

Maddox[9] has generalized Sunouchi's result which

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1) Bernstein [14]
2) Fine [32]
3) Alexits ([4], [5])
4) Alexits and Kralik [8]
5) Sun young sheng [121]
6) Alexits and Leindler [9]
7) Turan [142]
8) Sunouchi [12o]
9) Maddox [72]

concerns with the weakening of the hypothesis rather than strengthening of the conclusion by proving the following theorem :

Theorem B :- Let

$$\sum \frac{\lambda_n}{\lambda_{n+1}} C_n^2 < \infty$$

and suppose that for $k > o$, the sequence $\left\{C^k (\lambda_{n+1})\right\}$ corresponding to the orthogonal series (3.1.1) is summable $[R, \lambda, 1, 2]$ to $f(x)$ almost everywhere on $[a, b]$. Then for any sequence $\{\mu_m\}$ with $o < \inf \mu_m \leqslant \mu_m \leqslant 2$, the series (3.1.1) is $[R, \lambda, 1, \mu]$ summable to $f(x)$ almost everywhere on $[a, b]$.

Similar result for Nörlund summability was proved by Kantawala P.S.[1)] In this chapter we extend the results of Maddox for strong Euler and Strong $(\bar{N}, p_n)$ summability. Our theorem are as follows :-

Theorem 2 :- If (3.1.1) is $(\bar{N}, p_n)$ summable to $f(x)$ almost everywhere and the condition $np_n = O(P_n)$ is true then the condition (3.1.3) implies

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1) Kantawala [50]

$$\lim_{n \to \infty} \frac{1}{P_n} \sum_{m=o}^{n} p_m \left| S_m(x) - f(x) \right|^{\mu_m} = o$$

holds almost everywhere for any sequence $\{\mu_m\}$ with

$$o < \inf \mu_m \leq \mu_m \leq 2 .$$

Theorem 3 :- If the series (3.1.1) is (E, 1) summable to f(x) almost everywhere then the condition

$$(3.1.4) \qquad \sum_{n=1}^{\infty} C_n^2 \sqrt{n} < \infty$$

implies

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{m=o}^{n} \binom{n}{m} \left| S_m(x) - f(x) \right|^{\mu_m} = o$$

holds almost everywhere for any sequence $\{\mu_m\}$ with

$$o < \inf \mu_m \leq \mu_m \leq 2 .$$

For proving this theorems we need the following Lemmas :

Lemma 1 :-[1] If $\{p_n\} \in M^{\alpha}$, $\alpha > \frac{1}{2}$. then

$$\lim_{n \to \infty} \frac{n}{P_n^2} \sum_{k=o}^{n} \frac{p_k^2}{(k+1)^2} = \frac{1}{2\alpha - 1} .$$

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1) Meder J. [78].

Lemma 2 :-[2] If $\{p_n\} \in \bar{M}^{\alpha}$, $\alpha > \frac{1}{2}$, then under the condition

$$\sum_{n=1}^{\infty} c_n^2 n^{2\beta} < \infty \qquad (o < \beta < 1)$$

the relation,

$$t_n(x) - f(x) = o_x (n^{-\beta})$$

holds almost everywhere in (a, b).

Lemma 3 :-[1] Let $\{p_n\}$ be a nonnegative monotonic increasing or decreasing sequence of real numbers with $np_n = O(P_n)$ and $P_n \rightarrow \infty$ as $n \rightarrow \infty$ if the orthogonal series (3.1.1) is $(\bar{N}, p_n)$ summable almost everywhere to a function f(x) then

$$\lim_{n \rightarrow \infty} \frac{1}{P_n} \sum_{k=o}^{n} p_k [S_k(x) - f(x)]^2 = o$$

almost everywhere on [a, b].

Lemma 4 :-[3] If the series (3.1.1) with coefficients satisfying the condition (3.1.4), is summable by the

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1) Sharma [110]
2) Kantawala [50]
3) Patel R.K. [95]

method (E, 1) to a function $f(x)$. Then

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=o}^{n} \binom{n}{k} [S_k(x) - f(x)]^2 = o$$

holds almost everywhere on $[a, b]$.

Proof of Theorem 1 :-

$$| \bar{T}_n(x) - f(x) | = | \bar{T}_n(x) - \bar{T}_{2^n}(x) + \bar{T}_{2^n}(x) - t_{2^n}(x) + t_{2^n}(x) - f(x) |$$

$$\leqslant | \bar{T}_n(x) - \bar{T}_{2^n}(x) | + | \bar{T}_{2^n}(x) - t_{2^n}(x) | + | t_{2^n}(x) - f(x) |$$

Now,

$$\bar{T}_n(x) - t_n(x) = \frac{1}{P_n} \sum_{v=o}^{n} p_v S_v(x) - \frac{1}{P_n} \sum_{v=o}^{n} p_{n-v} S_v(x)$$

$$= \frac{1}{P_n} \sum_{v=o}^{n} (p_v - p_{n-v}) S_v(x)$$

$$= \frac{1}{P_n} \sum_{v=o}^{n} (p_v - p_{n-v}) \sum_{k=o}^{v} C_k \phi_k(x).$$

Now, by Schwarz inequality, we have

$$\int_a^b [\bar{T}_{2^n}(x) - t_{2^n}(x)]^2 dx \leqslant \frac{1}{P_{2^n}^2} \int_a^b \sum_{v=o}^{2^n} (p_{2^n-v} - p_v)^2 \sum_{v=o}^{2^n} \left\{ \sum_{k=o}^{v} C_k \phi_k(x) \right\}^2 dx$$

$$= \frac{1}{P_{2^n}^2} \sum_{v=0}^{2^n} (p_{2^n-v} - p_v)^2 \sum_{v=0}^{2^n} \sum_{k=0}^{v} C_k^2$$

$$= \frac{O(1)}{P_{2^n}^2} \sum_{v=0}^{2^n} (p_{2^n-v} - p_v)^2$$

$$= \frac{O(1)}{P_{2^n}^2} \sum_{v=0}^{2^n} p_v^2 .$$

Therefore,

$$\int_a^b \sum_{n=1}^{\infty} 2^{2n\beta} (\bar{T}_{2^n} (x) - t_{2^n} (x))^2 dx$$

$$= O(1) \sum_{n=1}^{\infty} \frac{2^{2n\beta}}{P_{2^n}^2} \sum_{v=0}^{2^n} p_v^2$$

$$= O(1) \sum_{n=1}^{\infty} \frac{2^{2n\beta}}{P_{2^n}^2} 2^n \sum_{v=0}^{2^n} \frac{P_v^2}{(v+1)^2} \frac{1}{2^n}$$

$$= O(1) \sum_{n=1}^{\infty} \frac{2^{2n\beta}}{2^n}$$

$$< \infty \quad \text{by Lemma 1,}$$

holds almost everywhere in (a, b).

Therefore,

(3.1.5) $$\bar{T}_{2^n}(x) - t_{2^n}(x) = o_x (2^{-n\beta}),$$

holds almost everywhere in (a, b).

By Lemma 2 we have,

$$t_n(x) - f(x) = o_x(n^{-\beta})$$

holds almost everywhere in (a, b).

i.e.

$$(3.1.6) \qquad t_{2^n}(x) - f(x) = o_x(2^{-n\beta})$$

holds almost everywhere in (a, b).

We have,

$$\bar{T}_n(x) - \bar{T}_{n-1}(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k S_k(x) - \frac{1}{P_{n-1}}\sum_{k=0}^{n-1} p_k S_k(x).$$

$$= \frac{1}{P_n}\sum_{k=0}^{n-1} p_k S_k(x) - \frac{1}{P_{n-1}}\sum_{k=0}^{n-1} p_k S_k(x) + \frac{p_n}{P_n} S_n(x)$$

$$= \left(\frac{1}{P_n} - \frac{1}{P_{n-1}}\right)\sum_{k=0}^{n-1} p_k S_k(x) + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{-p_n}{P_n P_{n-1}}\sum_{k=0}^{n-1} p_k \sum_{v=0}^{k} C_v \phi_v(x) + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{-p_n}{P_n P_{n-1}}\sum_{v=0}^{n-1} C_v \phi_v(x) \sum_{k=v}^{n-1} p_k + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{-p_n}{P_n P_{n-1}} \sum_{v=o}^{n-1} C_v \phi_v(x) \sum_{k=o}^{n-1} p_k + \frac{p_n}{P_n P_{n-1}} \sum_{v=o}^{n-1} C_v \phi_v(x) \sum_{k=o}^{v-1} p_k + \frac{p_n}{P_n} S_n(x)$$

$$= \frac{p_n}{P_n} (S_n(x) - S_{n-1}(x)) + \frac{p_n}{P_n P_{n-1}} \sum_{v=o}^{n-1} C_v \phi_v(x) P_{v-1}$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=o}^{n} C_v \phi_v(x) P_{v-1} .$$

Thus we have,

$$\int_a^b (\bar{T}_n(x) - \bar{T}_{n-1}(x))^2 dx = \frac{p_n^2}{P_n^2 P_{n-1}^2} \sum_{v=o}^{n} C_v^2 P_{v-1}^2$$

$$= \frac{p_n^2}{P_n^2 P_{n-1}^2} \sum_{v=o}^{n} C_v^2 \left(\sum_{i=o}^{v-1} p_i \right)^2$$

But by Schwarz inequality,

$$\left(\sum_{i=o}^{v-1} p_i \right)^2 \leqslant v \sum_{i=o}^{v-1} p_i^2$$

So we have,

$$\int_a^b (\bar{T}_n(x) - \bar{T}_{n-1}(x))^2 dx \leqslant \frac{p_n^2}{P_n^2 P_{n-1}^2} \sum_{v=o}^{n} P_v^2 \frac{v}{P_v^2} \left(\sum_{i=o}^{v} p_i \right)^2 C_v^2$$

$$= O(1)\, \frac{p_n^2}{P_n^2 P_{n-1}^2} \sum_{v=o}^{n} P_v^2\, C_v^2, \text{(by Lemma 1)}$$

$$= O(1)\, \frac{1}{n^2} \sum_{v=o}^{n} C_v^2$$

Hence,

$$\sum_{n=1}^{\infty} n^{2\beta+1} \int_a^b (\overline{T}_n(x) - \overline{T}_{n-1}(x))^2\, dx$$

$$= O(1) \sum_{n=1}^{\infty} \frac{n^{2\beta+1}}{n^2} \sum_{v=o}^{n} C_v^2$$

$$= O(1) \sum_{v=o}^{\infty} C_v^2 \sum_{n=v}^{\infty} n^{2\beta-1}$$

$$= O(1) \sum_{v=o}^{\infty} C_v^2$$

$$< \infty, \text{ if } \beta < \frac{1}{2} .$$

So by B. Levy's theorem,

$$\sum_{n=1}^{\infty} n^{2\beta+1} (\overline{T}_n(x) - \overline{T}_{n-1}(x))^2 < \infty$$

almost everywhere in (a, b).

Consequently for $2^m < n < 2^{m+1}$, we have

$$| \overline{T}_n(x) - \overline{T}_{2^m}(x)| = | \sum_{k=2^m+1}^{n} (\overline{T}_k(x) - \overline{T}_{k-1}(x) |$$

$$\leq \left\{ \sum_{k=2^m+1}^{2^{m+1}} k^{2\beta+1} \; (\bar{T}_k(x) - \bar{T}_{k-1}(x))^2 \; \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k^{2\beta+1}} \right\}^{\frac{1}{2}}$$

$$= o_x \, (2^{-\beta m})$$

$$= o_x \, (n^{-\beta})$$

holds almost everywhere in (a, b), i.e. the relation

$$(3.1.7) \qquad | \bar{T}_n(x) - \bar{T}_{2^m}(x) | = o_x \, (n^{-\beta})$$

holds for $2^m < n < 2^{m+1}$ almost everywhere in (a, b).

Therefore by (3.1.5), (3.1.6) and (3.1.7) we have the relation

$$| \bar{T}_n(x) - f(x) | = o_x \, (n^{-\beta})$$

holds almost everywhere in (a, b). This proves the theorem.

Proof of Theorem 2 :

By Lemma 3 we have,

$$\frac{1}{P_n} \sum_{m=o}^{n} p_m \, [\, S_m(x) - f(x) \,]^2 \longrightarrow o \quad \text{as } n \longrightarrow \infty .$$

almost everywhere on $[a, b]$.

Denote,

$$t_m = | S_m - f |$$

Then we have from hypothesis $o < C < \frac{\mu_m}{2} \leq 1$ for some C.

Hence,

$$t_m^{\mu_m} \leq t_m^{2} \quad \text{if} \quad t_m \geq 1$$

and

$$t_m^{\mu_m} \leq t_m^{2C} \quad \text{if} \quad t_m < 1 .$$

If $t_m < 1$, then by Hölders inequality, we have,

$$\frac{1}{P_n} \sum_{m=o}^{n} p_m \; t_m^{2C}$$

$$= \sum_{m=o}^{n} \left(\frac{p_m}{P_n}\right)^{C} t_m^{2C} \left(\frac{p_m}{P_n}\right)^{1-C}$$

$$\leq \left[\sum_{m=o}^{n} \left(\frac{p_m}{P_n}\right) t_m^{2} \right]^{C} \left[\sum_{m=o}^{n} \frac{p_m}{P_n} \right]^{1-C}$$

$$= \left[\sum_{m=o}^{n} \frac{p_m}{P_n} t_m^{2} \right]^{C}$$

$$\leq \left[\frac{1}{P_n} \sum_{m=o}^{n} p_m \, t_m^{2} \right]^{C}$$

Hence,

$$\frac{1}{P_n} \sum_{m=o}^{n} p_m \, t_m^{\mu_m} \longrightarrow o$$

as $n \longrightarrow \infty$ due to Lemma 3.

Also, if $t_m \geqslant 1$, this limit is obviously true.

Hence,

$$\frac{1}{P_n} \sum_{m=o}^{n} p_m \, | S_m(x) - f(x) |^{\mu_m} \to o \quad \text{as} \quad n \longrightarrow \infty,$$

almost everywhere in [a, b].

With this the theorem is proved.

<u>Proof of Theorem 3</u> :-

By Lemma 4 we have,

$$\frac{1}{2^n} \sum_{m=o}^{n} \binom{n}{m} (S_m(x) - f(x))^2 \longrightarrow o \quad \text{as } n \longrightarrow \infty$$

almost everywhere on [a, b].

Now write,

$$t_m = | S_m - f | .$$

Then, we have from hypothesis $o < C \leqslant \frac{\mu_m}{2} \leqslant 1$ for some C.

Hence,

$$t_m^{\mu_m} \leqslant t_m^2 \quad \text{if} \quad t_m \geqslant 1$$

and

$$t_m^{\mu_m} \leqslant t_m^{2C} \quad \text{if} \quad t_m < 1 .$$

i.e.

$$t_m^{\mu_m} \leq t_m^2 + v_m^C \tag{3.1.9}$$

where

$$v_m = 0 \quad \text{if} \quad t_m \geq 1$$

and

$$v_m = t_m^2 \quad \text{if} \quad t_m < 1.$$

By Hölders inequality, we have,

$$\frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m} v_m^C$$

$$\leq \frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m} t_m^{2C}$$

$$= \sum_{m=0}^{n} \left\{ \frac{\binom{n}{m}}{2^n} \right\}^C t_m^{2C} \left\{ \frac{\binom{n}{m}}{2^n} \right\}^{1-C}$$

$$\leq \left\{ \sum_{m=0}^{n} \frac{\binom{n}{m}}{2^n} t_m^2 \right\}^C \left\{ \sum_{m=0}^{n} \frac{\binom{n}{m}}{2^n} \right\}^{1-C}$$

$$= \left\{ \frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m} t_m^2 \right\}^C$$

So,

$$\frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m} v_m^C \leq \left\{ \frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m} t_m^2 \right\}^C \longrightarrow 0$$

as $n \longrightarrow \infty$ due to Lemma 4.

Hence from (3.1.9)

$$\frac{1}{2^n} \sum_{m=o}^{n} \binom{n}{m} t_m^{\mu_m} \longrightarrow o \qquad \text{as} \quad n \longrightarrow \infty$$

almost everywhere on [a, b].

i.e.

$$\frac{1}{2^n} \sum_{m=o}^{n} \binom{n}{m} \mid S_m(x) - f(x) \mid^{\mu_m} \longrightarrow o$$

as $n \longrightarrow \infty$ almost everywhere on [a, b].

This complets the proof of our theorem.